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‘Useful’ R-norm Information Measure and its Properties

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Abstract : *In the present communication, a new ‘useful’ R-norm information measure has been defined and characterized axiomatically. Its particular cases have been discussed. Properties of the new measure have also been studied.*
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I. Introduction

Let us consider the set of positive real numbers, not equal to 1 and denote this by $\Re^+$ defined as $\Re^+ = \{R : R \geq 0, R \neq 1\}$. Let Δ_n with $n \geq 2$ is the set of all probability distributions

$$P = \left\{(p_1, p_2, \ldots\ldots, p_n), \; p_i \geq 0 \text{, for each } i \text{ and } \sum_{i=1}^{n} p_i = 1\right\}.$$

[1] studied R-norm information of the distribution P defined for $R \in \Re^+$ by:

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^{R}\right)^{1/R}\right] \tag{1}$$

The R-norm information measure (1) is a real function $\Delta_n \to \Re^+$ defined on Δ_n ,where $n \geq 2$ and R^+ is the set of real positive numbers. The measure (1) is different from entropies of [2], [3], [4] and [5]. The main property of this measure is that when $R \to 1$ (1) approaches to Shannon’s entropy and when $R \to \infty$, $H_R(P) \to 1 - \max p_i$, where $i = 1,2,\ldots,n$.

The measure (1) can be generalized in so many ways. [6] Proposed and characterized the following parametric generalization of (1.1):

$$H_R^{\beta}(P) = \frac{R}{R+\beta-2}\left[1 - \left(\sum_{i=1}^{n} p_i^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}}\right], \; 0 < \beta \leq 1, \quad R(>0) \neq 1 \tag{2}$$

The above measure (2) was called generalized R-norm information measure of degree β and it reduces to (1) when β=1.

Further when R=1 (2) reduces to:

$$H_1^{\beta}(P) = \frac{1}{\beta-2}\left[1 - \left(\sum p_i^{\frac{1}{2-\beta}}\right)^{2-\beta}\right], \; 0 < \beta \leq 1 \tag{3}$$

In case $\gamma = \dfrac{1}{2-\beta}$ reduces to:

$$H^{\gamma}(P) = \frac{\gamma}{\gamma-1}\left[1 - \left(\sum_{i=1}^{n} p_i^{\gamma}\right)^{\frac{1}{\gamma}}\right] \quad , \frac{1}{2} < \gamma \leq 1 \tag{4}$$

This is an information measure which has been given by [7]. It can be seen that (4) also reduces to Shannon’s entropy when $\gamma \to 1$.

[8] Proposed and studied the following parametric generalization of (1):

$$H_R^{\alpha,\beta}(P) = \frac{R}{R+\beta-2\alpha}\left[1 - \left(\sum_{i=1}^{n} p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}\right], \alpha \geq 1, \, 0 < \beta \leq 1, \, R(>0) \neq 1, \, 0 < R+\beta \neq 2\alpha \tag{5}$$

They called (5) as the generalized R-norm information measure of type α and degree β. (5) reduces to (2) when $\alpha = 1$ and it further reduces to (1) when $\beta = 1$.Recently, [9] have applied (5) in studying the bounds of generalized mean code length.

In order to distinguish the events $E_1, E_2, \ldots, E_n$ with respect to a given qualitative characteristic of physical system taken into account, we ascribe to each event E_i a non- negative number $u(E_i) = u_i (> 0)$ directly proportional to its importance. We call u_i, the utility or importance of event E_i where probability of occurrence is p_i.In general u_i is independent of p_i (see [10]).

[11] characterized a quantitative-qualitative measure which was called 'useful' information by [10] of the experiment E and is given as:

$$H(P;U) = H(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n)$$

$$= -\sum u_i p_i \log p_i, \quad u_i > 0, \quad 0 < p_i \le 1, \quad \sum p_i = 1 \tag{6}$$

Later on [12] characterized the following measure of 'useful' information:

$$H(P;U) = \frac{-\sum u_i p_i \log p_i}{\sum u_i p_i} \tag{7}$$

Analogous to (1) we consider a measure of 'useful' R-norm information as given below:

$$H_R(P;U) = \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^R}{\sum u_i p_i}\right)^{1/R}\right], \tag{8}$$

where $U = (u_1, u_2, \ldots u_n)$ is the utility distribution and $u_i > 0$ is the utility of an event with probability p_i. It may be noted that if $R \to 1$, then (8) reduces to (7). Further let Δ_n^* be a set of utility distributions s.t. $U \in \Delta_n^*$ is utility distribution corresponding to $P \in \Delta_n$.

In the present paper we characterize the 'useful' R-norm information measure (8) axiomatically in section 2. In section 3 we study the properties of the new measure of 'useful' R-norm information measure.

II. Axiomatic Characterization

Let $S_n = \Delta_n \times \Delta_n^* \to R^+$, $n = 2,3,\ldots\ldots\ldots$ and G_n be a sequence of functions of $p_i{}'s$ and $u_i{}'s$, $i = 1,2,\ldots\ldots\ldots, n$, defined over S_n satisfying the following axioms:

Axiom 2.1. $G_n(P:U) = a_1 + a_2 \sum_{i=1}^{n} h(p_i, u_i)$, where a_1 and a_2 are non zero constants, and

$$p, u \in J = \{(0,1) \times (0,\infty)\} \cup \{(0,y); 0 \le y \le 1\} \cup \{(\infty, y') : 0 \le y' \le \infty\}.$$

This axiom is also called sum property.

Axiom 2.2. For $P \in \Delta_n, U \in \Delta_n^*, P' \in \Delta_m$, and $U' \in \Delta_m^*$, G_{mn} satisfies the following property:

$$G_{mn}(PP' : UU') = G_n(P:U) + G_m(P':U') - \frac{1}{a_1} G_n(P:U) G_m(P':U').$$

Axiom 2.3. $h(p,u)$ is a continuous function of its arguments p and u.

Axiom 2.4. Let all $p_i{}'s$ and $u_i{}'s$ are equiprobable and of equal utility of events respectively, then

$$G_n\left(\frac{1}{n}, \ldots\ldots\ldots, \frac{1}{n}, u, \ldots\ldots\ldots, u\right) = \frac{R}{R-1}\left(1 - n^{\frac{1-R}{R}}\right), \quad \text{where } n = 2,3,\ldots, \text{ and } R(>0) \ne 1$$

First of all we prove the following three lemmas to facilitate to prove the main theorem:

Lemma 2.1. From axiom 2.1 and 2.2, it is very easy to arrive at the following functional equation:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}h\left(p_i p'_j, u_i u'_j\right)=\left(\frac{-a_2}{a_1}\right)\sum_{i=1}^{n}h(p_i,u_i)\sum_{j=1}^{m}h\left(p'_j,u'_j\right), \tag{9}$$

where $(p_i,u_i), (p'_j,u'_j)\in J$ for $i=1,2,\ldots\ldots\ldots\ldots,n$ and $j=1,2,\ldots\ldots\ldots\ldots,m.$

Lemma 2.2. The continuous solution that satisfies (9) is the continuous solution of the functional equation:

$$h(pp',uu')=\left(\frac{-a_2}{a_1}\right)h(p,u)h(p',u'), \tag{10}$$

Proof: Let a,b,c,d and a',b',c',d' be positive integers such that $1\le a'\le a, 1\le b'\le b, 1\le c\le c'$, and $1\le d\le d'$.

Setting $n=a-a'+1=c'-c+1$ and $m=b-b'+1=d'-d+1,$

$p_i=\frac{1}{a}(i=1,2,\ldots,a-a'),\quad p_{a-a'+1}=\frac{a'}{a},$

$u_i=\frac{1}{c}(i=1,2,\ldots,c'-c),\ u_{c'-c+1}=\frac{c'}{c},$

$p'_j=\frac{1}{b}(j=1,2,\ldots,b-b'),\ p'_{b-b'+1}=\frac{b'}{b},$

$u'_j=\frac{1}{d}(j=1,2,\ldots,d'-d),\ u'_{d'-d+1}=\frac{d'}{d},$

From equation (9) we have:

$$(a-a')(b-b')h\left(\frac{1}{ab},\frac{1}{cd}\right)+(b-b')h\left(\frac{a'}{ab},\frac{c'}{cd}\right)+(a-a')h\left(\frac{b'}{ab},\frac{d'}{cd}\right)+h\left(\frac{a'b'}{ab},\frac{c'd'}{cd}\right)$$
$$=\left(\frac{-a_2}{a_1}\right)\left[(a-a')h\left(\frac{1}{a},\frac{1}{c}\right)+h\left(\frac{a'}{a},\frac{c'}{c}\right)\right]\left[(b-b'')h\left(\frac{1}{b},\frac{1}{d}\right)+h\left(\frac{b'}{b},\frac{d'}{d}\right)\right] \tag{11}$$

Taking $a'=b'=c'=d'=1$ in (11), we get:

$$h\left(\frac{1}{ab},\frac{1}{cd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{1}{a},\frac{1}{c}\right)h\left(\frac{1}{b},\frac{1}{d}\right). \tag{12}$$

Taking $a'=c'=1$ in (11) and using (12), we have:

$$h\left(\frac{b'}{ab},\frac{d'}{cd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{1}{a},\frac{1}{c}\right)h\left(\frac{b'}{b},\frac{d'}{d}\right). \tag{13}$$

Again taking $b'=d'=1$ in (11) and using (12), we get:

$$h\left(\frac{a'}{ab},\frac{c'}{cd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{1}{b},\frac{1}{d}\right)h\left(\frac{a'}{a},\frac{c'}{c}\right) \tag{14}$$

Now (11) together with (12), (13) and (13) reduces to:

$$h\left(\frac{a'b'}{ab},\frac{c'd'}{cd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{a'}{a},\frac{c'}{c}\right)h\left(\frac{b'}{b},\frac{d'}{d}\right) \tag{15}$$

Putting $\frac{a'}{a}=p,\ \frac{c'}{c}=u,\frac{b'}{b}=p',\frac{d'}{d}=u'$ in (15), we get the required results (10).

Next we obtain the general solution of (10).

Lemma2.3. One of the general continuous solution of equation (10) is given by:

$$h(p,u)=\left(\frac{-a_1}{a_2}\right)\left(\frac{p^{\mu}u^{\nu}}{pu}\right)^{1/\mu}, \text{ where } \mu\ne 0,\ \nu\ne 0 \tag{16}$$

and $h(p,u)=0$ (17)

Proof: Taking $g(p,u)=\left(\frac{-a_2}{a_1}\right)h(p,u)$ in (10), we have:

$$g(pp',uu')=g(p,u)g(p',u') \tag{18}$$

The most general continuous solution of (18) (refer to [13]) is given by:

$$g(p,u)=\left(\frac{p^{\mu}u^{\nu}}{pu}\right)^{1/\mu},\ \mu\neq 0 \text{ and } \nu\neq 0 \tag{19}$$

and

$$g(p,u)=0 \tag{20}$$

On substituting $g(p,u)=\left(\frac{-a_2}{a_1}\right)h(p,u)$ in (19) and (20) we get (16) and (17) respectively. This proves the lemma 2.3 for all rationals $p\in\,]0,1[$ and $u>0$, However, by continuity, it holds for all reals $p\in\,]0,1[$ and $u>0$.

Theorem2.1. The measure (8) can be determined by the axiom 2.1 to 2.4.

Proof: Substituting the solution (16) in axiom 2.1 we have:

$$G_n(P;U)=a_1\left[1-\sum_{i=1}^{n}\left(\frac{p_i^{\mu}u_i^{\nu}}{p_iu_i}\right)^{1/\mu}\right],\qquad \mu\nu\neq 0 \tag{21}$$

Taking $p_i=\frac{1}{n}$ and $u_i=u$ for each i in (21) we have:

$$G_n\left(\frac{1}{n},\ldots,\frac{1}{n},u,\ldots,u\right)=a_1\left(1-n^{\frac{1-\mu}{\nu}}u^{\frac{\nu-1}{\nu}}\right),\qquad n=2,3,\ldots, \tag{22}$$

Axiom (2.4) together with (22) gives:

$$a_1\left(1-n^{\frac{1-\mu}{\mu}}u^{\frac{\nu-1}{\nu}}\right)=\frac{R}{R-1}\left[1-n^{\frac{1-R}{R}}\right]$$

It implies

$$a_1=\frac{R}{R-1},\quad \mu=R,\quad \nu=1$$

Putting these values in (21) we have

$$G_n(P;U)=\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n}u_ip_i^{R}}{\sum_{i=1}^{n}u_ip_i}\right)^{1/R}\right]$$

$$=H_R(P;U)$$

Hence this completes the proof of theorem 2.1.

Particular cases:

(a) When utilities are ignored i.e. $u_i=1$ for each i, (8) reduces to (1).

(b) Further $R\to 1$, (1) reduces to Shannon's entropy [13].

III. Properties of 'useful' R-norm Information Measure

The 'useful' R-norm information measure $H_R(P;U)$ satisfies the following properties:

Property 3.1. $H_R(P;U)$ is symmetric function of their arguments provided that the permutation of $p_i's$ and $u_i's$ are taken together.

$$H_R(p_1,p_2,\ldots,p_{n-1},p_n;u_1,u_2,\ldots u_{n-1},u_n)=H_R(p_n,p_1,p_2,\ldots,p_{n-1};u_n,u_1,u_2,\ldots,u_{n-1})$$

Property 3.2. $H_R\left(\frac{1}{8},\frac{1}{8};1,1\right)=1$

Proof: $H_R(P;U)=\frac{R}{R-1}\left[1-\left(\frac{\sum u_ip_i^{R}}{\sum u_ip_i}\right)^{1/R}\right]$

for $i=1,2$

$$H_R(P;U)=\frac{R}{R-1}\left[1-\left(\frac{u_1p_1^{R}}{u_1p_1}+\frac{u_2p_2^{R}}{u_2p_2}\right)^{1/R}\right]$$

Taking $p_1 = \frac{1}{8}, \quad p_2 = \frac{1}{8}, \quad u_1 = 1, \quad u_2 = 1$ and $R = 2$

$$H_R\left(\frac{1}{8}, \frac{1}{8}; 1, 1\right) = \frac{2}{1}\left[1 - \left\{\frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}} + \frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}}\right\}^{1/2}\right] = 1$$

Property 3.3. Addition of two events whose probability of occurrence is zero or utility is zero has no effect on useful information, i.e.

$$H_R(p_1, p_2, \ldots, p_n, 0; u_1 u_2, \ldots u_{n+1}) = H_R(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n) = H_n(p_1, p_2, \ldots, p_{n+1}; u_1, u_2, \ldots, u_n, 0).$$

Proof: Let us consider

$$H_R(p_1, p_2 \ldots, p_n, 0; u_1, u_2, \ldots, u_{n+1}) == \frac{R}{R-1}\left[1 - \left\{\frac{u_1 p_1^R}{u_1 p_1} + \frac{u_2 p_2^R}{u_2 p_2} + \ldots + \frac{u_n p_n^R}{u_n p_n} + \ldots + \frac{0^R . u_{n+1}}{0 . u_{n+1}}\right\}^{1/R}\right]$$

$$= H_R(P; U)$$

Similarly we can prove that

$$H_n(p_1, p_2, \ldots, p_n, p_{n+1}; u_1, u_2, \ldots, u_n, u_{n+1}) = H_R(P; U)$$

Property 3.4. $H_R(P;U)$ satisfies the non-additivity of the following form:

$$H_R(P * Q; U * V) = H_R(P;U) + H_R(Q;V) - \frac{R-1}{R} H_R(P;U) H_R(Q;V)$$

where $P * Q = (p_1 q_1, \ldots, p_1 q_m, p_2 q_1, \ldots, p_2 q_m, p_n q_1, \ldots, p_n q_m)$, and

$U * V = (u_1 v_1, \ldots, u_1 v_m, u_2 v_1, \ldots, u_2 v_m, u_n v_1, \ldots, u_n v_m)$

Proof: R.H.S= $H_R(P;U) + H_R(Q;V) - \frac{R}{R-1} H_R(P;U) H_R(Q;V)$

$$= \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^R}{\sum u_i p_i}\right)^{1/R}\right] + \frac{R}{R-1}\left[1 - \left(\frac{\sum v_j q_j^R}{\sum v_j q_j}\right)^{1/R}\right] - \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^R}{\sum u_i p_i}\right)^{1/R}\right]\left[1 - \left(\frac{\sum v_j q_j^R}{\sum v_j q_j}\right)^{1/R}\right]$$

$$= \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^R}{\sum u \; p_i}\right)^{1/R} + 1 - \left(\frac{\sum v_j q_j^R}{\sum v_j q_j}\right)^{1/R} - 1 + \left(\frac{\sum u_i p_i^R}{\sum u_I p_i}\right)^{1/R} + \left(\frac{\sum q_j v_j^R}{\sum q_j v_j}\right)^{1/R} - \left(\frac{\sum u_i p_i^R}{\sum u_I p_i}\right)^{1/R}\left(\frac{\sum q_j v_j^R}{\sum q_j v_j}\right)^{1/R}\right]$$

$$= \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^R}{\sum u_I p_i}\right)^{1/R}\left(\frac{\sum q_j v_j^R}{\sum q_j v_j}\right)^{1/R}\right] = \frac{R}{R-1}\left[1 - \left(\frac{\sum_{i=1}^{n} \sum_{j=1}^{m} u_i v_j (p_i q_j)^R}{\sum_{i=1}^{n} \sum_{j=1}^{m} u_i v_j p_i q_j}\right)^{1/R}\right]$$

$= H_R(P * Q; U * V)$ = L.H.S.

Property 3.5. Let A_i, A_j be two events having probabilities p_i, p_j and utilities $u_{i,}, u_j$ respectively, then we define the utility u of the compound event $A_i \cap A_j$ as:

$$u(A_i \cap A_j) = \frac{u_i p_i + u_j p_j}{p_i + p_j} \tag{23}$$

Theorem 3.1 Under the composition law (23), the following holds:

$${}_{n+1}H_R(p_1, p_2, \ldots, p_{n-1}, p', p''; u_1 u_2, \ldots, u_{n-1}, u', u'') = {}_nH_R(P, U) + (p' + p'') H_R\left(\frac{p'}{p' + p''}, \frac{p''}{p' + p''}; u', u''\right),$$

where $p_n = p' + p''$ and $u_n = \frac{p'u' + p''u''}{p' + p''}$

Proof: ${}_{n+1}H_R(p_1, p_2, \ldots, p_{n-1}, p', p''; u_1, u_2, \ldots, u_{n-1}, u', u'')$

$$={}_nH_R\left(p_1,p_2,\ldots,p_{n-1};u_{1,}u_2\ldots,u_{n-1}\right)+\frac{R}{R-1}\left[1-\left\{\frac{u'p'^R}{u'p'}+\frac{u''p''^R}{u''p''}\right\}^{1/R}\right]$$

$$={}_nH_R+\frac{R}{R-1}\left[(p'+p'')-\left\{\frac{u'\left(\frac{p'}{p'+p''}\right)^R+u''\left(\frac{p''}{p'+p''}\right)^R}{u'\left(\frac{p'}{p'+p''}\right)+u''\left(\frac{p''}{p'+p''}\right)}\right\}^{1/R}(p'+p'')\right]$$

$$={}_nH_R+(p'+p'')H_R\left(\frac{p'}{p'+p''},\frac{p''}{p'+p''};u',u''\right)$$

This completes the proof of theorem 3.1.

IV. Conclusion

R- norm information measure is defined and characterized when the probability distribution P belong to R- norm vector space. This is a new addition to the family of generalized information measures.

In present paper we have considered that physical system has qualitative characterization in addition to quantitative and have defined and characterized a new measure R-norm information measure . This measure can further be generalized in many ways and can be applied in source coding when source symbols have utility also in addition to probability of occurrence.

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