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The Non-Existence of Perfect Odd Numbers

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***Abstract**: The aim of this article is to solve a problem opened more than 2 millennia ago (300 BC) : There is no odd perfect number.*

***Résumé:** Le but de cet article est de résoudre un problème ouvert il y' a plus de 2 millénaires (300 av. J.-C) à savoir qu'il n'existe aucun nombre **parfait impair**.*

***Key words:** perfect number, number theory.*

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I. Introduction :

A perfect number ([1,3]) is a natural number equal to half of the sum of its divisors or is the sum of its proper divisors (Example : 6=1+2+3, 12=1+2+3+6). Pythagoras then Euclid the discovered more than 2 millennia ago (300 BC), but we ignores almost everything about them.

Despite the simplicity of the definition, two essential questions not yet found an answer from Euclid. The first: is there a infinity of perfect numbers? The second: are there numbers perfect odd?

Recently, fourteen researchers from Brigham Young University in Provo (Utah, United States) have uploaded to the open access site arXiv [2] an article giving an account of new insights and ways of research concerning the second riddle, whether there are numbers perfect odd - even if they admit that the resolution is still " out of range "[1].

The purpose of this article is to answer their question by proving that there is no perfect odd number.

Theorem: There is no odd perfect number.

Proposition 1: If N is a non-zero natural number $N\in\mathbb{N}^*$ with $N=\prod_{i=1}^{n}P_i^{\alpha_i}$ and the P_i are prime disjoint two by two, then: N is **perfect** is equivalent to:

$$2\prod_{i=1}^{n}P_i^{\alpha_i}=\prod_{i=1}^{n}(1+\sum_{j=1}^{\alpha_i}P_i^j)$$

Proof : Deduced from the fact that the term of the line is the sum of all the divisors of N.

Corollary 1:

Let $N\in 2\mathbb{N}^*+1$. If N is perfect then $N=\prod_{i=1}^{n}P_i^{\alpha_i}$ with:

- The P_i are disjoint prime numbers two by two.
- $\alpha_1\in 2\mathbb{N}^*+1$
- $\alpha_i\in 2\mathbb{N}\ \ if\ \ i\neq 1$

Proof: It results from the **proposition 1** above and the fact that $N\in 2\mathbb{N}^*+1$ (odd).

Corollary 2:

Let $M\in 2\mathbb{N}^*+1$. If $N=M^2$ then N is not perfect.

Proof : It results from the **Corollary 1.**

Proof of the theorem:
Theorem: There is no odd perfect number.
Proof :
Let N be an **odd** integer.

If N is **perfect**, then from the **corollaries** 1 and 2 we deduce that: $N=p^{2k+1}M^2$, with p a prime number not dividing M.
N is assumed to be perfect, therefore:

$$N=p^{2k+1}M^2=\sum_i d_i(M^2)+(1+\sum_{j=1}^{2k+1}p^j)+\sum_{j=1}^{2k+1}p^j(\sum d_i(M^2))+M^2+\sum_{j=1}^{2k}p^jM^2 .$$

With $d_i(M^2)$ a divisor of M^2 distinct from 1 and M^2 .
We deduce that in $\mathbb{Z}/p\mathbb{Z}$ we have : $0=\sum_i d_i(M^2)+1+M^2$.

Let $\sum_i d_i(M^2)+1+M^2=p^lK$ with $p\wedge K=1$.

First K is odd, because if it is even, from the equality $\sum_i d_i(M^2)+1+M^2=p^lK$ with $p\wedge K=1$, we deduce that 2 divides $\sum_i d_i(M^2)$, so $\sum_i d_i(M^2)$ is even.
But M^2 has an odd number of divisors (because it is a square), moreover the divisors of M^2 are all odd, so $\sum_i d_i(M^2)$ must be odd. Which is contradictory.
By substituting in the above equality, we will have:

$$p^{2k+1}M^2=p^lK+\sum_{j=1}^{2k+1}p^j+\sum_{j=1}^{2k+1}p^j(\sum d_i(M^2))+\sum_{j=1}^{2k}p^jM^2$$

So $p^{2k+1}M^2=p^lK+\sum_{j=1}^{2k+1}p^j+\sum_{j=1}^{2k+1}p^j(p^lK-1-M^2)+\sum_{j=1}^{2k}p^jM^2$

And $p^{2k+1}M^2=p^lK+p\frac{p^{2k+1}-1}{p-1}+p\frac{p^{2k+1}-1}{p-1}(p^lK-1-M^2)+p\frac{p^{2k}-1}{p-1}M^2$

We deduce that: $p^{2k}M^2=p^{l-1}K+\frac{p^{2k+1}-1}{p-1}+\frac{p^{2k+1}-1}{p-1}(p^lK-1-M^2)+\frac{p^{2k}-1}{p-1}M^2$

And $p^{2k}M^2=p^{l-1}K+\frac{p^{2k+1}-1}{p-1}(p^lK-M^2)+\frac{p^{2k}-1}{p-1}M^2$

And finally $2p^{2k}M^2=p^{l-1}K+p^lK\frac{p^{2k+1}-1}{p-1}=p^{l-1}K(1+p\frac{p^{2k+1}-1}{p-1})$

Since K is odd, then $l-1=2k$. And we will have:

$$2M^2=K(1+p\frac{p^{2k+1}-1}{p-1})=K(\frac{p^{2k+2}-1}{p-1})=K(\frac{p^{k+1}-1}{p-1})(p^{k+1}+1)$$

If $p=3\ mod(4)$, from the above equality, we deduce that $2M^2=0\ mod(4)$, and M and therefore N will be even, which is impossible.
The remaining and most delicate case is the one where $p=1\ mod(4)$ which I will demonstrate by the following stratagem:
From the equality :

$N=p^{2k+1}M^2=\sum_i d_i(M^2)+(1+\sum_{j=1}^{2k+1}p^j)+\sum_{j=1}^{2k+1}p^j(\sum d_i(M^2))+M^2+\sum_{j=1}^{2k}p^jM^2$, we deduce:

$$M^2=\frac{\sum d_i(M^2)}{p^{2k+1}}+\frac{1}{p^{2k+1}}+\frac{\sum_{j=1}^{2k+1}p^j}{p^{2k+1}}+\frac{\sum_{j=1}^{2k+1}p^j}{p^{2k+1}}(\sum d_i(M^2))+\frac{\sum_{j=1}^{2k}p^j}{p^{2k+1}}M^2+\frac{M^2}{p^{2k+1}}$$

So :

$$M^2=\sum d_i(M^2)\frac{(\frac{1}{p})^{2k+2}-1}{\frac{1}{p}-1}+\frac{1}{p^{2k+1}}+\frac{1}{p^{2k}}\frac{p^{2k+1}-1}{p-1}+\frac{M^2}{p}\frac{(\frac{1}{p})^{2k+1}-1}{\frac{1}{p}-1}$$

And :

$$M^2(1-\frac{1}{p}\frac{(\frac{1}{p})^{2k+1}-1}{\frac{1}{p}-1})-(\sum d_i(M^2))\frac{(\frac{1}{p})^{2k+2}-1}{\frac{1}{p}-1}=\frac{1}{p^{2k+1}}+\frac{1}{p^{2k}}\frac{p^{2k+1}-1}{p-1}$$

As $1-\frac{1}{p}\frac{(\frac{1}{p})^{2k+1}-1}{\frac{1}{p}-1}=\frac{(\frac{1}{p})^{2k+2}-1}{\frac{1}{p}-1}$, Then we have :

$$M^2-\sum d_i(M^2)=\frac{\frac{1}{p}-1}{(\frac{1}{p})^{2k+2}-1}(\frac{1}{p^{2k+1}}+\frac{1}{p^{2k}}\frac{p^{2k+1}-1}{p-1})\leq 2(\frac{1}{p^{2k+1}}+\frac{1}{p^{2k}}\frac{p^{2k+1}-1}{p-1})\leq 2(1/5+2)\leq 5$$

.

So : $M^2=\sum d_i(M^2)+\mu$ with $\mu\in\{0,2,4\}$ (because $M^2-\sum d_i(M^2)$ is an even integer), but N is perfect, so :

$$\begin{aligned} N&=p^{2k+1}M^2\\ &=p^{2k+1}(\sum d_i(M^2)+\mu)\\ &=p^{2k+1}+\sum_{j=0}^{2k}p^j+p^{2k+1}(\sum d_i(M^2))+(\sum d_i(M^2))(\sum_{j=0}^{2k}p^j)+(\sum_{j=0}^{2k}p^j)M^2 \end{aligned}$$

.

So : $(\mu-1)p^{2k+1}=(\sum_{j=0}^{2k}p^j)(1+M^2+\sum d_i(M^2))$.

Let us distinguish two cases:

If $k\neq 0$, we have an immediate contradiction by using the above equality.

If $k=0$: by using the equality :

$$M^2-\sum d_i(M^2)=\frac{\frac{1}{p}-1}{(\frac{1}{p})^{2k+2}-1}(\frac{1}{p^{2k+1}}+\frac{1}{p^{2k}}\frac{p^{2k+1}-1}{p-1})$$

We deduce that $M^2-\sum d_i(M^2)=1$ which is absurd (because $M^2-\sum d_i(M^2)$ is an even integer).

Hence the Theorem.

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