IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728 p-ISSN: 2319–765X
PP 16-22
www.iosrjournals.org

Poisson equation in infinite networks

Sujith S
Department of Science, Sree Narayana Gurukulam College of Engineering. Kadayiruppu, Kolenchery, Ernakulam, Kerala.

Abstract: *Under some restrictions, a global solution u of the Poisson equation $\Delta u = f$ can be constructed in Euclidean Spaces, Riemann surfaces, Brelot harmonic spaces and also in finite electrical networks where the conductance $c(x, y)$ is symmetric in the vertices x and y. We discuss here some situations where the discrete Poisson equation has a solution in infinite networks or in infinite trees, without assuming the symmetry of the conductance.*
2000 Mathematics Subject Classification: *31C20, 31B30.*
Keywords and Phrases: *Bipotential infinite networks, Singular vertices in infinite trees, discrete Poisson equation.*

Introduction

Brelot[4] shows that if w is an open set in $\mathbb{R}^n$, $n \geq 2$, and if $\mu \geq 0$ is a Radon measure on w, then there exists a superharmonic function u on w such that $\Delta u = -\mu$ in the sence of distributions. To prove this he makes use of a Runge-type approximation theorem for harmonic functions in $\mathbb{R}^n$. A similar approximation theorem for harmonic functions in a Riemann surface is proved in Pfluger[6]. Consequently, following the method used by Brelot in the Eucledean case, we can prove that $\Delta u = -\mu$ has a solution in any Riemann surface also. In the axiomatic case of Brelot[4], De la Pradelle[5] proves a similar approximation theorem by introducing the axiome d'an analyticite' in the adjoint harmonic space. (This axiom of analyticity in a Brelot harmonic space Ω states that if h is a harmonic function in a domain ω vanishing in a neighbourhood of a point in ω, then h is identically 0). Using this result Anandam[1] shows that if $\mu \geq 0$ is a Radon measure on ω, then there exists a superharmonic function u in ω with associated measure μ in a local Rieze representation.

Now Kirchhoff in a finite electrical network X (see for example, Anandam[1])proves that if f is a real-valued function on X, then there exists a real-valued function u such that $\Delta u = f$ on X if and only if $\sum_{x \in X} f(x) = 0$. (Here Δ is the discrete Laplace operator represented by a symmetric square matrix). If T is an infinite tree without any terminal vertices Bajunaid et al. [2] and [3] show that if h is a harmonic function in $B_n = \{x : x \in T, |x| \leq n\}$ where $|x| = d(x, e)$ is the distance of x from a fixed vertex e, then there exists a harmonic function H on T such that $H = h$ on B_n. Consequently for any real valued function f on T, it is possible by the above method to choose a real-valued function u on T such that $\Delta u = f$.

Thus the existence of a solution to the Poisson equation $\Delta u = f$ depends on certain restrictions on the base space ω and the function f. In this note we investigate restrictions that may be necessary to solve the Poisson equation $\Delta u = f$ in an arbitrary infinite network without symmetric conductances.

Preliminaries

Let X be an infinite network (that is, a graph with countable infinite vertices and countable infinite edges), connected (that is, any two vertices can be joined by a path), locally finite (that is, any vertex has only a finite number of adjacent vertices called neighbours) and without self loops(that is any vertex is not considered as its own neighbour). To any pair of vertices x, y in X is associated a non-negative real number $t(x,y)$ such that $t(x,y) > 0$ if and only if x and y are neighbours (denoted by $x \sim y$). Consequently $t(x) = \sum_{y \in X} t(x,y) = \sum_{y \sim x} t(x,y) > 0$ for any vertex x in X . We do not place the restriction that $t(x,y) = t(y,x)$ for every pair x , y (that is, $t(x,y)$ need not be symmetric). We write $p(x,y) = \frac{t(x,y)}{t(x)}$ for any pair x, y in X . Then $\sum_y p(x,y) = 1$ for every x . Given any real-valued function u on X , write $\Delta u(x) = \sum_y p(x,y)[u(y) - u(x)]$ for $x \in X$. If E is an arbitrary set in X , then we say that $z \in E$ is an interior point of E if all the neighbours of x also lie in E . Let us denote by $\overset{\circ}{E}$ the set of all interior points of E . $\partial E = E \setminus \overset{\circ}{E}$ is termed the boundary of E . If u is a real-valued function on E , and if

$$\Delta u(x) = \sum_y p(x,y)[u(y) - u(x)] \le 0$$

for every $x \in \overset{\circ}{E}$ then we say that u is superharmonic on E ; if $\Delta u(x) = 0$ for every $x \in \overset{\circ}{E}$, u is said to be harmonic on E .

Let us fix a vertex e in X . If x is any vertex in X and if $e = x_0, x_1, \ldots x_n = x$ is a path connecting e to x , then we say that this path has a length n . There may be many such paths with corresponding lengths, connecting e to x . The shortest length is called the distance between e and x and denoted by $|x| = d(e,x)$. Remark that in an infinite tree there is only one path with the shortest length. (We use the term infinite tree to denote an infinite network which does not have any closed path.)

Poisson Equation in an infinite network

In a general network X without any assumption it is difficult to solve the Poisson equation. Hence we require some extra conditions. In the following section we assume that the network is a bipotential network. In a network X , given $z \in X$, there always exists a superharmonic function $q_z(x)$ on X such that $\Delta q_z(x) = -\delta_z(x)$ [7]. Consequently, if f is any real-valued function on X that vanishes outside a finite set, then

$$u(x) = -\sum_{z \in X} f(z) q_z(x)$$

is a well-defined function on X such that $\Delta u(x) = f(x)$ for any $x \in X$.

In particular, if F is a finite set in X and if f is a real-valued function on F , then there exists u on F such that $\Delta u(x) = f(x)$ for every $x \in \overset{\circ}{F}$. Generally of course it is clear that if $\phi \ge 0$ is a real-valued function on X such that $v(x) = \sum_{z \in X} \phi(z) q_z(x)$ is finite at each vertex x , then $v(x)$ is superharmonic on X such that $\Delta v(x) = -\phi(x)$. For, $v(x) = \lim_{n \to \infty} \sum_{z \in E_n} \phi(z) q_z(x)$ where $\{E_n\}$ is an exhaustion of X by finite set in the sense that each E_n is a finite set, $E_n \subset \overset{\circ}{E}_{n+1}$ and $X = \bigcup_n E_n$ and we know that the limit of superharmonic functions is superharmonic if the limit function is finite.

Definition 3.1 *A function* b *in an arbitrary set* E *in an infinite network* X *is said to be biharmonic if there exists a harmonic function* h *on* E *such that* $\Delta b(x) = h(x)$ *for each* $x \in \overset{\circ}{E}$.

Properties:

1. Assume that $t(x) = \sum_{y:x} t(x, y)$ is bounded for every $x \in X$. Then if b is biharmonic on X and bounded, $\Delta b = h$ is also bounded.

Proof. Since b is biharmonic $\Delta b = h$. That is,

$$\begin{aligned}
|h(x)| &= |\sum_{y\sim x} t(x, y)[b(y) - b(x)]| \\
&= |\sum_{y\sim x} t(x, y)b(y) - t(x)b(x)| \\
&\leq \sum_{y\sim x} t(x, y)|b(y)| + t(x)|b(x)| \\
&\leq Mt(x) + Mt(x).
\end{aligned}$$

That is, h is bounded.

2. If b_n is a sequence of biharmonic functions converging to b on X , then b is biharmonic.

Proof. Since b_n is biharmonic, there exists h_n harmonic on X such that $\Delta b_n = h_n$.

Now,

$$\begin{aligned}
h_n &= \Delta b_n \\
&= \sum_{y} t(x, y)[b_n(y) - b_n(x)] \\
&= \sum_{y} t(x, y)b_n(y) - b_n(x) \\
&\to \sum_{y} t(x, y)b(y) - b(x) \quad \textit{since } b_n \to b. \\
&= \Delta b(x)
\end{aligned}$$

Now, if $b_n \to b$ then $\Delta b_n \to \Delta b$. That is $h_n \to \Delta b$. Since h_n is harmonic and $h_n \to \Delta b$,we deduce that Δb is also harmonic. Hence b is biharmonic.

Definition 3.2 *A hyperbolic network* X *is said to be a bipotential network if there exists a pair of positive potentials* p *and* q *on* X *such that* $(-\Delta)q = p$.

Remark 3.3 *It can be shown that if* X *is a bipotential network and if* $p \geq 0$ *is a potential with finite harmonic support, then there exist a unique potential* $q \geq 0$ *on* X *such that* $(-\Delta)q = p$ *on* X *. For, if* $G_y(x)$ *is the Green potential on* X *, then* $q(x) = \sum_{y} p(y)G_y(x)$.

Remark 3.4 *In particular, if* $G_y(x)$ *is the Green potential with point harmonic support* $\{y\}$ *in a bipotantial network* X *, then there exist a unique potential* $Q_y(x)$ *on* X *such that* $(-\Delta)Q_y(x) = G_y(x)$ *. The potential* $Q_y(x)$ *is called a biharmonic Green potential with point harmonic support* $\{y\}$ *.*

Notation: We have defined that a function u is biharmonic if $\Delta u = h$ where h is harmonic. We can actually represent the biharmonic function u as a pair (u,h) or $(u,(-\Delta)u)$. Similarly we can represent a bisuperharmonic function as $(u_1,(-\Delta)u_1)$ where $(-\Delta)u_1$ is a superharmonic function. We say that $(u_1,(-\Delta)u_1)\geq 0$ is a non-negative bisuperharmonic function if $u_1\geq 0$ and $(-\Delta)u_1\geq 0$ is a non-negative superharmonic function.

Theorem 3.5 *If* $u\geq 0$ *is a bisuperharmonic function on a subset* E *in a bipotential network* X *, then* u *is the sum of a bipotential and a biharmonic function on* E *.*

Proof. Given $u\geq 0$ that is $(u,(-\Delta)u)\geq 0$. Since $(-\Delta)u\geq 0$, that is u is superharmonic, it can be written as the sum of a potential p and a harmonic function h. That is $(-\Delta)u(x)=p(x)+h(x)$.

Also $u\geq 0$ is superharmonic, that is

$$\begin{aligned} u(x) &= \sum_{y\in X} G_y(x)(-\Delta)u(y)+a \quad harmonic \quad function \quad h_1(x) \\ &= \sum_{y\in X} G_y(x)[p(y)+h(y)]+h_1(x) \\ &= \sum_{y\in X} G_y(x)p(y)+\sum_{y\in X} G_y(x)h(y)+h_1(x), \end{aligned}$$

where $Q(x)=\sum_{y\in X} G_y(x)p(y)$ is a potential such that $(-\Delta)Q(x)=p(x)$; and hence $Q(x)$ is a bipotential and $H(x)=\sum_{y\in X} G_y(x)h(y)+h_1(x)$ is biharmonic since $\Delta H(x)=\Delta[\sum_{y\in X} G_y(x)h(y)]=-h(x)$.

Thus we have

$$\begin{aligned} (u,(-\Delta)u) &= (Q+H,p+h) \\ &= (Q,p)+(H,h) \end{aligned}$$

where (Q,p) is a bipotential and (H,h) is a biharmonic function.

Theorem 3.6 *Let* X *be a bipotential network. Let* f *be a real-valued function on* X *such that* $|f|\leq s$ *where* s *is a potential with finite harmonic support in* X *. Then there exists* u *on* X *such that* $\Delta u=f$ *. If* $f\leq 0$ *, then* u *can be chosen as a potential on* X *.*

Proof. Since X is a bipotential network, $(-\Delta)q=p$ on X, for a pair of positive potentials in X. Let s have harmonic support on the finite set A. Then we can choose a constant $\alpha>0$ such that $s\leq\alpha p$ on A. Consequently by the Domination Principle[7] $s\leq\alpha p$ on X. If $G_y(x)=G(x,y)$ in the Green function on X,

$$\begin{aligned} \alpha q(x) &= \sum_{y\in X}(-\Delta)\alpha q(y)G_y(x) \\ &= \sum_{y\in X}\alpha p(y)G_y(x) \\ &\geq \sum_{y\in X} s(y)G_y(x) \\ &\geq \sum_{y\in X}|f(y)|G_y(x) \end{aligned}$$

Hence $u(x) = -\sum_{y \in X} f(y) G_y(x)$ is finite for each $x \in X$ and $\Delta u(x) = f(x)$. Note that u is the difference of two potentials on X.

If $f \leq 0$, then u is a non-negative superharmonic function and hence the sum of a potential g and a harmonic function. Then $\Delta g = \Delta u = f$.

Theorem 3.7 (Riquier Problem). *Let F be a finite subset of a network X. Let f and g be two functions defined on ∂F. Then there exists a biharmonic function (b, h) on F such that $\Delta b = h$, $\Delta h = 0$ on $\overset{\circ}{F}$, and $b = g$, $h = f$ on ∂F.*

Proof. Replace E by $\overset{\circ}{F}$ in the Generalised Dirichlet problem [7] to obtain h on F such that $h = f$ on ∂F and $\Delta h(x) = 0$ at each $x \in \overset{\circ}{F}$. Extend h by 0 outside F, say h_0. Let v be the function on X defined by $v = -\sum_y G_y(x) h_0(y)$. Then $v(x)$ is finite and $\Delta v(x) = h_0(x)$. Now take a function H harmonic on F such that $\Delta H = 0$ on $\overset{\circ}{F}$ and $H = g - v$ on ∂F. Let $b = v + H$ on F. Then on ∂F, $b = v + g - v = g$ and $\Delta b = \Delta v = h_0 = h$ on $\overset{\circ}{F}$.

Poisson equation in an infinite tree

In a general network X we have required the existence of bipotentials on X to prove the Poisson equation $\Delta u = f$. Now we are relaxing the condition and trying to prove the same in the case of an infinite tree T. Anyway in some cases we need the infinite tree T should be a bipotential tree. But in most of the cases it is not necessary that bipotentials be defined on T.

Definition 4.1 *A non-terminal vertex x, $|x| = n \geq 1$, in an infinite tree T is said to be singular if the set $A = \{y \in X : y : x, |y| = n+1\}$ consists of only terminal vertices.*

Note that if $x \in \partial B_n$ then x has only one neighbour in B_n and all the other neighbours are outside B_n.

Note 4.2 *Since X is infinite, all the neighbours of e cannot be terminal vertices. We can take e as a non-singular vertex.*

Definition 4.3 *An infinite tree T is called a smooth tree if the set of all singular vertices in T is a finite set.*

Example 4.4 *Trees without terminal vertices (such as binary trees, homogeneous trees etc.) are smooth. Trees in which every non-terminal vertex has at least two non-terminal vertices as neighbours are also smooth. (Note that since X is infinite, every non-terminal vertex should have at least one non-terminal vertex as neighbour.)*

Theorem 4.5 *In a smooth tree T with or without potentials, given any real valued function f on T there exists a real-valued function u on T such that $\Delta u = f$.*

Proof. Let $B = \{x : |x| \leq m\}$ be a finite set containing all the singular points in $\overset{\circ}{B}$. Then $u(x) = -\sum_{a \in B} f(a) q_a(x)$ is a real valued function T such that $\Delta u = f$ for each $x \in \overset{\circ}{B}$. Henceforth we suppose that u is defined only on B and extend it to the whole of X such that $\Delta u = f$ on X.

Take a point b on ∂B. b is a non-singular vertex by the choice of B. The vertex b has only one neighbour a in B. Outside B the vertex b has atleast one non-terminal and possibly some terminal vertices as neighbours. Let $z_i : b$ be the set of terminal vertices on $T \setminus B$. Take $u(z_i) = \lambda_i$ so that at z_i,

$$\Delta u(z_i) = t(z_i, b)[u(b) - \lambda_i] = f(z_i).$$

Here λ_i is the only unknown and we can find its value easily. By a similar procedure, if $z \in T \setminus B$ is any terminal vertex that is a neighbour of b, we can define the value of $u(z)$ so that $\Delta u(z) = f(z)$. Since b is not singular we know that b has at least one non-terminal vertex $y_1 \sim b$ on $T \setminus B$. Consider now all the non-terminal vertices y_j in $T \setminus B$ which are neighbours of b. Let $u(y_j) = \mu$ a constant for all j where μ is chosen so that at the vertex b

$$\Delta u(b) = \sum_{x_i \sim b} t(b, x_i)[u(x_i) - u(b)] = f(b)$$

$$f(b) = \sum_{y_j \notin B, y_j \sim b} t(b, y_j)[u(y_j) - u(b)] + \sum_{z_i \sim b, z_i \in B} t(b, z_i)[u(z_i) - u(b)]$$

$$= \sum_{y_j \notin B, y_j \sim b} t(b, y_j)[\mu - u(b)] + \sum_{z_i \sim b, z_i \in B} t(b, z_i)[u(z_i) - u(b)].$$

The value μ is the only unknown in the above equation and we calculate its value. This method extends u to all the neighbours of b outside B.

We repeat this procedure with respect to each of the vertices on ∂B. Then we get an extended function u defined on $v(B)$ (which consists of B and all the neighbours of B) so that $\Delta u(x) = f(x)$ for every $x \in B$. By recurrence we extend u to the whole of X so that $\Delta u(x) = f(x)$ for every $x \in X$.

Corollory 4.6 *If h is harmonic on an arbitrary subset E of a smooth tree T then there exists a biharmonic function b on E generated by h (that is, $\Delta b(x) = h(x)$ for $x \in \overset{\circ}{E}$).*

Proof. Extend h by 0 outside E. Denote the new function by h_0. Since T is smooth, there exists a real-valued function b on T such that $\Delta b = h_0$. In particular, b is biharmonic on E such that $\Delta b(x) = h(x)$ if $x \in \overset{\circ}{E}$.

Theorem 4.7 *Let b be biharmonic outside a finite set in a bipotential smooth tree T. Then there exists a unique biharmonic function B on T such that for a potential p on T, $(b - B)$ and $\Delta(b - B)$ are bounded by p outside a finite set.*

Proof. Let E be a subset of T such that $T \setminus E$ is a finite set and b is biharmonic on E. That is there exists a harmonic function h on E such that $\Delta b = h$ on $\overset{\circ}{E}$. Extend h by 0 on $T \setminus E$. Denote this function by h_0.

Let

$$u(x) = h_0(x) - \sum_{y \in \partial E} (-\Delta) h_0(y) G_y(x).$$

Then u is harmonic on T and $h = u - p_1^{'} + p_2^{'}$ where $p_1^{'}$ and $p_2^{'}$ are potentials with finite harmonic support in T. Since T is assumed smooth, we can chose q_1, q_2 and B_0 such that $(-\Delta) q_1 = p_1^{'}$,

$(-\Delta)q_2 = p_2^{'}$, $(-\Delta)B_0 = u$, $b = B_0 + q_1 - q_2 + v$ outside a finite set, where v is harmonic outside a finite set. Note that $v = H + p_1 - p_2$ where H is harmonic on T and p_1 and p_2 are potentials with finite harmonic support in X [8]. Write $B = B_0 + H$. Then $b = B + q_1 - q_2 + p_1 - p_2$ outside a finite set. Note that since T is a bipotential network, (by Theorem3.6) q_1 and q_2 can be chosen as potentials in X. Then outside a finite set $|b - B| \le q_1 + q_2 + p_1 + p_2$ and $|\Delta(b - B)| \le p_1^{'} + p_2^{'}$. Hence writing $p = q_1 + q_2 + p_1 + p_2 + p_1^{'} + p_2^{'}$ we have proved the theorem.

To prove the uniqueness of B, suppose $B^{'}$ is another biharmonic function on T. Then outside a finite set $|b - b^{'}| \le p^{'}$ and $|\Delta(B - B^{'})| \le p^{'}$, where $p^{'}$ is a potential on T. Let $f = B - B^{'}$ which is biharmonic on T. Then $|f| \le |b - B| + |b - B^{'}| \le p + p^{'}$ outside a finite set. Also $|\Delta f| = |\Delta B - \Delta B^{'}| \le |\Delta(b - B)| + |\Delta(b - B^{'})| \le p + p^{'}$, outside a finite set. Since Δf is harmonic on X and is majorised by the potential $p + p^{'}$ outside a finite set,then $\Delta f = 0$ on X, that is the function f is harmonic on X. Also, since f is bounded by the potential $p + p^{'}$ outside s finite set, $f = 0$. Hence $B = B^{'}$.

References

[1] Victor Anandam, Harmonic functions and potentials on finite or infinite networks, Springer-Verlag Lecture Notes of the Unione Matematica Italiana 12, 2011.

[3] I. Bajunaid, J.M. Cohen, F. Colonna and D. Singman, Trees as Brelot Spaces, Adv. in Appl. Math., 30 (2003); 706-745.

[3] I. Bajunaid, J.M. Cohen, F. Colonna and D. Singman, Corrigendum to Trees as Brelot Spaces, Adv. in Appl. Math., (2010); doi:10.1016/j.aam.2010.09.004

[4] M. Brelot, Axiomatique des fonctions harmoniques, Les Presses de $\ell^{'}$ Université de Montréal, 1966.

[5] De La Pradelle, Approximation et caractère de quasi-analycité dans la théorie axiomatique des fonctions harmoniques, Ann. Inst. Fourier, 17 (1967); 383-399.

[6] A. Pfluger, Theorie der Riemannscher Fl $\ddot{a}$ chen, Springer-Verlag, 1957.

[8] Sujith Sivan, Madhu Venkataraman, Parahyperbolic Networks, Mem. Fac. Sci. and Eng., Shimane University, 44 (2011); 1-16.

[8] Sujith Sivan, $\alpha(x)$-Potentials in an infinite network, Arab Journal of Mathematical Sciences, 17 (2011);101-113.