IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 1 Ver. II. (Feb. 2014), PP 01-07
www.iosrjournals.org

Topological 3- Rings

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Abstract: *In this paper we study the 3- rings ,Idempotent of 3-ring and some other theorems .In the second section we introduce Ideals on 3-rings,center of 3-rings and theorems, Topological 3-rings and their properties: the set of open neighbourhoods of 0, its properties in topological 3-rings, Every topological 3- ring is a homogeneous algebra and other theorem.*
Key words: *Hausdorff space **,** Ring, p-ring, Topological space.*

I. Introduction

D. Van Dantzig firstly introduced the concept of topological ring in his thesis. Later N. Jacobson , L.S. Pontryagin , L.A. Skornjakov Small and S. Warner developed and studied various properties :Connected topological rings, Totally disconnected topological rings, Banach algebras, Ring of P-addict integers, locally compact fields, locally compact division rings and their structure. McCoy and Montgomery introduced the concept of a *p*-ring (*p* prime) as a ring *R* in which *x p* = *x* and *p x* = 0 for all *x* in *R*. Thus, Boolean rings are simply 2-rings (*p* = 2).,Koteswararao.P in his thesis developed the concept of 3-rings,3-rings generates A*-algebras and their equivalence. With this as motivation ,I introduce the concept of Topological 3-rings.

1. Prelimanaries

1.1 Definition: A commutative ring (R,+,.1) such that x^3=x, 3x=0 for all x in R is called a 3-ring.

1.2 Note: (1) x + x= -x for all x in a 3-ring R

(2) Here after R-stands for a 3-ring.

1.3 Example: 3 = {0, 1, 2} . Then (3, +, . ,1) is a 3-ring where

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

1.4 Example 2: Suppose X is a non empty set .Then (3^x, +,. 0,1) is a 3-ring with
(a) (f + g)(x) = f(x) +g(x).
(b) (f . g)(x) = f(x).g(x).
(c) 0(x) =0.
(d) 1(x) = 1 for all x∈X, f, g∈3^x.

1.5 Definition: Let R be a 3-ring. An element a∈R is called an idempotent if a^2=a .

1.6 Lemma: An element a∈R is an idempotent iff 1-a is an idempotent.

Proof: Suppose a is an idempotent
Claim : (1-a) is an idempotent
$(1-a)^2$= 1+a^2-2a =1+a-2a (∵ a is idempotent)
=1-a
∴ (1-a) is an idempotent

Conversely suppose that (1-a) is an idempotent
We have to show that a is an idempotent:

∵ (1-a) is an idempotent
1-(1-a) is an idempotent (By above)
⇒a is an idempotent element.

1.7 Lemma: For any element a in a 3-ring R, a^2 is an idempotent.
Proof: Suppose $a \in R$
∵ R is a 3-ring , $a^3=a$
$(a^2)^2= a^2 . a^2 = a^3. a = a. a = a^2$.
∴ a^2 is an idempotent for every $a \in R$.

II. Main Results

2.1 Definition : A non empty subset I of a 3-ring R is said to be ideal
if (i) $a, b \in I \Rightarrow a + b \in I$, (ii) $a \in I, r \in R \Rightarrow a\,r$ □ $ra \in I$.
Note: A non empty sub set I of R is said to be a right ideal (left ideal) of R ,if (i) $a , b \in I \Rightarrow a + b \in I$ (ii)$a \in I , r \in R \Rightarrow a\,r \in I$ ($r\,a \in I$)

2.2 Note : Suppose $a \in R$ then there is minimal left ideal (right ideal) exists containing a which is called the principal right (left) ideal denoted by $(a)l$ ($(a)r$) is the set of all ra (ar), $r \in R$.
i.e, $(a)r = \{ar / r \in R\}$ and $(a)l= \{ra / r \in R\}$.

2.3 Note : The set of all right ideals form a partially ordered set with respect to set theoretical inclusion $I \subseteq J$. This set has a minimum element:
0= (0)and a maximum one : R = (1)r.

2.4 Note (1) **:** For any set of ideals I1, I2 ..... ∃ a maximal ideal
I such that $I \subseteq I1, I2$, ... and $I1 \cap I2 \cap$.... is the maximal ideal contained in every ideal I1, I2, ... and it is denoted by glb {I1,I2, ...}.

(2)For any set of ideals I1, I2, ... ∃ a minimal ideal I such that
$I \subseteq I1, I2$, ... and it is denoted by lub {I1, I2, ...}.

2.5 Note : For the ideals I1, I2; glb {I1, I2} is denoted by $I1 \wedge I2$. and lub {I1, I2} is denoted by $I1 \vee I2$.
Thus the set of right ideals form a lattice with $\wedge,\vee$ Zero (0), unit R.

2.6 Definition: The center of a 3-ring R is the set $C = \{a \in R / ax = xa , \forall x \in R\}$. C is a commutative ring with unit 1.

2.7 Theorem: If a,b are the idempotent elements in C,then ab an idempotent and $ab \in C$ and also $(a) \wedge (b) = (ab)$
Proof:Let R be a 3-ring .
Suppose $a,b \in R$ and a,b are idempotents.
$(ab)^2 =ab.ab = a^2.b^2 = ab$.
Therefore ab is an idempotent.
Let $x \in R \Rightarrow (ab)x = a(bx) =a(xb) =(ax)b =(xa)b =x(ab)$
Therefore (ab)x =x(ab)
∴ $ab \in C$. ab=ba ∈a and also belongs to b
∴ $(ab)_* \subseteq (a)_*(b)_* \Rightarrow (ab)_* = (a)_* \wedge (b)_*$
Let $x \in (a)_* \wedge (b)_* \Rightarrow ax =bx =x$
∴ $abx = x$ ∴ $x \in (ab)_*$
∴ $(a)_* \wedge (b)_* \subseteq (ab)_*$
∴ $(a)_* \wedge (b)_* = (ab)_*$

2.8 Theorem: If a,b are idempotents in C, then $a+b-ab \in C$, idempotent and also $(a)_* \vee (b)_* = (a+b-ab)_*$
Proof: a+b-ab = 1-(1-a)(1-b).
Since $a,b \in C \Rightarrow (1-a),(1-b) \in C$ and are idempotent.
$\Rightarrow 1-(1-a)(1-b) \in C$ and idempotent.
∴ a + b-a b is an idempotent and belongs to C.

$(a)_* \vee (b)_* = ((a)_*^{\,*} \vee (b)_*^{\,*})^* = (1-(1-a)(1-b))_* = (a-b-ab)_*$
$\therefore (a)_* \vee (b)_* = (a + b - ab)_*$

2.9 Theorem: Center of a 3-ring C is a 3-ring
Proof: Let $a \in C \Rightarrow a \in R$
Since R is a 3-ring and $a \in R$ then $a^3 = a$ and $3a = 0$
We have
$3(a\,x) = (3a)x$
$= 0x$ (since R is a 3-ring)
$= 0.$
Therefore $3(a\,x) = 0, \forall a \in R, x \in R$.
Let $z \in R$
$z.\,a\,x = z\,a.\,x$
$= x.za$
$= x\,a\,z$
$= x\,a.\,z$
$= a\,x\,.z.$
$\therefore a\,x \in C$
Therefore C is 3-ring.

2.10 Definition: A set R is said to be a topological 3- ring if

1. R is a 3- ring.

2. R is a topological space.

3. The operations +, **.**, –, (–)* are continuous.

2.11 Note : For any subsets U, V ⊆ R, we define

$U + V = \{u + v / u \in U, v \in V\}$

$U\,.\,V = \{uv / u \in U, v \in V\}$.

$-U = \{-u / u \in U\}$.

$U^* = \{u^* / u \in U\}$

2.12 Note : 1) + : R × R ⟶ R is continuous means, for every neighbourhood W of a + b, a, b ∈ R there exist neighbourhoods U of a, V of b such that U + V ⊆ W.

2) **.** : R × R ⟶ R is continuous, for every neighbourhood W of ab, a, b ∈ R, there exist neighbourhoods U of a, V of b such that
U **.** V ⊆ W.

3) – : R ⟶ R is continuous, if for every neighbourhood W of –a, there exist a neighbourhood U of a such that – U ⊆ W.

4) (–)* : R ⟶ R* is continuous means, for every neighbourhood W of a*, there exist a neighbourhood U of a such that U* ⊆ W.

2.13 Lemma : Suppose R is a topological 3- ring. If c ∈ R, then

i) The map $x \longrightarrow c + x$, is homeomorphism.

ii) The maps $x \longrightarrow cx$, $x \longrightarrow xc$ are continuous.

Proof : The subspace {c} × R of R × R is clearly homeomorphic to R via (c, b) ⟶ b and the restriction of + to {c} × R to R is continuous and clearly bijection.

$x \longrightarrow (c, x) \longrightarrow c + x$ is continuous and bijective.

$x \longrightarrow c + x$ is continuous and bijective. And its inverse $x \longrightarrow -c + x$ is continuous and bijective. $x \longrightarrow c + x$ is homeomorphism.

The subspace $\{c\} \times R$ of $R \times R$ is clearly homeomorphic to R via $(c, b) \longrightarrow b$ and the restriction of **.** to $\{c\} \times R \longrightarrow R$ is continuous. Similarly $x \longrightarrow xc$ is continuous.

2.14 Note : 1) R is a topological 3- ring. Since $- : R \longrightarrow R$ is clearly homeomorphism. So U is open, – U is also open.

2) Since $x \longrightarrow x + c$ is homeomorphic, then for any open $U \subseteq R$, $c \in R$, then $U + c = \{u + c/u \in U\}$ is open. If U, V are open, then U + V is open.

3) If U is open neighbourhood of c iff U – c is an open neighbourhood of 0. So,the topology of R is completely determined by the open neighbourhoods of 0.

2.15 Definition : Let X be a topological space. If $x \in X$, then a fundamental system of neighbourhoods of x is a non-empty set M of open neighbourhoods of x with the property that, if U is open and $x \in U$, then there is $V \in M$ with $V \subseteq U$.

2.16 Definition : Let R be a 3- ring. A non-empty set N of subsets of R is fundamental if it satisfies the following conditions.

(a) Every element of N contain 0.
(b) If $U, V \in N$, then there is $W \in N$ with $W \subseteq U \cap V$.
(0). For $U \in N$ and $c \in U$, there exist $V \in N$ such that $c + V \subseteq U$.
(1). For each $U \in N$ there exist $V \in N$ such that $V + V \subseteq U$.
(2). $U \in N$ then $-U \in N$.
(3). If $U \in N$ there exist $V \in N$ such that $V^* \subseteq U$.
(4). For $c \in R$ and $U \in N$ there is $V \in N$ such that $cV \subseteq U$ and $Vc \subseteq U$.
(5). For each $U \in N$ there is $V \in N$ such that $V.V \subseteq U$.

2.17 Theorem : Suppose R is a topological 3- ring. Then the set N of open neighbourhoods of 0 satisfies.

(0). For $U \in N$ and $c \in U$, $\exists V \in N$ such that $c + V \subseteq U$.
(1). For each $U \in N$, there exist $V \in N$ such that $V + V \subseteq U$.
(2). If $U \in N$, then $-U \in N$.
(3). If $U \in N$, then $\exists V \in N \ni V^* \subseteq U$.
(4). For $c \in R$ and $U \in N$, there is $V \in N$ such that $cV \subseteq U$ and $Vc \subseteq U$.
(5). For each $U \in N$ there is $V \in N$ such that $V.V \subseteq U$.

Conversely, if R is a regular ring and N a non-empty set of subsets of R which satisfies N0, N1, N2, N3, N4 and N5 has the property that (a) every element of N contains 0 and (b) if $U, V \in N$, then there is $W \in N$ such that $W \subseteq U \cap V$,then there is a unique topology on R making R into a topological3-ring in such away that N is a fundamental system of neighbourhoods of 0.

Proof :

N0 . Let $U \in N$ and $c \in U$, then U – c is a neighbourhood of 0.
$\Longrightarrow \exists$ a neighbourhood V of 0 such that $V \subseteq U - c \Longrightarrow V + c \subseteq U$

N1. Let $U \in N \Longrightarrow 0 \in U \Longrightarrow (0, 0) \in +^{-1}(U)$
$\because$ + is continuous, so $+^{-1}(U)$ is open and $(0, 0) \in +^{-1}(U)$, $\exists$ open sets, V1, V2 with $(0, 0) \in V1 \times V2 \subseteq +^{-1}(U)$.
Let $V = V1 \cap V2 \Longrightarrow (0, 0) \in V \times V \subseteq +^{-1}(U) \Longrightarrow V + V \subseteq U$.

N2. Let $U \in N \Longrightarrow U$ neighbourhood of 0.
$- : R \rightarrow R$ is homeomorphic and U is open $\Longrightarrow -U$ is open.
$\because 0 \in U \Longrightarrow 0 \in -U$. $\therefore -U \in N$.

N3. $\because * : R \longrightarrow R$ is continuous and U is a neighbourhood of 0, then $*^{-1}(U)$ is open and $0 \in *^{-1}(U)$. $\Longrightarrow \exists$ a neighbourhood V of 0 such that $V \subseteq *^{-1}(U) \Longrightarrow V^* \subseteq U$.

N4. Let U ∈ N and c ∈ R ⟹ U is neighbourhood of 0. $x \subseteq cx$ is continuous and U is neighbourhood of 0, then ∃ a neighbourhood V
of 0 such that cV ⊆ U (By taking $x = 0$). Similarly Vc ⊆ U.
N5. Let U ∈ N ⟹ U is a neighbourhood of 0.
∵ multiplication **.** is continuous, $.^{-1}$ (U) is open and contains (0, 0).
⟹ ∃ neighbour hoods V1, V2 of 0 such that (0, 0) ∈ V1 × V2 ⊆ $.^{-1}$ (U).
Let V = V1 ∩ V2 ⟹ (0, 0) ∈ V × V ⊆ $.^{-1}$ (U) ⟹ V**.**V ⊆ U.
Conversely suppose R is a regular ring and N be a non-empty set of
subsets of R with the given properties. We define a subset U of R to be open if for every x ∈ U, ∃ W ∈ N x + W ⊆ U.
Clearly this is a topology.
For : Clearly ∅, R are open.
Suppose {Uα / α ∈ Δ} is a family of open sets.
Let U = ∪U
α∈Δ
Let x ∈ U ⟹ x ∈∪U
α∈Δ
⟹ x ∈ Uα for some α ∈ Δ ⟹ ∃ V ∈ N ∋ V + x ⊆ Uα

⟹ V + x ⊆ ∪ U
α∈Δ

⟹ V + x ⊆ U.
∴ U is an open set.
Let U1, U2 be two open sets. Let U = U1 ∩ U2.
Let x ∈ U ⟹ x ∈ U1 and x ∈ U2 ⟹ ∃ V1, V2 ∈ N ∋ V1 + x ⊆ U1, V2+ x ⊆ U2.Then ∃ V ∈ N ∋ V ⊆ V1 ∩ V2. Then V + x ⊆ U1, V + x ⊆ U2.
∴ V + x ⊆ U1 ∩ U2. ∴ U1 ∩ U2 is open.
Let U ∈ N and x ∈ R. Then ∃ V ∈ N ∋ V + x ⊆ U (By N0)
∴ The sets in N are also open sets containing 0.
Claim : If U is open, then for c∈ R, c + U is open.
Let b∈ c + U⟹ b – c ∈ U.
∴ ∃ V ∈ N ∋ b – c + V ⊆ U (By (N0) ⟹ b + V ⊆ c + U.
∴ c + U is open set.
Claim : + : R × R → R is continuous.
Let U be an open set. Let (c, d) ∈ $+^{-1}$ (U) ⟹ c + d ∈ U.
∴ ∃ W ∈ N ∋ c + d + W ⊆ U. ∵ W ∈ N, by N1, ∃ Q ∈ N ∋ Q + Q ⊆ W.
∴ c+ d + Q + Q ⊆ c + d + W ⊆ U ⟹ (c, d) ∈ (c + Q) × (d + Q) ⊆ $+^{-1}$ (U).
∴ $+^{-1}$ (U) is open. ∴ + is continuous.
Claim : – : R → R is continuous.
Let U be an open set. Let b ∈ – U⟹ – b∈ U.
From N0, ∃ V∈ N ∋ –b + V ⊆ U ⟹ b – V ⊆ – U.
∴ – U is open. (∵V ∈ N ⟹ –V ∈ N by N2). ∴ – is continuous
.**Claim :** The map θ : R ⟶ R by x ⟶ cx is continuous.

Let U be an open set. Let x ∈ θ^{-1}(U) ⟹ θ(x) ∈ U ⟹ cx ∈ U
⟹ ∃ W∈ N ∋ cx + W ⊆ U (By N0). Q W ∈ N, ∃ V ∈ N ∋ cV ⊆ W.
Then x + V ⊆ θ^{-1} (U). ∴ θ^{-1} (U) is open
∴ θ i.e., x ⟶ c x is continuous. Similarly x ⟶ x c is continuous.
Claim : m : R × R ⟶ R is continuous, where m(a, b) = a **.**b
Let U be an open set and (c, d) ∈ m^{-1} (U).
The maps θ : R ⟶ R and Ψ : R ⟶ R where θ (x) = c x, Ψ(x) = x c are
continuous.
∵ (c, d) ∈ m^{-1} (U) ⟹ m(c, d) ∈ U ⟹ c d ∈ U
⟹ ∃ W ∈ N ∋ c d + W ⊆ U (By N0).
∵ W ∈ N, ∃ Q ∈ N ∋ Q + Q ⊆ W (By N1)
.∵ Q ∈ N, ∃ V ∈ N ∋ V**.**V ⊆ Q (By N5).
∵ Q∈ N, ∃ P ∈ N ∋ P + P ⊆ Q (By N1).

Let $Pd = \Psi^{-1}(P) \cap Q \cap V$, $Pc = \theta^{-1}(P) \cap Q \cap V$.
Then $(c, d) \in (c + Pd) \times (d + Pc) \subseteq m^{-1}(U)$. ∴ m is continuous.
Claim : $* : R \longrightarrow R$ is continuous. Let U be an open set.
Let $x \in *^{-1}(U) \Rightarrow x^* \in U \Rightarrow U - x^*$ is a neighbourhood of 0
.∴ $\exists\ V \in N \ni V^* \subseteq U - x^* \Rightarrow x^* + V^* \subseteq U$.
Suppose □□ is another topology on R for which N is a fundamental system
of neighbourhoods of 0 in this topology. Then the topology □□ and the topology
defined above have same open base.
∴ The topology □□ must agree with the topology we have defined above.
∴ The topology is unique.
∴ N generates a unique topology on R for which N is a fundamental system of neighbourhoods of 0.

2.18 Note : If R is a 3- ring then R has no non-zero nilpotent elements, every prime ideal is maximal and Jacobson radical of R is {0}.

2.19 Theorem : Suppose R is a topological 3- ring. S, T are subsets of R
.Then a) ST, S + T are compact whenever S, T are compact.
b) – S, S* are compact whenever S is compact.
c) ST, S + T are connected sets whenever S, T are connected sets.
d) –S, S* are connected whenever S is connected.

Proof :
a) Since continuous image of a compact set is compact.
+, **.** : $R \times R \longrightarrow R$ are continuous, S, T are compact sets, then
.$(S \times T) = ST$,
$+(S \times T) = S + T$ are compact.
b) ∵ $- : R \longrightarrow R$ and $* : R \longrightarrow R$ are continuous and S in compact, then –S, S* are compact.
c) ∵ continuous image of a connected set in connected, **.** : $R \times R \longrightarrow R$ and $+ : R \times R \longrightarrow R$ are continuous,
. $(S \times T) = ST$, $+(S \times T) = S + T$ are connected sets.
d) ∵ $- : R \longrightarrow R$, $* : R \longrightarrow R$ are continuous and S is connected, so –S, S* are connected.

2.20 Theorem : The union of all connected subsets contain 0 is a topological Sub 3- ring.
Proof : Suppose $\{S_i / i \in I\}$ is a class of all connected sets containing 0.
Let $S = \bigcup_{i \in I} S_i$ contain 0
∵ $0 \in S \Rightarrow 0 \in S_i$ for some $i \in I \Rightarrow 1 - 0 \in - S_i \Rightarrow 1 \in - S_i$

∵ S_i is connected, $- S_i$ is also connected.
∴ $1 \in S$. Let $a \in S \Rightarrow a \in K_i$ for some $i \Rightarrow - a \in - K_i \Rightarrow - a \in S$.
∴ $a \in S \Rightarrow - a \in S$
Suppose $a, b \in S \Rightarrow a \in S_i$, $b \in S_j \Rightarrow a + b \in S_i + S_j$
∴ $a + b \in S$ (∵ $S_i + S_j$ is connected)
∴ S is a topological sub 3- ring of R.

2.21 Theorem : Suppose R is a topological 3- ring and I is ideal of R. Then
$\bar{I}$ is also an ideal of R.
Proof : Suppose I is an ideal of R. $\bar{I} = \{a \in R /$ every neighbourhood of a intersects I$\}$
Claim : $\bar{I}$ is an ideal. Let $a, b \in \bar{I}$
⟹ Every neighbourhood of a, every neighbourhood of b intersects I.
Suppose W is a neighbourhood of a + b.
⟹ ∃ neighbour hood U of a, neighbourhood V of b such that $U + V \subseteq W$.
∵ U intersects I, V intersects I so U + V intersects I, then W intersects I.
∴ $a + b \in \bar{I}$. Let $a \in \bar{I}$, $b \in R$.
Claim : $a b \in \bar{I}$. ∵ $a \in \bar{I}$ ⟹ Every neighbourhood of a intersects I.
Let W be a neighbourhood of ab. then ∃ neighbourhood U of a, neighbourhood V of b ∋ $UV \subseteq W$.
∵ $U \cap I \neq \emptyset$ ∃ $a \in U \ni a \in I$. Let $a b \in UV \Rightarrow a b \in I$ (∵ $a \in I$)
∴ $UV \cap I \neq \emptyset$. ∴ UV intersects I.
∵ $UV \subseteq W$, so W intersects I. ∴ $a b \in \bar{I}$
Similarly $ba \in \bar{I}$. ∴ $\bar{I}$ is an ideal of R.

2.22 Theorem : Every maximal ideal M of a topological 3- ring R is closed.
Proof : Clearly $M \subseteq \overline{M}$. $\because \overline{M}$ is ideal, so $M = \overline{M}$ $\therefore$ M is closed.

2.23 Theorem : If a topological 3- ring is T2 space then it is a Hausdorff space.
Proof : Suppose R is a T2 space and $a, b \in R$ and $a \neq b$.
$\because$ R is a T2 space $\exists$ neighbourhood U of a and neighbourhood V of b $\ni a \notin V, b \notin U$. Suppose $U \cap V \neq$
Let $W = U \cap V$. Let $c \in W \Rightarrow W - c$ is neighbourhood of 0.
Let $K = W - c \Rightarrow K$ is neighbourhood of 0.
$\Rightarrow K + a$ and $K + b$ are neighbourhoods of a and b respectively and $(K + a) \cap (K + b) =$.
$\therefore$ R is Hausdorff space.

2.24 Theorem : Every topological 3- ring is a homogeneous algebra.
ie., for every p, q ($p \neq q$) there is a continuous map $f : R \rightarrow R$ such that $f(p)=q$.
Proof : R is a topological 3- ring. Let $c = q - p$, then the function $f : R \rightarrow R$ by $f(x) = c + x$ is continuous and $f(p) = c + p = q - p + p = q$.

2.25 Theorem : Suppose R is a topological 3- ring and X = spec R. R* is a complete Boolean algebra. Suppose M is a subset of Spec R = X.
Denote QM ,the set of elements $e \in R^*$ for which $M \subseteq X_e$. Then $X_{\wedge QM} \subseteq \overline{M}$. In particular if M is nowhere dense in Spec R, then $\wedge QM = 0$.
Proof : Let $x \in X_{\wedge QM}$.
Suppose $x \notin \overline{M}$. $\Rightarrow \exists$ a neighbourhood X_e of the point $\ni X_e \cap M = \varnothing$.
$\Rightarrow M \subseteq X_{e'}$ $(e' = 1 - e) \Rightarrow e' \in QM$
$\therefore e \wedge (\wedge QM) \subseteq e \wedge e1 = 0$
i.e., $e \wedge (\wedge QM) = 0 \Rightarrow X_{e \wedge (\wedge QM)} = \varnothing \Rightarrow X_e \cap X_{\wedge QM} = \varnothing$
It is a contradiction (Q $x \in X_{\wedge QM}$ and $x \in$ X e). $\therefore x \in \overline{M}$.
$\therefore X_{\wedge QM} \subseteq \overline{M}$. . Suppose M is nowhere dense.
$\Rightarrow$ $\overline{M}$ contains no non-empty open subset.
But $X_{\wedge QM} \subseteq \overline{M} \Rightarrow X_{\wedge QM} = \varnothing$ $\Rightarrow \wedge QM = 0$.

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