IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 1 Ver. II. (Feb. 2014), PP 08-10
www.iosrjournals.org

Uniqueness of Meromorphic Functions Sharing One Value and a Small Meromorphic Function

Subhas.S.Bhoosnurmath[1], K.S.L.N.Prasad[2]
[1]*Department of Mathematics,Karnatak University,Dharwad-580003-INDIA*
[2]*Department of Mathematics, Karnatak Arts College, Dharwad-580001-INDIA*

Abstract: *In this paper we prove a uniqueness theorem for a meromorphic function which is sharing one value and a small meromorphic functionwith its derivatives.*

I. Introduction

Let f and g be two non-constant meromorphic functions defined on the complex plane. If f and g have the same a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities).

We wish to list few results which are already proved.

Rubel and C. C. Yang have proved the following result.

Theorem A[1]: Let f be a non constant entire function. If f and f' share two finite, distinct values CM, then $f \equiv f$

Later, Mues and Steinmetz improved Theorem A with the following result.

Theorem B [2]: Let f be a non constant entire function. If f and f' share two finite distinct values IM, then $f \equiv f'$.

Further, Jank, Mues and Volkmann proved the following two results in [3]

Theorem C: Let f be a non constant meromorphic function and let $a \neq 0$ be a finite constant. If f, f' and f'' share the value a CM, then $f \equiv f'$.

Theorem D: Let f be a non constant entire function and let $a \neq 0$ be a finite constant. If f and f' share the value a IM and if $f''(z) = a$, whenever $f(z) = a$ then $f \equiv f'$.

We wish to consider a slightly different case where a meromorphic function share one value and a small meromorphic function.

Our main result is the following.

Theorem: Let f be a non constant meromorphic function with $N(r,f)+N\left(r,\frac{1}{f}\right)=S(r,f)$. Let χ be a small meromorphic function satisfying $T(r, \chi) = o\{T(r, f)\}$.

If f and f ' share ∞ and χ CM and satisfies the equation

$$k f' - f - (k-1)\chi = 0 \tag{1}$$

for $k \neq 0$, then $f \equiv f'$.

Further, if μ and λ are two small meromorphic functions satisfying $T(r,\mu)=o\{T(r,f)\}$ and $T(r,\lambda)=o\{T(r,f)\}$ $(\chi \neq \mu,\ \chi \neq \lambda)$ satisfying

$$\overline{N}\left(r,\frac{1}{f-\mu}\right)+N\left(r,\frac{1}{f-\lambda}\right)+\overline{N}(r,f)=S(r,f), \text{then, } \frac{f-\mu}{\chi-\mu}=\frac{f'-\lambda}{\chi-\lambda}.$$

We require the following Lemmas to prove our result.

Lemma 1 [4] Let f be a non constant meromorphic function. Then,

for $n \geq 1$,

$$N\left(r,\frac{1}{f^{(n)}}\right) \leq 2^{n-1}\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)\right]+N\left(r,\frac{1}{f}\right)+S(r,f).$$

Lemma 2 [4]: Let f_1 and f_2 be two non constant meromorphic functions and $\alpha_1 \not\equiv 0,\ \alpha_2 \not\equiv 0$ be two small meromorphic functions satisfying $T(r,\alpha_i)=o\{T(r,f)\}(i=1,2)$, where $T(r,f)=\text{Max}\{T(r,f_1),T(r,f_2)\}$.

If $\alpha_1 f_1 + \alpha_2 f_2 \equiv 1$, then, $T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + o\{T(r, f)\}$.

II. Proof of the Theorem

From (1), we have $kf' - f - (k-1)\chi = 0$

Therefore, $\frac{f-\chi}{f'-\chi} = k$, where k is a non zero constant.

Put $f_1 = \frac{1}{\chi} f,\ f_2 = k,\ f_3 = \frac{-k}{\chi} f'$ $(\text{where } \chi \neq 0)$ so that $f_1 + f_2 + f_3 \equiv 1$ (2)

If $k \neq 1$, we get, $\frac{1}{\chi(1-k)} f - \frac{k}{\chi(1-k)} f' \equiv 1$

Then, by Lemma 2, we have

$$T(r, f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}(r, f) + S(r, f)$$

and $T(r, f') < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}(r, f') + S(r, f)$.

Using Lemma 1 and noting that $N\left(r, f^{(k)}\right) = N(r, f) + k\overline{N}(r, f)$,
we get ,

$$T(r, f) \leq 3N\left(r, \frac{1}{f}\right) + 2N(r, f) + S(r, f) \tag{3}$$

and $T(r, f') \leq 3N\left(r, \frac{1}{f}\right) + 3N(r, f) + S(r, f)$ (4)

Adding (3) and (4) we get

$$T(r, f) + T(r, f') \leq 6N\left(r, \frac{1}{f}\right) + 5N(r, f) + S(r, f)$$

$$\leq 6\left[N(r, f) + N\left(r, \frac{1}{f}\right)\right] + S(r, f)$$

This gives $T(r, f) + T(r, f') \leq S(r, f)$ in view of the hypothesis.

Or $1 \leq \frac{S(r, f)}{T(r, f) + T(r, f')} \to 0$, as $r \to \infty$

Or $1 \leq 0$, which is a contradiction.

This contradiction proves that k = 1.

Therefore, $\frac{f-\chi}{f'-\chi} = 1$

Or $f - \chi = f' - \chi$

Or $f \equiv f'$.

Further, $f - \mu = (f' - \lambda) + (\lambda - \mu)$.

If $\lambda \neq \mu$ then $\frac{f-\mu}{\lambda-\mu} - \frac{f'-\mu}{\lambda-\mu} = 1$ (5)

Since $T(r, f) \leq T(r, f - \mu) + o\{T(r, f)\}$.

By Lemma 6, we have

$$T(r,f) < \overline{N}\left(r,\frac{1}{f-\mu}\right)+\overline{N}\left(r,\frac{1}{f'-\lambda}\right)+\overline{N}(r,f)+S(r,f) \qquad (6)$$

and $$T(r,f') < \overline{N}\left(r,\frac{1}{f-\mu}\right)+\overline{N}\left(r,\frac{1}{f'-\lambda}\right)+\overline{N}(r,f)+S(r,f). \qquad (7)$$

Now, $f'-\lambda = f-\lambda$

Hence, zeros of $f'-\lambda$ occur only at the zeros of $f-\lambda$.

Therefore, $N\left(r,\frac{1}{f'-\lambda}\right) = N\left(r,\frac{1}{f-\lambda}\right)$

Therefore, from (6) and (7), we have

$$T(r,f)+T(r,f') < 2\left[\overline{N}\left(r,\frac{1}{f-\mu}\right)+N\left(r,\frac{1}{f-\lambda}\right)+\overline{N}\,(r,f)\right]+S(r,f)$$

Hence using hypothesis, we have

$$T(r,f)+T(r,f') < S(r,f)$$

or $$1 \le \frac{S(r,f)}{T(r,f)+T(r,f')} \to 0 \quad \text{as} \quad r \to \infty$$

Thus, $1 \le 0$ which is a contradiction.

This contradiction proves that $\lambda = \mu$.

Therefore, $\frac{f-\mu}{\chi-\mu} = \frac{f'-\lambda}{\chi-\lambda}$

Hence the Theorem.

References

[1]. L. RUBEL and C. C. YANG, (1976) : Values shared by an entire function and its derivative in "Complex Analysis, Kentucky" (Proc. Conf.), Lecture notes in Mathematics, Vol. 599, pp. 101-103, Springer Verlag, Berlin 1977.

[2]. E. MUES and N. STEINMETZ (1979) : Mermorphic functionen, die mit ihrer Ableitung wert teilen. Manuscripta Math 29 (1979), 195-206.

[3]. G. JANK, E. MUES and L. VOLKMANN (1986) : Meromorphic functionen, die mit ihrer ersten und zweiten ableitung einen endlichen wert teilen, Complex variables 6, 51-71.

[4]. INDRAJIT LAHIRI (1998) : 'Uniqueness of meromorphic functions sharing the same 1-points'. Bull. Korean Math. Soc. 35, No. 2, pp. 375-385.