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Approximation of Function Belonging To The Lip$(\psi(t), p)$ Class By Matrix-Cesaro Summability Method

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Abstract: *In this paper, we have established a theorem on approximation of function belonging to Lip$(\psi(t), p)$ class by Matrix-Cesaro summability method of Fourier series.*
Keywords: *Degree of approximation, Lip$(\psi(t), p)$class of function, Matrix-Cesaro summability method, Fourier series, Lebesgue integral.*

I. Introduction

Bernstein[3] used (C,1) means to obtain the degree of approximation function f by Lip1 class. Jackson[6] determined the degree of approximation by using (C,δ) method in Lipα class for 0<α<1. Alexits[1], Chandra[5], Sahney and Goel[7], Sahney and Rao[8], Alexits and Leindler[2] studied the degree of approximation of function $f \in Lip\alpha$ and obtained the results which are not satisfied for n=0,1 or α=1. Binod Prasad Dhakal[4] studied the degree of approximation of function $f \in Lip\alpha$ considering cases 0<α<1 and α=1 separately using Matrix-Cesaro summability method.

In this paper we have extended this result by obtaining the degree of approximation of function f belonging to a generalized class Lip(α).

II. Definitions And Notations

Let$f$ be a periodic function with period 2π and integrable in the Lebesgue sense. Let its Fourier series be given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad (2.1)$$

The degree of approximation of a function f:R→R by a trigonometric polynomial t_n of order is defined by

$E_n(f) = \|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$

A function $f \epsilon \text{Lip}\alpha$ if

$|f(x+t) - f(x)| = O(|t|^\alpha)$, for$0 < \alpha \le 1$

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose n^{th} partial sum is given by

$$s_n = \sum_{k=0}^{n} u_k$$

Cesaro means (C,1) of sequence $\{s_n\}$ is given by

$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k$.

If $\sigma_n \to s$ as $n \to \infty$ then the sequence $\{s_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesaro means (C,1) to s.

Let T = $(a_{n,k})$ be an infinite lower triangular matrix satisfying the Silverman-Toeplitz conditions of regularity i.e. $\sum_{k=0}^{n} a_{n,k} \to 1$ as $n \to \infty$, $a_{n,k} = 0$, for $k > n$ and $\sum_{k=0}^{n}|a_{n,k}| \le M$, a finite constant.

Matrix-Cesaro means $T(C_1)$ of the sequence $\{s_n\}$ is given by

$t_n = \sum_{k=0}^{n} a_{n,n-k}\, \sigma_{n-k}$

$= \sum_{k=0}^{n} a_{n,n-k} \frac{1}{n-k+1} \sum_{r=o}^{n-k} s_r$

If $t_n \to s$ as $n \to \infty$ then the sequence $\{s_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Matrix-Cesaro means $T(C_1)$ to s.

Important cases of Matrix-Cesaro means are:

(i) $(N, p_n)C_1$ means when$a_{n,n-k} = p_k/P_n$, where $P_n = \sum_{k=0}^{n} p_k \neq 0$

(ii) $(N, p_n)C_1$ means when$a_{n,n-k} = p_{n-k}/P_n$

(iii) $(N, p, q)C_1$ means when$a_{n,n-k} = p_k q_{n-k}/R_n$,where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$

We shall use following notation:

$$\phi(t) = f(x+t) + f(x-t) - f(x)$$

$$K(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)\,t/2}{\sin^2(t/2)}$$

III. Main Theorem

Let f is a 2π-periodic function, Lebesgue integrable on [$-\pi,\pi$] and $f \in$ Lip ($\psi(t)$,p) class and if

$$\left\{\int_0^{1/n+1} \left(\frac{\psi(t)}{t^{1\backslash p}}\right)^p dt\right\}^{1/p} = O\left(\psi\left(\frac{1}{n+1}\right)\right) \tag{3.1}$$

And

$$\left\{\int_{1/n+1}^{\pi} \left(\frac{\psi(t)}{t^{1\backslash p+2}}\right)^q dt\right\}^{1/q} = O\left((n+1)^2\psi\left(\frac{1}{n+1}\right)\right) \tag{3.2}$$

Then the degree of approximation of f by the Matrix-Cesaro T(C_1) summability method of its Fourier series is given by

$$\|t_n - f\|_\infty = O\left((n+1)^{1/p}\psi\left(\frac{1}{n+1}\right)\right)$$

For the proof of our theorem following lemmas are required:

Lemma:1 For 0<t <$(n+1)^{-1}$ and $\frac{1}{sint} \le \frac{\pi}{2t}$for 0<$t$ <$\frac{\pi}{2}$

$$K(n,t) = O(n+1)$$

Proof: $K(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)t/2}{\sin^2(t/2)}$

$= \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}(n-k+1)$

$\le \frac{\mathrm{n}+1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}$

$= \frac{\mathrm{n}+1}{2\pi}$

$= O(n+1)$

Lemma: 2 For$(n+1)^{-1}$<t <π

$$K(n,t) = O\left(\frac{1}{(n+1)t^2}\right)$$

Proof: $K(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)t/2}{\sin^2(t/2)}$

$\le \frac{1}{2\pi}\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\pi^2}{t^2}$

$= \frac{\pi}{2\mathrm{t}^2}\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1}$

$= \frac{\pi}{2\mathrm{t}^2} O\left(\frac{1}{(n+1)}\right)$

$= O\left(\frac{1}{(n+1)t^2}\right)$

IV. Proof Of Main Theorem

The n[th] partial sum of series $s_n(x)$of the series (2.1) is given by

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t) \frac{\sin(n+1/2)t}{\sin(t/2)}\,\mathrm{dt}$$

The (C,1) transformσ_n ofs_n is given by

$$\frac{1}{n+1}\sum_{k=o}^{n} s_n(x) - f(x) = \frac{1}{2(\mathrm{n}+1)\pi}\int_0^{\pi} \frac{\phi(t)}{\sin(t/2)}\sum_{\mathrm{k}=0}^{\mathrm{n}} \sin(k+1/2)\,\mathrm{t\,dt}$$

$$\sigma_n(x) - f(x) = \frac{1}{2(\mathrm{n}+1)\pi}\int_0^{\pi} \phi(t) \frac{\sin^2(n+1)\,t/2}{\sin^2(t/2)} dt$$

The matrix means of the sequence $\{\sigma_n\}$is given by

$$\sum_{k=0}^{n} a_{n,k}\,(\sigma_n(x) - f(x)) = \int_0^{\pi} \phi(t) \frac{1}{2\pi}\sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{1}{(\mathrm{k}+1)} \frac{\sin^2(k+1)\,t/2}{\sin^2(t/2)} dt$$

Or $\sum_{k=0}^{n} a_{n,n-k}\,(\sigma_{n-k}(x) - f(x)) = \int_0^{\pi} \phi(t) \frac{1}{2\pi}\sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{1}{(\mathrm{n}-\mathrm{k}+1)} \frac{\sin^2(n-k+1)t/2}{\sin^2(t/2)} dt$

$t_n(x) - f(x)$=$\int_0^{\pi} \phi(t)K(n,t)dt$

$= \int_0^{\frac{1}{n+1}} \phi(t)K(n,t)dt + \int_{\frac{1}{n+1}}^{\pi} \phi(t)K(n,t)dt$

$= I_1 + I_2$ (4.1)

Now I_1= $\int_0^{\frac{1}{n+1}} \phi(t)K(n,t)dt$

$$|I_1| \le \int_0^{\frac{1}{n+1}} \frac{\psi(t)}{t^{1\backslash p}} K(n,t)dt$$

$$= \left\{\int_0^{1/n+1} \left(\frac{\psi(t)}{t^{1\backslash p}}\right)^p dt\right\}^{1/p} \left\{\int_0^{1/n+1} (K(n,t))^q dt\right\}^{1/q}$$

$$= O\left(\psi\left(\frac{1}{n+1}\right)\right) O(n+1) \left\{\int_0^{1/n+1} dt\right\}^{1/q}$$

$$= O\left(\psi\left(\frac{1}{n+1}\right)\right) O\left((n+1)^{1-\frac{1}{q}}\right)$$

$$= O\left((n+1)^{1/p}\psi\left(\frac{1}{n+1}\right)\right) \tag{4.2}$$

And $I_2 = \int_{\frac{1}{n+1}}^{\pi} \phi(t)K(n,t)dt$

$$|I_2| \le \int_{\frac{1}{n+1}}^{\pi} \frac{\psi(t)}{t^{1\backslash p}} K(n,t)dt$$

$$= \left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\psi(t)}{t^{1\backslash p}}\right)^p dt\right\}^{1/p} \left\{\int_{\frac{1}{n+1}}^{\pi} (K(n,t))^q dt\right\}^{1/q}$$

$$= \left\{\int_{1/n+1}^{\pi} \left(\frac{\psi(t)}{t^{\frac{1}{p}+2}}\right)^p dt\right\}^{1/p} O\left(\frac{1}{n+1}\right)$$

$$= O\left(\frac{1}{n+1}\right) O\left((n+\ 1)^2\psi\left(\frac{1}{n+1}\right)\right) O\left(\frac{1}{(n+1)^{\frac{1}{q}}}\right)$$

$$= O\left(\left(\frac{1}{(n+1)^{\frac{1}{q}-1}}\right)\psi\left(\frac{1}{n+1}\right)\right)$$

$$= O\left((n+1)^{1/p}\psi\left(\frac{1}{n+1}\right)\right) \tag{4.3}$$

Now combining (4.1),(4.2) and (4.3), we get

$$\|t_n - f\|_\infty = \sup\left|(CE)_n^q(x) - f(x)\right|$$
$$= O\left((n+1)^{1/p}\psi\left(\frac{1}{n+1}\right)\right)$$

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