

Application Of Laguerre Polynomial Equation In Solving Schrodinger Equation Radial Section Of Helium Ion

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Abstract:

The helium ion is one of the ions included in the hydrogenic atom because it has only one electron in its outermost orbital, so the problem in the helium ion can be solved using the Schrodinger equation approach at spherical coordinates. This study was conducted to examine the solution of helium ions in the representation of positional space using the Schrödinger equation on quantum numbers $n \leq 4$. The type of research used is basic research, namely the development of existing theories. The results obtained in the form of Laguerre polynomials are $L_n(r) = e^r \frac{d^n}{dr^n} (r^n e^{-r})$ and associated Laguerre polynomials is $L_n^k(r) = e^r r^{-k} \frac{d^n}{dr^n} (e^{-r} r^{n+k})$ then applied to the Schrodinger wave function of the radial part of the Helium ion, namely $R_{nl} = -\left(\frac{2z}{na_0}\right)^{\frac{3}{2}} \sqrt{\frac{(n-\ell-1)!}{2n[(n+1)!]^3}} e^{-\left(\frac{zr}{na}\right)} (\xi)^\ell L_{n-\ell}^{2\ell+1}(\xi)$.

Key Word: Laguerre; Polynomial; Schrodinger; Radial Section; Helium Ion

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I. Introduction

The physics that developed until the end of the nineteenth century was known as classical physics and had two main branches, namely Newtonian classical mechanics and Maxwellian electromagnetic field theory. Classical mechanics is characterized by the presence of particles as something confined within space. The term confined can simply be said to mean the existence of a clear boundary between matter and something outside itself or its environment [1]. While electromagnetic is characterized by the quantity of the field of waves that propagate in space. The terrain is scattered in the fog-like space of different thicknesses and thus until it finally completely disappears [2].

The development of physics in the classical era continued until new discoveries were obtained continuously. However, some of the physical phenomena that occurred in the late nineteenth century could not be explained by classical physical theory. One of them is the formulation of Rayleigh and James Jeans which does not match the results of observations in the low wavelength region in the ultraviolet region. This is because the results of integration are infinitely valuable, so this event is known as an ultraviolet disaster [3]. Therefore, quantum physics appears to be a complement to the shortcomings of classical physics in the investigation of physics [4].

The behavior of light like particles is supported by Planck's hypothesis, Einstein's quantum theory of Light as well as Compton's scattering theory. This phenomenon assumes that light is a particle [4]. In 1924 Louis de Brogile proposed the hypothesis that particles should have wave properties taking into account the symmetry properties of nature [5]. The wavelength is obtained by Planck's constant divided by mass multiplication and wave rate. De Brogile's hypothesis explains waves have properties as particles and particles have wave properties [6].

The Schrödinger equation is an equation introduced by Erwin Schrödinger to discuss the description of particle waves in atomic dimensions that satisfy the principles and laws of physics [4]. The simplest form of the Schrödinger equation is the equation for a free particle or a particle affected by a constant potential, $V(x) = C$ [7]. This equation is very important as well as Newton's equation which became the basis for the development of physical science, so that the Schrödinger equation itself became the basis for the development of modern physics related to quantum mechanics [8].

The 3-dimensional stable Schrödinger equation in spherical coordinates is commonly used in solving the problem of hydrogenic atoms. This equation is a partial differential equation of order 2 that depends on 3 variables, namely: $\Psi(r, \theta, \phi)$. So that the variable separation method, by stating $\Psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$ where $R(r)$ is a radial function and $Y(\theta, \phi)$ is an angular function (Purwanto, 2016), often used. Aziz &

Abdullah (2015) stated that the variable separation method in solving the Schrodinger equation of hydrogen atoms can use the angular momentum operator method, potential operator and can use the Laplacian operator. The wave function of hydrogen atoms can be represented in terms of position and momentum representations.

The wave function of a hydrogenic atom is a solution of the Schrödinger equation of a hydrogenic atom [10]. A hydrogenic atom is an atom that gives up its electrons leaving only one electron in its outermost orbit [11][12]. In the hydrogenic atom there is a charged nucleus Ze and an electroton surrounding the nucleus. Helium is a type of noble gas atom that has two electrons in its orbitals, two protons and two neutrons [13]. If one of the two electrons in a helium atom ions, it will be a helium ion that has characteristics similar to a hydrogenic atom [7]. The helium ion system is similar to the hydrogen atomic system, so the hydrogenic atomic stage can be used to solve the helium ion schrodinger equation [14].

The wave function of the helium ion can be determined by solving the Schrödinger equation for hydrogenic atoms [15]. Schrodinger completion can be done separately into radial section and angle section functions [16]. If a system contains radial functions or angular functions then the Schrodinger equation is reduced to a second-order differential equation. This equation has a special solution by containing polynomials of special functions such as hermit, hypergeometry, lagendre and Laguerre [17].

One important component in the analysis and modeling of the Schrödinger equation is the Laguerre polynomial. The Laguerre polynomial is a mathematical function that serves as a solution to differential equations. The Schrödinger equation for hydrogenic atoms in spherical coordinates also yields differential equations in some cases when the system has spherical symmetry such as hydrogen atoms and hydrogen-like atoms [18]. Laguerre's polynomials helped and formulated exact solutions of the Schrödinger equation for such systems and provided valuable insights in quantum and atomic physics [19].

II. Material And Methods

The Schrödinger equation plays a role what is very important in the development of modern physics, especially quantum mechanics, is logically analogous to the laws of classical physics [20]. The Schrödinger equation is a second-order partial differential equation used to provide information about the waveform of a particle.

The steady Schrödinger equation for hydrogenic atoms in cartesian coordinates is generally formulated as follows:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

(1)

or:

$$\nabla^2\Psi + \frac{2m}{\hbar^2}(E - V)\Psi = 0$$

(2)

Using the help of the Laplace transform, the Schrödinger equation can also be expressed in spherical coordinates, namely:

$$\frac{\hbar^2}{2m_0} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Psi(r, \theta, \phi) + [E - V]\Psi(r, \theta, \phi) = 0$$

(3)

The variable separation method can be done by introducing $\Psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$ and multiply equation

(3) by $\frac{1}{R(r)Y(\theta, \phi)}$ The following equation will be obtained:

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{\partial R}{\partial r} \right) + \frac{2m_0 r^2}{\hbar^2} [E - V]R + \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right] = 0$$

(4)

or

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{\partial R}{\partial r} \right) + \frac{2m_0 r^2}{\hbar^2} [E - V]R = - \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right] = \lambda$$

(5)

Based on equation (5) above, the radial section Schrodinger equation can be simplified to:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{\partial R}{\partial r} \right) + \frac{2m_0}{\hbar^2} [E - V]R = \frac{\lambda}{r^2} R$$

(6)

By introducing the parameters $V(r) = -\frac{\text{konstanta}}{r} = -\frac{A}{r}$ $\lambda = l(l + 1)$ and, equation (6) can be expressed in the form:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{\partial R}{\partial r} \right) + \left\{ \frac{2m_0}{\hbar^2} \left[E + \frac{A}{r} \right] - \frac{l(l+1)}{r^2} \right\} R = 0$$

(7)

And using parameter (i) ; (ii) $\xi = \beta r \beta^2 = \frac{8\mu|E|}{\hbar^2}$; and (iii) $\gamma = \frac{2\mu A}{\beta \hbar^2} = \frac{A}{\hbar} \left(\frac{\mu}{2|E|}\right)^{1/2}$ obtain:

$$\xi \frac{d^2 R}{d\xi^2} + 2 \frac{dR}{d\xi} + \left[\gamma - \frac{\xi}{4} - \frac{l(l+1)}{\xi} \right] R = 0 \tag{8}$$

This research is a descriptive qualitative research in the form of a literature study to solve the Schrodinger equation of the radial part of the helium ion using the Laguerre equation [21].

$$r \frac{d^2 R}{dr^2} + (1 - r) \frac{dR}{dr} + \alpha R = 0 \quad (\alpha = \text{constan}) \tag{9}$$

Which has a solution in the form of a polynomial function, namely:

$$R(r) = \sum_{\lambda=0}^{\infty} a_{\lambda} r^{k+\lambda} \quad (a_0 \neq 0) \tag{10}$$

By substituting equation (10), the first derivative is

$$\frac{dR}{dr} = \sum_{\lambda=0}^{\infty} (k + \lambda) a_{\lambda} r^{k+\lambda-1} \tag{11}$$

And derivatives of both,

$$\frac{d^2 R}{dr^2} = \sum_{\lambda=0}^{\infty} (k + \lambda)(k + \lambda - 1) a_{\lambda} r^{k+\lambda-2} \tag{12}$$

into equation (9). So the Laguerre equation above becomes:

$$[k(k - 1) + k] a_0 r^{k-1} + \sum_{\lambda=0}^{\infty} \{[(k + \lambda + 1)(k + \lambda) + k + \lambda + 1] a_{\lambda+1} + (\alpha - k - \lambda) a_{\lambda}\} r^{k+\lambda} = 0 \tag{13}$$

The above equation gives the meaning that [22]:

$$\begin{aligned} \text{(i)} & [k(k - 1) + k] a_0 = 0 \quad \text{and} \\ \text{(ii)} & a_{\lambda+1} = \frac{(\alpha - k - \lambda)}{(k + \lambda + 1)(k + \lambda) + k + \lambda + 1} a_{\lambda} \end{aligned} \tag{14}$$

Given that the recursion formula is obtained as follows: $a_0 \neq 0, k = 0$

$$a_{\lambda+1} = -\frac{(\alpha - \lambda)}{(\lambda + 1)^2} a_{\lambda} \tag{15}$$

So the solution of the Laguerre equation (10) above, can be expressed in terms of power series functions as follows: $R(r) = \sum_{\lambda=0}^{\infty} a_{\lambda} r^{k+\lambda} \quad (a_0 \neq 0)$

$$R(r) = a_0 \left[1 - \frac{\alpha}{1^2} r + \frac{\alpha(\alpha-1)}{(2!)^2} r^2 + \dots + \frac{(-1)^j \alpha(\alpha-1)\dots(\alpha-j+1)r^j}{(j!)^2} \right] + \dots \tag{16}$$

Equation (16) becomes finite (polynomial of degree n) when or (equal to a positive integer). So that the finite solution can be stated: $\alpha - j + 1 = 0, \alpha = j - 1$

$$\begin{aligned} R(r) &= a_0 \sum_{\lambda=0}^n \frac{(-1)^{\lambda} \alpha(\alpha-1)\dots(\alpha-\lambda+1)r^{\lambda}}{(\lambda!)^2} \\ R(r) &= a_0 \sum_{\lambda=0}^n \frac{(-1)^{\lambda} \alpha! r^{\lambda}}{(\alpha-\lambda)!(\lambda!)^2} \equiv L_n(r) \end{aligned} \tag{17}$$

Where Rodrigues' formula for the Laguerre polynomial is [23].

$$L_n(r) = e^r \frac{d^n}{dr^n} (r^n e^{-r}) \tag{18}$$

Laguerre polynomials for range n are obtained as presented in table 1.

Tabel 1. Polinomial Laguerre $L_n(r)$

n	$L_n(r)$
1	1
2	$1 - r$
3	$2 - 4r + r^2$
4	$6 - 18r + 9r^2 - r^3$

The k differential of the Laguerre equation to r obtained:

$$r \frac{d^{k+2}}{dr^{k+2}} L_n(r) + (k + 1 - r) \frac{d^{k+1}}{dr^{k+1}} L_n(r) + (n - k) \frac{d^k}{dr^k} L_n(r) = 0 \tag{19}$$

or

$$r \frac{d^2 L_n^k(r)}{dr^2} + (k + 1 - r) \frac{dL_n^k(r)}{dr} + (n - k)L_n^k(r) = 0$$

where

$$L_n^k(r) = \frac{d^k}{dr^k} L_n(r) \tag{20}$$

and n and $towards$ is a constant (positive integer). Associated Laguerre polynomials with degrees represented by $(n - k)L_n^k(r)$ [24]. By using the generating function for associated Laguerre polynomials [25],

$$\frac{(-t)^k \exp(-tr/(1-t))}{(1-t)^{k+1}} = \sum_{n=k}^{\infty} \frac{L_n^k(r)t^n}{n!} \tag{21}$$

and Rodrigues' formula is

$$L_n^k(r) = e^r r^{-k} \frac{d^n}{dr^n} (e^{-r} r^{n+k}) \tag{22}$$

III. Result

The solution of the radial part of the Schrodinger equation (8) is done by considering the equation at the time so that the equation is obtained: $\xi \rightarrow \infty$

$$\left\{ \frac{d^2}{d\xi^2} - \frac{1}{4} \right\} R_{nl}(\xi) = 0 \tag{23}$$

And it has a solution,

$$R_{nl}(\xi) \sim e^{-\frac{\xi}{2}}. \tag{24}$$

At the time $\xi \rightarrow 0$, equation (8) becomes:

$$\frac{d^2}{d\xi^2} \{ \xi R_{nl}(\xi) \} - \frac{l(l+1)}{\xi^2} \{ \xi R_{nl}(\xi) \} = 0 \tag{25}$$

Has a shaped solution,

$$R_{nl}(\xi) \sim \xi^l. \tag{26}$$

So the solution to the Schrodinger equation of the radial part of the hydrogenic atom is,

$$R_{nl}(\xi) = N_{nl} e^{-\frac{\xi}{2}} \xi^l L_n^k(\xi) \tag{27}$$

or

$$R_{nl}(r) = N_{nl} e^{-\frac{zr}{na_0}} \left(\frac{2zr}{na_0} \right)^l L_{n+1}^{2l+1} \left(\frac{2zr}{na_0} \right) \tag{28}$$

where

$$N_{nl} = - \left(\frac{2z}{na_0} \right)^{3/2} \sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}} \tag{29}$$

is a normalization constant. So the solution of the radial function of the normalized hydrogenic atom is:

$$R_{nl} = - \left(\frac{2z}{na_0} \right)^{3/2} \sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-\left(\frac{zr}{na}\right)} (\xi)^l L_{n+l}^{2l+1}(\xi) \tag{30}$$

where $\xi = \frac{2zr}{na_0}$.

Using the associated Laguerre function and Laguerre function, the radial wave function of the Helium ion, ${}^4_2\text{He}^+$ at quantum numbers is $(n, l) = (1, 0)$

(i) $L_{n+l}^{2l+1}(\xi) = L_1^1(\xi) = \frac{d}{d\xi} L_1(\xi)$

Because $L_1(\xi) = e^\xi \frac{d}{d\xi} (\xi e^{-\xi}) = 1 - \xi$

So $L_1^1(\xi) = \frac{d}{d\xi} L_1(\xi) = -1$

(ii) $N_{10} = - \left(\frac{2.2}{a_0} \right)^{3/2} \sqrt{\frac{(1-0-1)!}{2[(1+0)!]^3}} = -2 \left(\frac{2}{a_0} \right)^{3/2}$

(iii) $R_{10}(r) = 2 \left(\frac{2}{a_0} \right)^{3/2} e^{-\frac{2r}{a_0}}$

For quantum numbers $(n, l) = (2, 0)$, the radial wave function is

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_2^1(\xi) = \frac{d}{d\xi} L_2(\xi)$$

$$\text{Because } L_2(\xi) = e^\xi \frac{d^2}{d\xi^2} (\xi^2 e^{-\xi}) = 2 - 4\xi + \xi^2$$

$$\text{So } L_2^1(\xi) = \frac{d}{d\xi} L_2(\xi) = 2\xi - 4$$

$$(ii) N_{20} = -\left(\frac{2.2}{2a_0}\right)^{3/2} \frac{\sqrt{(2-0-1)!}}{\sqrt{4[(2+0)!]^3}} = -\left(\frac{2}{a_0}\right)^{3/2} \frac{1}{4\sqrt{2}}$$

$$(iii) R_{20}(r) = \left(\frac{1}{a_0}\right)^{3/2} e^{-\frac{r}{a_0}} \left(2 - \frac{2r}{a_0}\right)$$

In the same way, the wave function of Helium ions on quantum numbers $(n, l) = (2, 1)$ is,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_3^3(\xi) = \frac{d}{d\xi} L_3(\xi)$$

$$\text{Because } L_3(\xi) = e^\xi \frac{d^3}{d\xi^3} (\xi^3 e^{-\xi}) = 6 - 18\xi + 9\xi^2 - \xi^3$$

$$\text{So } L_3^3(\xi) = \frac{d}{d\xi} L_3(\xi) = -6$$

$$(ii) N_{21} = -\left(\frac{2.2}{2a_0}\right)^{3/2} \frac{\sqrt{(2-1-1)!}}{\sqrt{4[(2+1)!]^3}} = -\left(\frac{2}{a_0}\right)^{3/2} \frac{1}{12\sqrt{6}}$$

$$(iii) R_{21}(r) = \frac{1}{\sqrt{3}(a_0)^{3/2}} e^{-\left(\frac{r}{a_0}\right)} \left(\frac{2r}{a_0}\right)$$

And on quantum numbers $(n, l) = (3, 0)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_3^1(\xi) = \frac{d}{d\xi} L_3(\xi)$$

$$\text{Because } L_3(\xi) = e^\xi \frac{d^3}{d\xi^3} (\xi^3 e^{-\xi}) = 6 - 18\xi + 9\xi^2 - \xi^3$$

$$\text{So } L_3^1(\xi) = \frac{d}{d\xi} L_3(\xi) = -18 + 18\xi - 3\xi^2$$

$$(ii) N_{30} = -\left(\frac{2.2}{3a_0}\right)^{3/2} \frac{\sqrt{(3-0-1)!}}{\sqrt{2(3)[(3+0)!]^3}} = -\left(\frac{4}{3a}\right)^{3/2} \sqrt{\frac{2}{6(6)^3}}$$

$$(iii) R_{30}(r) = \frac{8}{\sqrt{2}(3a_0)^{3/2}} e^{-\left(\frac{2r}{3a_0}\right)} \left(1 - \frac{4r}{3a_0} + \frac{8r^2}{27a_0^2}\right)$$

And on quantum numbers $(n, l) = (3, 1)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_4^3(\xi) = \frac{d}{d\xi} L_4(\xi)$$

$$\text{Because } L_4(\xi) = e^\xi \frac{d^4}{d\xi^4} (\xi^4 e^{-\xi}) = 24 - 96\xi + 72\xi^2 - 16\xi^3 + \xi^4$$

$$\text{So } L_4^3(\xi) = \frac{d^3}{d\xi^3} L_4(\xi) = 24\xi - 96$$

$$(ii) N_{31} = -\left(\frac{2.2}{3a_0}\right)^{3/2} \frac{\sqrt{(3-1-1)!}}{\sqrt{6[(3+1)!]^3}} = -\left(\frac{4}{3a}\right)^{3/2} \sqrt{\frac{1}{6[24]^3}}$$

$$(iii) R_{31}(r) = \frac{2\sqrt{2}}{9} \left(\frac{2}{3a_0}\right)^{3/2} \left(\frac{4r}{a_0} - \frac{4r^2}{3a_0^2}\right) e^{-\left(\frac{2r}{3a_0}\right)}$$

And on quantum numbers $(n, l) = (3, 2)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_5^5(\xi) = \frac{d}{d\xi} L_5(\xi)$$

$$\text{Because } L_5(\xi) = e^\xi \frac{d^5}{d\xi^5} (\xi^5 e^{-\xi}) = 120 - 600\xi + 600\xi^2 - 200\xi^3 + 25\xi^4 - \xi^5$$

$$\text{So } L_5^5(\xi) = \frac{d^5}{d\xi^5} L_5(\xi) = 120$$

$$(ii) N_{32} = -\left(\frac{2.2}{3a_0}\right)^{3/2} \frac{\sqrt{(3-2-1)!}}{\sqrt{6[(3+2)!]^3}} = -\frac{1}{180\sqrt{5}(3a_0)^{3/2}}$$

$$(iii) R_{32}(r) = \frac{4}{27\sqrt{10}} \left(\frac{2}{3a_0}\right)^{3/2} e^{-\frac{2r}{3a_0}} \left(\frac{4r^2}{a_0^2}\right)$$

And on quantum numbers $(n, l) = (4, 0)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_4^1(\xi) = \frac{d}{d\xi} L_4(\xi)$$

$$\text{Because } L_4(\xi) = e^\xi \frac{d^4}{d\xi^4} (\xi^4 e^{-\xi}) = 24 - 96\xi + 72\xi^2 - 16\xi^3 + r\xi^4$$

$$\text{So } L_4^1(\xi) = \frac{d}{d\xi} L_4(\xi) = -24$$

$$(ii) N_{40} = -\left(\frac{2.2}{3a_0}\right)^{3/2} \sqrt{\frac{(4-0-1)!}{8[(3+2)!]^3}} = -\left(\frac{1}{a_0}\right)^{3/2} \frac{\sqrt{6}}{240\sqrt{2}}$$

$$(iii) R_{40}(r) = \frac{1}{\sqrt{2}} \left(\frac{1}{a_0}\right)^{3/2} \left(1 - \frac{3r}{2a_0} + \frac{r^2}{2a_0^2} - \frac{r^3}{24a_0^3}\right) e^{-\left(\frac{r}{2a_0}\right)}$$

And on quantum numbers $(n, l) = (4, 1)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_4^1(\xi) = \frac{d}{d\xi} L_4(\xi)$$

$$\text{Because } L_5(\xi) = e^\xi \frac{d^5}{d\xi^5} (\xi^5 e^{-\xi}) = 120 - 600\xi + 600\xi^2 - 200\xi^3 + 25\xi^4 - \xi^5 e^{-\xi}$$

$$\text{So } L_5^3(\xi) = \frac{d^5}{d\xi^5} L_5(\xi) = -1200 + 600\xi - 60\xi^2$$

$$(ii) N_{41} = -\left(\frac{2.2}{3a_0}\right)^{3/2} \sqrt{\frac{(4-0-1)!}{8[(3+2)!]^3}} = -\left(\frac{1}{a_0}\right)^{3/2} \frac{\sqrt{6}}{240\sqrt{2}}$$

$$(iii) R_{41}(r) = \frac{1}{2\sqrt{30}} \left(\frac{1}{a_0}\right)^{3/2} \left(\frac{r^2}{4a_0^2} - \frac{5r}{2a_0} + 5\right) \left(\frac{r}{a_0}\right) e^{-\left(\frac{r}{2a_0}\right)}$$

And on quantum numbers $(n, l) = (4, 2)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_5^2(\xi) = \frac{d^2}{d\xi^2} L_5(\xi)$$

$$\text{Because } L_6(\xi) = e^\xi \frac{d^6}{d\xi^6} (\xi^6 e^{-\xi}) = 720 - 4320\xi + 5400\xi^2 - 2400\xi^3 + 450\xi^4 - 36\xi^5 + \xi^6$$

$$\text{So } L_6^4(\xi) = \frac{d^4}{d\xi^4} L_6(\xi) = -4320\xi + 720\xi$$

$$(ii) N_{42} = -\left(\frac{2.2}{4a_0}\right)^{3/2} \sqrt{\frac{(4-2-1)!}{8[(4+2)!]^3}} = -\left(\frac{1}{a_0}\right)^{3/2} \frac{1}{720\sqrt{720}\sqrt{8}}$$

$$(iii) R_{42}(r) = \frac{1}{24\sqrt{10}} \left(\frac{1}{a_0}\right)^{3/2} e^{-\left(\frac{r}{2a_0}\right)} \left(6 - \frac{r}{a_0}\right) \left(\frac{r}{a_0}\right)^2$$

And on quantum numbers $(n, l) = (4, 3)$ are,

$$(i) L_{n+l}^{2\ell+1}(\xi) = L_7^3(\xi) = \frac{d^3}{d\xi^3} L_7(\xi)$$

$$\text{Because } L_7(\xi) = e^\xi \frac{d^7}{d\xi^7} (\xi^7 e^{-\xi}) = 5040 - 35280\xi + 52920\xi^2 - 29400\xi^3 + 7350\xi^4 - 882\xi^5 + 49\xi^6 - \xi^7$$

$$\text{So } L_7^7(\xi) = \frac{d^7}{d\xi^7} L_7(\xi) = -5040$$

$$(ii) N_{43} = -\left(\frac{2.2}{4a_0}\right)^{3/2} \sqrt{\frac{(4-3-1)!}{8[(4+3)!]^3}} = -\left(\frac{4}{4a_0}\right)^{3/2} \sqrt{\frac{1}{8.5040^3}}$$

$$(iii) R_{43}(r) = \frac{1}{24\sqrt{70}} \left(\frac{1}{a_0}\right)^{3/2} e^{-\left(\frac{r}{2a_0}\right)} \left(\frac{r}{a_0}\right)^3$$

The radial wave function of a hydrogenic atom expresses the movement of electrons propagating and spreading from the center of the atom in all directions and depending on the distance (r) of origin. The radial wave function of a hydrogenic atom contains a principal quantum number (n) and an orbital quantum number (l). The principal quantum number represents the energy level of the electron orbital while the orbital quantum number determines the shape of the electron orbital. So the radial function of the helium ion wave in the principal quantum number $n \leq 4$ is as follows:

Table 2. Radial wave function of $Z = 2$

n	ℓ	$R_{n,\ell}$
1	0	$2 \left(\frac{2}{a_0}\right)^{3/2} e^{-\frac{2r}{a_0}}$
2	0	$\left(\frac{1}{a_0}\right)^{3/2} e^{-\frac{r}{a_0}} \left(2 - \frac{2r}{a_0}\right)$
	1	$\frac{1}{\sqrt{3} \left(a_0\right)^{3/2}} e^{-\left(\frac{r}{a_0}\right)} \left(\frac{2r}{a_0}\right)$
3	0	$\frac{8}{\sqrt{2} \left(3a_0\right)^{3/2}} e^{-\left(\frac{2r}{3a_0}\right)} \left(1 - \frac{4r}{3a_0} + \frac{8r^2}{27a_0^2}\right)$
	1	$\frac{2\sqrt{2}}{9} \left(\frac{2}{3a_0}\right)^{3/2} \left(\frac{4r}{a_0} - \frac{4r^2}{3a_0^2}\right) e^{-\left(\frac{2r}{3a_0}\right)}$
	2	$\frac{4}{27\sqrt{10}} \left(\frac{2}{3a_0}\right)^{3/2} e^{-\frac{2r}{3a_0}} \left(\frac{4r^2}{a_0^2}\right)$

n	ℓ	R _{n,ℓ}
4	0	$\frac{1}{\sqrt{2}} \left(\frac{1}{a_0}\right)^{3/2} \left(1 - \frac{3r}{2a_0} + \frac{r^2}{2a_0^2} - \frac{r^3}{24a_0^3}\right) e^{-\left(\frac{r}{2a_0}\right)}$
	1	$\frac{1}{2\sqrt{30}} \left(\frac{1}{a_0}\right)^{3/2} \left(\frac{r^2}{4a_0^2} - \frac{5r}{2a_0} + 5\right) \left(\frac{r}{a_0}\right) e^{-\left(\frac{r}{2a_0}\right)}$
	2	$\frac{1}{24\sqrt{10}} \left(\frac{1}{a_0}\right)^{3/2} e^{-\left(\frac{r}{2a_0}\right)} \left(6 - \frac{r}{a_0}\right) \left(\frac{r}{a_0}\right)^2$
	3	$\frac{1}{24\sqrt{70}} \left(\frac{1}{a_0}\right)^{3/2} e^{-\left(\frac{r}{2a_0}\right)} \left(\frac{r}{a_0}\right)^3$

In determining the normalized radial function of helium ions using the bohr radius $a_0 = 0.529 \times 10^{-10}$ meters.

Figure 1a is a graph of the results of the radial function of the hydrogenic atomic wave used as a validation of the graph simulation for the principal quantum number $n = 2$ in the reference used [26].

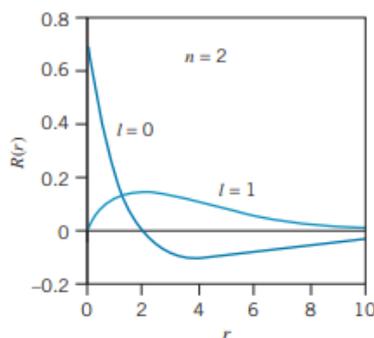


Figure 1a

Based on figure 1a it can be explained that the ordinate axis states the position of electrons (r) measured from the nucleus in the hydrogen atom in units (a_0) and the abscissa axis which expresses the peak of $R(r)$ in units of ammunition (Å).

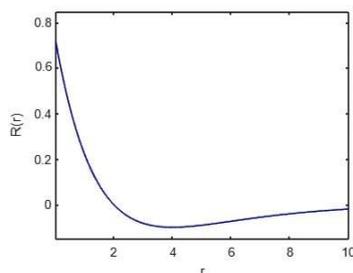


Figure 1b

Figure 1b is the result of the graph obtained by the researcher has similarity with the validation graph, so that figure 1b can be stated to be in accordance with the literature.

Based on the results of table 2, there are ten different forms of radial functions in the principal quantum number $n = 4$. To visualize each form of radial function obtained is by graphic form. The following is a graph of the radial wave function for the value of the main quantum number $n = 4$ sequentially and the orbital quantum number (l) where in the ordinate part the position of the electron is at a distance (r) from the center of the atom with an interval of up to 30. The part of the abscissa that expresses the radial wave function of ions in units of Armstrong (Å).

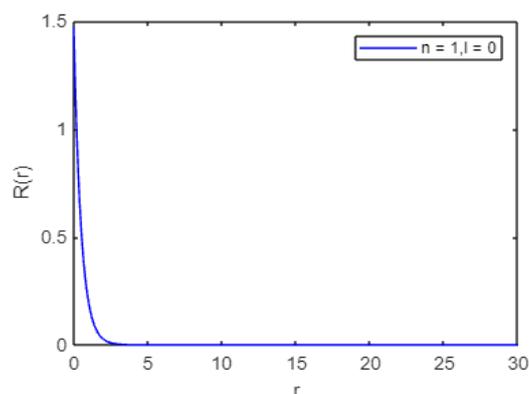


Figure 2

Based on figure 2 as a graph of the radial wave function on the principal quantum number $n = 1$ and the orbital quantum number $l = 0$ waveform is obtained. Where in this condition, there is a maximum peak at 1.470 \AA . and a minimum peak at $0,008 \text{ \AA}$.

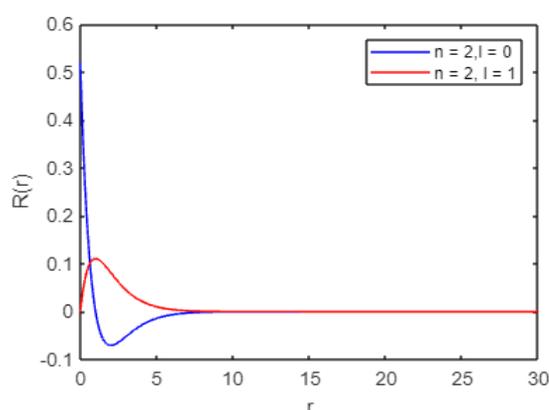


Figure 3

Figure 3 shows a graph of the radial wave function at the principal quantum number $n = 2$ and the orbital quantum number $l = 0$ and $l = 1$ obtained 2 waveforms. At $n = 2$ and $l = 0$, there is a maximum peak at $0,519 \text{ \AA}$. And a minimum peak at -0.069 \AA . At $n = 2$ and $l = 1$, there is a maximum peak at 0.108 \AA and a minimum peak at 0.003 \AA .

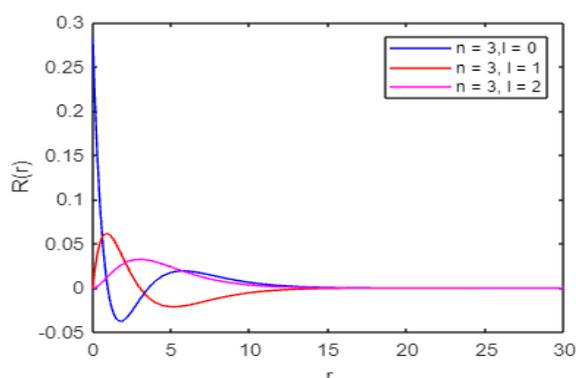


Figure 4

Figure 4 shows a graph of radial wave functions at major quantum numbers $n = 3$ and orbital quantum numbers $l = 0, 1$ and 2 obtained 3 waveforms. At $n = 3$ and $l = 0$, there is a maximum peak at 0.282 \AA and a minimum peak at -0.037 \AA . At $n = 3$ and $l = 1$, there is a maximum peak at 0.059 \AA and a minimum peak at -0.021 \AA . At $n = 3$ and $l = 2$, there is a maximum peak at 0.032 \AA and a minimum peak at 0.001 \AA .

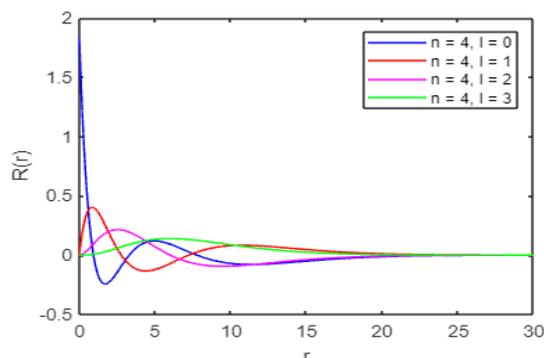


Figure 5

Figure 5 shows a graph of the radial wave function at the main quantum number $n = 4$ and the orbital quantum number $l = 0, 1, 2$ and 3 obtained 4 waveforms. At $n = 4$ and $l = 0$, there is a maximum peak at 1.837Å and a minimum peak at -0.243Å . At $n = 4$ and $l = 1$, there is a maximum peak at 0.395Å and a minimum peak at -0.133Å . At $n = 4$ and $l = 2$, there is a maximum peak at 0.214Å and a minimum peak at -0.088Å . At $n = 4$ and $l = 3$, there is a maximum peak at 0.138Å and a minimum peak at 0.001Å .

Based on the graph of the radial wave function on the principal quantum number $n \leq 4$ it appears that there are ten different waveforms depending on the value of the orbital quantum number and the peak point gets smaller as the orbital quantum number increases. The results of the recapitulation of peak point data from garfik can be seen in table 3.

Tabel 3. Peak point of the graph

n	ℓ	Maximum peak point (Å)	Minimum breaking point (Å)
1	0	1,470	0,008
2	0	0,519	-0,069
	1	0,108	0,003
3	0	0,282	-0,037
	1	0,059	-0,021
	2	0,032	0,001
4	0	1,837	-0,243
	1	0,395	-0,133
	2	0,214	-0,088
	3	0,138	0,001

The crest point formed shows an exponential decrease that occurs because the deviation of the radial wave function is inversely proportional to the exponential function.

IV. Conclusion

Based on the results obtained, it can be concluded that the solution of the Helium Ion Schrodinger Equation on the main quantum numbers $n \leq 4$ in the representation of positional space has 10 radial functions. As the main quantum number increases with the orbital quantum number, the distance from the peak point value will decrease. When viewed from each quantum number, for the state of quantum numbers the distance from the peak point value increases.

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