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A note on a Comparative Analysis of Nonrelativistic Limit of the Klein-Gordon and Dirac Equations.

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Abstract
In this note we have carefully presented a comparative analysis of nonrelativistic limit of both Klein- Gordon and Dirac equations. For the duo, their free particle solutions were deduced from where it was observed that they had similar results in some aspects especially in energy of the particle. Further analysis was on their behavior when it comes to nonrelativistic limit where it was observed that in nonrelativistic limit, Klein-Gordon equation approximate to Schrodinger equation for electromagnetic potential whose wave function can invariably be transformed into Lorentz transformation in accordance to the transformation laws of wave function that gives a solution that is in line with a wave function characterizing a spin-0 particle while Dirac equation on the other hands was transformed into the Pauli's equation that presents the proper nonrelativistic wave equation for spin- $1/2$ *particles that exists at both low and as well as at high velocities respectively.*

Keywords; *Note, Comparative analysis, Klein-Gordon equation, Dirac Equation, particle Nonelativistic, limit, Schrodinger Equation, Wave function, Particle's Spin, Transformation.*

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I. Introduction

Klein-Gordon and Dirac equations were introduced in quantum mechanical study as support to Schrodinger equation. The duo especially Klein-Gordon could be labeled Schrodinger relativistic wave equation that were proposed by so many physicists at the early days of introduction of quantum mechanics [1] and many authors have used different methods to study both exactly and quasi-exactly solvable Schrödinger, Klein-Gordon and Dirac equations in the presence of variable mass even as all of them do not present the same structure. In course of application of the Klein-Gordon equation, some anomalies were pointed out based on the result of solution which off course was notably based on the difference in structure of the equation as it was found that KG is second order derivative both in time and position rendered it difficult in probabilistic interpretation, but it was expected the an ideal relativistic wave equation is the one that is first order derivative in time with positive definite probabilistic interpretation [2]. These observations prompted Dirac to construct his own form of relativistic wave equation by circumventing all those factors that made Klein-Gordo equation's solution inefficient in local probability interpretation of a particle as expected [3]. However, despite the faults as pointed out by Dirac, Klein-Gordon equation is not to be completely condemned in its entirety because it is applicable and acceptable when it comes to study of motion of massive spineless particle and more so it resurfaces in quantum field theory[4];[5].A comparative study of free Klein-Gordon particle and free classical particle has been carried out and from the analysis of the study, it has been observed that the results are similar [6]; [7]. In any case in some aspect of results, from both Dirac and Klein-Gordon equations, it was observed that they exhibited also the same characteristics especially when it comes to their display of positive and negative values in energy [8].The use of the duo in the study of particle in external electromagnetic field and the behavior of the particles when subjected to extremely restricted potential[9] using anyof the equations had been carried out.[10] coupled with the analysis of their spectral convergence characteristic[11] in which both had been studied.[12];[13].A mathematical proof bad been conducted for the Klein-Gordon and Dirac equations within the domain of relativistic quantum mechanics where it was observed that both have the correct nonrelativistic limits with further study on the proof as it applies to their physical behavior in an external electromagnetic field for which it was noted that there is clear cut differential characteristic particularly when it comes to the case of their restrictions on the size of the field [10] especially when considering only the free-field. On applying transformations to facilitate study of the nonrelativistic limiting behavior, both equations as carried out by some researchers, there must caution as everything has to be presented in operator form in order for solution to be favorable as required. The study of relativistic effects is always useful in some quantum mechanical system

Therefore, the Dirac equation has become the most appealing relativistic wave equation for spin-1/2 particles as allegedly compared to Klein-Gordon equation. For example, in the relativistic treatment of nuclear study their relativistic effects are always of interest quantum mechanically. [14];[15];[16] [17] ; [18] and as such we are poised to analyzed the behavior of the Dirac equation as compared to its counterpart Klein- Gordon equation as it concern their nonrelativistic limit in order to ascertain the comparability and their limitations within that limit.

II. Mathematical Supplement of Klein-Gordon and Dirac Equations

In this section, we first of all present the mathematical analysis of the two equations as regards the way each showcased its concept concerning free particles.

2.1 Particle Consideration based on Klein-Gordon Equation

From the free particle equation of Klein-Gordon equation, we write

$$\hat{P}^{\mu}\hat{P}_{\mu}\psi = m_o^2c^2\psi \qquad (1)$$

From where the contravariant four vectors transform relating to the four momentum operators are related to as

$$\hat{P}^{\mu}\hat{P}_{\mu} = -\hbar\, {\partial}/{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}} = -\hbar\left[\frac{1}{c^2}{\partial^2}/{\partial t^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^{2a}}\right)\right]$$

$$\equiv -\hbar^2\nabla^2 = \hbar^2\left({1}/{c^2}\frac{\partial}{\partial t^2}^{2} - \Delta\right) \qquad (2)$$

Substituting (2) in (1) yields $\left(\nabla^2 + \frac{m_o^2c^2}{\hbar^2}\right)\psi = \left[\frac{\partial^2}{c\partial t^2}+\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{m_o^2c^2}{\hbar^2}\right]\psi = o$ (3)

Using Lorentz covariance on Klein-Gordon equation, the free particle solution is of the form

$$\psi = \exp\left(-\frac{i}{\hbar}p_{\mu}x^{\mu}\right) = \exp\left\{-\frac{i}{\hbar}\left(p_o x^o - px\right)\right\}$$

$$= \exp\left[+\frac{i}{\hbar}\left(p.x - Et\right)\right] \qquad (4)$$

Substituting equation (4) in (3), we obtain

$$\hat{P}\mu\hat{P}^{\mu}\psi = m_o c^2\psi \rightarrow \hat{P}\mu\hat{P}^{\mu}\exp\left(\frac{i}{\hbar}p_{\mu}x^{\mu}\right)$$

$$= m_o c^2\exp\left(-\frac{i}{\hbar}p_{\mu}x^{\mu}\right) = \frac{E^2}{c^2} - P.P = m_o c^2$$

The above expression results to $E = \pm\sqrt{m_o^2c^2 + p^2}$ (5)

This indicates that E has two distinct values of energy, one in positive and the other in negatives domain respectively

In Fesbach-Villar representation where Φ is the eigenstate of the momentum operator, the solution Klein-Gordon equation is of the form

$$i\hbar\frac{\partial\Phi}{\partial t} = \hat{\tau}_3 E_p\Phi \qquad (6)$$

Gives two different solutions of momentum whereby one is positive (+ E_p) and the other negative (- E_p)

$$\Phi = \exp\left((ip.{r}/{\hbar}\right)\Theta, \Theta = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \qquad (7)$$

And since $E_p = \sqrt{m_o^2c^4 + p^2c^2}$

By inserting the matrix, $\hat{\tau}_3$ we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}=E_p\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \quad (8)$$

This result to

$$i\hbar\nu_1=E_p\nu_1, i\hbar\nu_2=-E_p\nu_2 \quad (9)$$

The integration of Θ_1 and Θ_2 give the expression of the form

$$\nu_1=N_1\exp\left(-\frac{i}{\hbar}E_pt\right) \text{ and } \nu_2=N_2\exp\left(+\frac{i}{\hbar}E_pt\right) \quad (10)$$

Where N_1 and N_2 are normalization constants which are determined by normalization condition

$$\int\phi^{\dagger}\hat{\tau}\phi d^3r=\int\Theta^{\dagger}\hat{\tau}\Theta d^3r=1 \quad (11)$$

Yielding $\left|N_1\right|^2=\left|N_2\right|^2=\frac{1}{\nu}$

This results to two independent solutions such as

$$\Phi^{(+1)}=\frac{1}{\sqrt{\nu}}\begin{pmatrix}1\\0\end{pmatrix}\exp\left[\frac{i}{\hbar}\left(p.r-E_pt\right)\right], charg\,e+1 \quad (12)$$

And

$$\Phi^{(-1)}=\frac{1}{\sqrt{\nu}}\begin{pmatrix}0\\1\end{pmatrix}\exp\left[\frac{i}{\hbar}p.r+E_pt\right], charg\,e-1 \quad (13)$$

Each linear combination; $n_1\Phi^{(+)}+n_2\Phi^{(-1)}$ with $\left|n_1\right|^2-\left|n_2\right|^2=1$

Is a normalized eigenfunction of the momentum p with charge+1, and each linear combination with $\left|n_1\right|^2-\left|n_2\right|^2=-1$ being a normalized solution with charge -1

This X-rayed the fact that there are two particles as there are two values of energies observed as obtained from equation (6)

These ongoing solutions invariably indicated a link to antiparticle as inferred in nature.

In actual sense the relativistic limit of Klein-Gordon equation can be analyzed by recalling equation (1) from where an ansatz resulting to

$$\psi(r,t)=\varphi(r,t)\exp\left(-\frac{i}{\hbar}m_oc^2t\right) \quad (14)$$

As deduced after splitting the time dependence of ν into two terms, one containing the rest mass from where it was clearly revealed that in the relativistic limit, the difference between the total energy E of the particle and the energy associated to the rest mass m_oc^2 is small and this led to the definition $E'=E-m_oc^2$

Where E' is the nonrelativistic energy explaining that $E \ll m_oc^2$ which imply that

$$\left|i\hbar\frac{\partial\varphi}{\partial t}\right| \approx E' \ll m_oc^2\varphi \quad (15)$$

Equation (15) is valid for the energy and using this with equation (14), we obtain

$$\frac{\partial\varphi}{\partial t}=\left(\frac{\partial\varphi}{\partial t}-i\frac{m_oc^2}{\hbar}\varphi\right)\exp\left(-\frac{i}{\hbar}m_oc^2t\right)\approx -i\frac{m_oc^2}{\hbar}\exp\left(-\frac{i}{\hbar}m_oc^2t\right)$$

$$\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t}-i\frac{m_oc^2}{\hbar}\varphi\exp\left(-\frac{i}{\hbar}m_oc^2\right)\right)$$

$$=\left[-i\frac{2mc}{\hbar}\frac{\partial\varphi}{\partial t}-i\frac{m_oc^2}{\hbar}-\frac{m_o^2c^4}{\hbar^2}\varphi\right]\exp\left(-\frac{i}{\hbar}m_oc^2t\right)$$

Now we substitute this into equation (1) yields

$$-\frac{1}{c^2}\left[\frac{2m_oc^2}{\hbar}\frac{\partial\varphi}{\partial t}+\frac{m_oc^2\varphi}{\hbar}\right]\exp\left(-\frac{i}{\hbar}m_{oa}c^2\right)$$

$$=i\hbar\frac{\partial\varphi}{\partial t}=\frac{i\hbar^2}{2m_o}\varphi=\frac{\hbar^2}{2m_o}\nabla^2\varphi \qquad (16)$$

This equation (16) exactly presents the free spineless particle in accordance with Schrodinger equation which is independent upon whether the particle is relativistic or nonrelativistic in nature, but however it was at that early stage it was only by inference that it was assumed that Klein-Gordon equation describes spin-0 particle. Invariably the confirmation by the solution of equation (1) in conjunction with result of negative and positive values of momentum was subsequently deduced by normalization of the either of the type corresponding to the (-) and (+) charges

With this, the general solution of Klein-Gordon for (+), (-) charges respectively and spin-0 particle emerges as thus

$$\varphi_{(+)}=\sum_n a_n\varphi_n(+)=\sum_n a_n\sqrt{\frac{m_oc^2}{L^3E_{p_n}}}\exp\left(\frac{i}{\hbar}p_nx-E_{p_n}t\right) \qquad (17)\text{ a}$$

$$\varphi_{(-)}=\sum_n b_n\varphi_n(-)=\sum_n b_n\sqrt{\frac{m_oc^2}{L^3E_{p_n}}}\exp\left(\frac{i}{\hbar}p_nx+E_{p_n}t\right) \qquad (17)\text{ b}$$

2.2 Nonrelativistic limit of Klein-Gordon Equation

The nonrelativistic limit of Klein-Gordon equation can be regenerated by the solution obtained from the interaction of aspin-0 particle with electromagnetic field which is made possible by coupling the Klein-Gordon as expressed in equation(1) into gauge invariance which yields explicitly

$$\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}-eA_o\right)^2\varphi=\left[\left(\sum_{i=1}^{3}+i\hbar\frac{\partial}{\partial x^i}+\frac{e}{c}A_i\right)^2+m_oc^2\right]\varphi$$

$$=\left[\left(+i\hbar\nabla+\frac{e}{c}A_i\right)^2+m_oc^2\right]\varphi \qquad (18)$$

By transforming equation (18), we have

$$\varphi(x,t)=\varphi(x,t)\exp\left(-\frac{i}{\hbar}m_oct\right) \qquad (19)$$

In this case, $\varphi(x,t)$ stands for nonrelativistic part of the wave function for which the relation

\ $\left|i\hbar\frac{\partial\varphi}{\partial t}\right| \ll m_oc^2|\varphi|, |eA_o\varphi| \ll m_oc^2$ must be valid with the first signifying the low magnitude of nonrelativistic energy $i\hbar\frac{\partial}{\partial t}$ as compared to the rest energy, while the second part is the potential which has to flatten in order to compare with the rest energy since any variation in the potential would invariably result to increase in the binding energies that may in turn lead to spontaneous pair creation that would then interfere with the nonrelativistic limit.

2.3 Free particle consideration based on Dirac Concept

Dirac in his formulation of his free particle equation looked at the inconsistence that existed at the onset of formulation of Klein-Gordon equation which was occasioned by nonsymmetrical characteristic of the earlier works of Schrödinger and Klein-Gordon equations made their idea completely relativistically invariant in space and time derivative and this was what Dirac opted to circumvent.

Dirac, from his own approach wrote Hamiltonian in the form

$$H=c\vec{\alpha}.p+Bm_oc^2 \qquad (20)$$

Where $\vec{\alpha}=\begin{pmatrix}0 & \vec{\alpha}\\ \alpha & 0\end{pmatrix}$, $\beta=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

When momentum operator in the coordinate basis is applied, the free particle equation in terms of Dirac reads

$$\left[-i\hbar c\vec{\alpha}.\nabla+\beta mc^2\right]\varphi(r)=i\hbar\frac{\partial\varphi(r,t)}{\partial t} \quad (21)$$

However, as a free particle, the parameters α and β are independent of position, time, momentum and energy otherwise the Hamiltonian would contain space-time dependent energies which would violet his earlier approach coupled with the fact that this might give rise to forces.
This then lead the above equation (21) to become

$$\left[-i\hbar c.\nabla+\beta mc^2\right]\varphi(r,t)=0 \quad (22)$$

Where $\varphi(r,t)=\begin{pmatrix}\varphi_1(r,\mathrm{t}\\ \varphi_2(r,\mathrm{t}\\ \varphi_3(r,\mathrm{t}\\ \varphi_4(r,\mathrm{t}\end{pmatrix}$ (23)

This equation is known as four components Dirac spinor with whose help led to the eigenvalue problem the results in

$$\varphi(r,t)=\varphi(r)\exp-{}^{i}/_{\hbar}Et \quad (24)$$

That when put in equation (22) gives

$$\left[-i\hbar c\vec{\alpha}\nabla+\beta mc^2\right]\varphi(r)=E\varphi(r) \quad (25)$$

The free particle plane wave solution of this is of the form

$\varphi_p(r)=U_p\exp ip.{}^{r}/_{\hbar}$ .This was used in transforming equation (25) to be in the form

$$\left[c\vec{\alpha}.p+\beta c^2\right]U_p=EU_p \quad (26)$$

U_p Is a four- component vector and from this equation the matrix on the left side of (26) can be rightly expressed in term of 2x2 blocks thus

$$U_p=\begin{pmatrix}\phi_p\\ \chi_p\end{pmatrix} \quad (27)$$

This would lead to writing correctly the matrix in equation (26) in the form

$$\begin{pmatrix}mc^2 & c\vec{\alpha}.p\\ c\vec{\alpha} & mc^2\end{pmatrix}\begin{pmatrix}\phi_p\\ \chi_p\end{pmatrix}=\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}\begin{pmatrix}\phi_p\\ \chi_p\end{pmatrix}$$

$$\begin{pmatrix}E-mc^2 & c\vec{\alpha}.p\\ -c\vec{\alpha}.p & E+mc^2\end{pmatrix}\begin{pmatrix}\phi_p\\ \chi_p\end{pmatrix}=0 \quad (28)$$

This equation for the case becomes

$$\chi_p=\frac{c\alpha.p}{E+mc^2}\phi_p \quad (29)$$

while the second case becomes

$$\phi_p=\frac{c\vec{\alpha}.p}{E-mc^2}\chi_p \quad (30)$$

From equation (29), a single equation in terms of ϕ_p is deduced

$$\left(E-mc^2\right)\left(E+mc^2\right)\phi_p-c^2\left(\vec{\alpha}-p\right)^2\phi_p=0 \quad (31)$$

With $\left(\vec{\alpha}.p\right)^2=\left(\vec{\alpha}.p\right)\left(\vec{\alpha}.p\right)=p.p+i\vec{\alpha}\left(pxp\right)=p^2$

A situation that is possible when $\left[E^2-\left(mc^2+c^2p^2\right)\right]\phi_p=0$

And since, $\phi\neq 0$ then

$$E = E_p = \pm\sqrt{p^2c^2 + m^2c^2} \qquad (32)$$

Equation (32) as obtained here still reveal that fact that the eigenvalue is either positive or negative which is in agreement with the result obtained from Klein-Gordon equation that has also their domain of positive and negative energy eigenvalue respectively

2.4 Nonrelativistic limit in Dirac Equation

In order to probe in the nonrelativistic limiting case in Dirac concept, we consider a case of a particle such as electron at rest for which it is appropriate to obtain Dirac equation using

$\hat{p}\varphi = 0$ which in turn would turn Dirac equation to be in the form

$$i\hbar\frac{\partial\varphi}{\partial t} = \vec{\beta}m_o c^2 \qquad (33)$$

Where $\hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

With the solution that yielded four eigenfunctions given as

$$\varphi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp\left[-i\left(m_o\, c^2/\hbar\right)t\right],\ \varphi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \exp\left[-i\left(m_o\, c^2/\hbar\right)t\right], \qquad (34)a$$

$$\varphi^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \exp\left[+i\left(m_o\, c^2/\hbar\right)t\right],\ \varphi^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \exp\left[+i\left(m_o\, c^2/\hbar\right)t\right], \qquad (34)\ b$$

We note that in equation (34), the first one that is (34) represents the forward propagating solutions corresponding to positive particle solution in spin-$\frac{1}{2}$ particle propagating with an energy equal to the rest mass energy while the last two (34)b describe particle moving backward in time which signifies the negative energy solution that indicates the presence of an anti-particle.

Thus Dirac equation did not only describe the particle and anti-particle but also went further to describe the spin of the particle.

Nonrelativistic limit where E>0,

$$E_p \approx mc^2 + \frac{p^2}{2m} \qquad (35)$$

So that we have

$$\chi_p = \frac{c\vec{\alpha}.p}{2mc^2 + \frac{p^2}{2m}}\phi_p \qquad (36)$$

As $mc^2 \gg \frac{p^2}{2m}$, it follows that $\chi_p \ll \phi_p$

Therefore $\phi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} or \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and with this result, the eigenfunction reduces to

$$\varphi_1(r) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp ip.\frac{r}{\hbar} \ \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \exp ip.r/\hbar \qquad (37)$$

Showing the eigenfunction corresponding to those of a free particle with spin eigenfunction $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ for

$m_o = \frac{\hbar}{2} or - \frac{\hbar}{2}$ respectively

2.5 Nonrelativistic Limit Dirac Equation based on electromagnetic Potential.

Dirac also considered the fact that the amenability of its equation to nonrelativistic concept can also be confirmed by taking into consideration the two components of Pauli's equation in nonrelativistic limit which could be achieved by introducing the electromagnetic four-potentials.

$$A^{\mu} = \left[A_o(x), A(x)\right] \quad (38)$$

Into the Dirac equation so that it could be completely guided by the electromagnetic potentials.

$$c\left(i\hbar\frac{\partial}{\partial t} - \frac{e}{c}A_o\right)\varphi = \left[c\vec{\alpha}.\left(\hat{p} - \frac{e}{c}A\right) + \hat{\beta}m_o c^2\right]\varphi \quad (39)a$$

or

Thus $$i\hbar\frac{\partial\varphi}{\partial t} = \left[c\vec{\alpha}\left(\hat{p} - \frac{e}{c}A\right) + eA_o + \hat{\beta}m_o c^2\right]\varphi \quad (39)b$$

Equation (39) contains the interaction with the electromagnetic field whose Hamiltonian is given in the form

$$\hat{H} = -\frac{e}{c}c\vec{\alpha}.A + eA_o = \frac{e}{c}\hat{\upsilon}.A + eA_o \quad (40)$$

where $$\hat{\nu} = \frac{d\hat{x}}{dt} = c\vec{\alpha} = \frac{c^2\hat{p}\hat{\Lambda}}{c\sqrt{\hat{p}^2 + m_o^2c^2}} \quad (41)$$

Equation (41) is the relativistic velocity operator which contains the Zitterbewegung while equation (40) corresponds to classical interaction of a moving charged point-like particle with the electromagnetic field.
Effectively, the nonrelativistic limiting case for Dirac equation is efficiently X-rayed in the representation

$$\varphi = \begin{pmatrix}\vec{\phi}\\\vec{\chi}\end{pmatrix}$$

Where the four-component spinor $\vec{\phi}$ is decomposed into two component spinor $\vec{\phi}$ and $\vec{\chi}$. This will convert equation (34) a to

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\vec{\phi}\\\vec{\chi}\end{pmatrix} = \begin{pmatrix}c\hat{\sigma}.\hat{\prod}\vec{\chi}\\c\hat{\sigma}.\hat{\prod}\vec{\phi}\end{pmatrix} + eA_o\begin{pmatrix}\vec{\phi}\\\vec{\chi}\end{pmatrix} + m_o c^2\begin{pmatrix}\vec{\phi}\\-\vec{\chi}\end{pmatrix} \quad (42)$$

And then by considering the rest mass energy, $m_o c^2$ as the largest energy in the system being separated by

$\begin{pmatrix}\vec{\phi}\\\vec{\chi}\end{pmatrix} = \begin{pmatrix}\phi\\\chi\end{pmatrix}\exp\left[=i\left(m_o c^2 / \hbar\right)t\right]$ results to solution of form

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{\left(\hat{p} - \frac{e}{c}\right)^2}{2m_o c^2 - \frac{e\hbar}{2m_o c^2}}\hat{\sigma}\right]\phi \quad (43)$$

This equation is the Pauli equation with the two components of ϕ describing the spine degrees of freedom where free Dirac wave equation highlighted.

The observation from the ongoing analysis is that in the nonrelativistic limit, Dirac equation transform s into the Pauli equation which is the proper nonrelativistic wave equation for spin-$\frac{1}{2}$ particle which found to be in existence at both higher and low velocities, This really indicates that Dirac equation explains particle with spin-

$\frac{1}{2}$ contrary to Klein-Gordon equation that is valid only for spin 0 particles. In any case, spin come with the linearization of the second-order differential equation.

III. Analysis and Conclusion

From the comparative analysis of Klein- Gordon and Dirac equation it was generally known that their free particle solutions seem to be the same thing; the result of the Klein-Gordon solution in free particle aspect and in Feshbach representation yielded two momentum, one positive and the other negative just as in the of the energy. In the duo, the positive and negative values represent positive charge and negative charge respectively. In interpretation of Klein-Gordon solution, it was inferred that that particle described by the wave equation does not depend up on whether the particle is nonrelativistic or relativistic and as such it was by inference that it was assumed that Klein-Gordon equation described spine-zero particles. [19]

In Dirac case, two energies and momentum solutions also emerged corresponding to the negative and positive charges respectively. However, their interpretation appeared different. Dirac interpretation of his solution faulted the idea of single particle solution as it leads to Zitterbewegung [8] which is as a result of the unexpected interference that would occur between the positive and negative energy of the wave packets. Based on this his idea, he strongly stressed the impossibility of single particle solution unless by approximation.

In the nonrelativistic limit of the duo, the energy value in the case of Klein- Gordon equation where the kinetic energy value was inferred to be smaller as compared to the rest energy coupled with that of the shallow potentials that is found paramount in order not to increase the binding energy that might on the other hand lead to the impossibility of actualizing the nonrelativistic limit while in Dirac case, the kinetic energy and the potential energy of the particle are greater than the rest energy in order to achieve the nonrelativistic limit. The nonrelativistic wave function obtained by applying the assumption as considered by Dirac in his proposition which said that kinetic and potential energies are small as compared to the rest energy culminated in the deduction of the Pauli equation that finally turned out to consist of two components of wave equation describing the spin degrees of freedom. This then clarifies the essential truth that in the nonrelativistic limit, Dirac equation could [7] transform into the Pauli's equation as in equation (43) that presents the proper nonrelativistic wave equation for spin- $1/2$ particles with its existence at low and as well as at high velocities respectively. Evidentially, this clarifies the fact that Dirac describes particle with spin- $1/2$ as compared to the Klein-Gordon equation whose claim and speculation in description of spin-0 particle is based on inference. This fact was because in nonrelativistic limit, Klein-Gordon equation could be transformed to the Schrodinger equation for electromagnetic potential whose wave function can invariably under Lorentz transformation in accordance to the transformation laws wave of function confirmed it to resemble a wave function that characterizes aspin-0 particle as described by Klein- Gordon equation[1];[8].Thus solution of the free Dirac equation is, component-wise, is a solution of the free Klein–Gordon equation. The equation does not form the basis of a consistent quantum relativistic one-particle theory. There is no known such theory for particles of any spin. For full reconciliation of quantum mechanics with special relativity, quantum field theory is needed, [4] in which the Klein–Gordon equation reemerges as the equation obeyed by the components of all free quantum fields. In quantum field theory, the solutions of the free (noninteracting) versions of the original equations turned out to play a role.

Reference

[1]. Feshbach H and Villars F (1958) Elementary Relativistic Wave Mechanics of Spin 0 and Spin 1/2 Particles Rev. Mod. Phys. 30, 24-45

[2]. Cirincione, R. J.(1979) A mathematical study of the non-relativistic limit in quantum Mechanics.Ph.D. Thesis University of California, Berkeley

[3]. Dirac P A M (1978) Mathematical Foundations of Quantum Theory Ed Marlow A R (New York: Academic). (See pp.3,4).

[4]. Barut A O (1965) Electrodynamics and Classical Theory of Fields and Particles (New York.

[5]. Sterman G 1993 An Introduction to Quantum Field Theory (Cambridge: University Press).P.7.

[6]. Bjorken J D and Drell S D (1965) Relativistic Quantum Fields (New York: McGraw).(See Chapter 12.) MacMillan). See pp. 102,211.

[7]. Chernoff, P. R. (1977) Schrodinger and Dirac operators with singular potentials and Hyperbolic equations. Pac. J. Math. 72, 361-382

[8]. Greiner, W. (2000). Relativistic Quantum Mechanics. Wave Equations (https://books.google.com/ooks?id=2DAInxwvlHYC&pg=PA1&dq=Relativistic+quantum+mec Mechanics)(3rd ed.),Springer Verlag. ISBN 3-5406-74578

[9]. Veselic, K (1970) Potential. Nucl. Phys. A **147,** 215-224

[10]. ANDREW Y. SCHOENE, **(**1979) **On** the Nonrelativistic Limits of the Klein-Gordon and Dirac Equations JOURNAL OF MATHEMATICAL ANALYSIS AND PPLICATIONS 71, 36-47

[11]. Ross J. Cirincione and Paul R. Chernoff (1981) Dirac and Klein-Gordon Equations: Convergence of Solutions in the Nonrelativistic Limit Commun. Math. Phys. 79, 33-46

[12]. Nikolic, H. (2009) Time in Relativistic and Nonrelativistic Quantum Mechanics. *International Journal of Quantum Information* **7** (3): 595-602

[13]. Nikolic, H. (2006) Relativistic Bohmian interpretation of quantum mechanics. Conference on the present status of Quantum Mechanics. *AIP Conference Proceedings* **844**: 272-280

[14]. Eshghi M, Hamzavi M and Ikhdair S M 013 Chin. Phys. B **22**

[15]. Wang I C and Wong C Y (1988) Phys. Rev. D **38** 3248

[16]. Mao G 2003 Phys . Rev. C **67** 044318

[17]. Feshbach H and Villars F (1958) Elementary Relativistic Wave Mechanics of Spin 0 and Spin 1/2 Particles Rev. Mod. Phys. 30, 24-45

[18]. Alberto P, Lisboa R, Malheiro M and de Castro A. S., (2005) Phys. Rev. C **71** 034313

[19]. Comay E. (2004) Difficulties with the Klein-Gordon Equation. Apeiron Vol.11 (3) 1-13