IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 1 Ver. III. (Feb. 2014), PP 22-25
www.iosrjournals.org

A Class of SαG*- Open Sets in Topological Spaces

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Abstract: *In this paper we introduce the concept of γ-sαg* -open sets and discuss some of their basic properties.*
Key words: *γ-sαg*-open sets and γ-sαg*-regular operation. AMS Classification: 54 A 05*

I. Introduction

The study of semi open set and semi continuity in topological space was initiated by Levine[14]. Bhattacharya and Lahiri[3] Introduced the concept of semi generalized closed sets in the topological spaces analogous to generalized closed gets introduced by Levine[15]. Further they introduced the semi generalized continuous functions and investigated their properties. Kasahara[11] defined the concept of an operation on topological spaces and introduced the concept of α - closed graphs of a function. Jankovic[10] defined the concept of α - closed sets. Ogata [21] Introduced the notion of τ_γ which is the collection of all γ -open sets in the topological space
(X, τ) and investigated the relation between γ - closure and τ_γ - closure.

In this paper, we introduce the concept of γ-sαg*-open sets and discuss some of their basic properties.

II. Premilinaries

Throughout this paper (X, τ) represent non-empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) denote the closure and interior of A respectively. The intersection of all α-closed sets containing a subset A of (X, τ) is called the α-closure of A and is denoted by αcl(A).

Definition 2.1 [11]: Let (X, τ) be a topological space. An operation γ on the topology τ is a mapping from τ on to power set P(X) of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denote the value of γ at V. It is denoted by $\gamma: \tau \to P(X)$.

Definition 2.2 [21]: A subset A of a topological space (X, τ) is called γ -open set if for each $x \in A$ there exists a open set U such that $x \in U$ and $U^\gamma \subseteq A$.

τ_γ denotes set of all γ -open sets in (X, τ).

Definition 2.3[21]: The point $x \in X$ is in the γ - closure of a set $A \subseteq X$ if $U^\gamma \cap A \neq \phi$ for each open set U of x. The γ - closure of set A is denoted by $cl_\gamma(A)$.

Definition 2.4[21]: Let (X, τ) be a topological space and A be subset of X then τ_γ-$cl(A) = \cap \{F : A \subseteq F, X - F \in \tau_\gamma\}$

Definition 2.5 [21]: Let (X, τ) be topological space. An operation γ is said to be regular if, for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Definition 2.6 [21]: A topological space (X, τ) is said to be γ - regular, where γ is an operation of τ, if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that U^γ contained in V.

Remark 2.7 [21]: Let (X, τ) be a topological space, then for any subset A of X, $A \subseteq cl(A) \subseteq cl_\gamma(A) \subset \tau_\gamma$-$cl(A)$.

Definition 2.8[24]**:** A subset A of (X, τ) is said to be a γ-semi open set if and only if there exists a γ -open set U such that $U \subseteq A \subseteq cl\,\gamma(U)$.

Definition 2.9[24]: Let A be any subset of X. Then τ_γ-int(A) is defined as
τ_γ- int(A) = $\cup\{U : U$ is a γ-open set and $U \subseteq A\}$

Definition 2.10[24]: A subset A of X is said to be γ-semi closed if and only if X – A is γ-semi open.

Definition 2.11[24]: Let A be a subset of X. Then, τ_γ-scl(A) = $\cap\{F : F$ is γ-semi closed and $A \subseteq F\}$.

Definition 2.12[20]: A subset A of (X, τ) is said to be a strongly αg*- closed set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in (X,τ).

Definition 2.13[20]: If a subset A of (X, τ) is a strongly αg*-open set then X – A is a strongly αg*-closed set.

Definition 2.14[20]: A space (X, τ) is said to be a $_{s*}T_c$ space if every strongly αg*- closed set of (X, τ) is closed in (X, τ).

III. . γ -SαG* - Open Sets

Definition 3.1: A subset A of a topological space (X, τ) is called γ -sαg* -open set of (X,τ) if for each x $\in$A, there exists a sαg*-open set U such that x $\in$U and $U^{\gamma} \subseteq A$.

$\tau_{\gamma s*}$ denotes the set of all γ-sαg* -open sets in (X, τ)

Example 3.2: Let X = {a, b, c} and let τ = {ϕ, X, {a}, {b}, {a, b}, {a, c}} be a topology on X. For b $\in$ X, we define an operation γ: $\tau \rightarrow$ P(X) by $\gamma(A) = A^{\gamma} = A$ if b $\in$ A, γ(A) = cl(A) if b$\notin$A. Then, the γ-sαg*-open sets are {ϕ, X, {b}, {c}, {a, b}, {a, c}}.

Remark 3.3: The concept of sαg*-open sets and γ -sαg*-open sets are independent.

In Example 3.2, the set {a} is sαg*-open but it is not γ-sαg*-open. Also the set {c} is γ-sαg*-open but not sαg*-open.

Proposition 3.4: Every γ-open set of a topological space (X, τ) is γ-sαg*-open.

Proof: Let A be a γ-open set in X. Let x$\in$A, then there exists an open set G containing x such that $G^{\gamma} \subseteq A$. But every open set is sαg*-open. Therefore, A is a γ -sαg* -open set in X. Thus, $\tau_{\gamma} \subseteq \tau_{\gamma s}$*.

The converse of the above theorem is not true always as seen from the following example.

Example 3.5: Let X = {a, b, c} and let τ = {ϕ, X, {b}, {a, b}}. Let γ: $\tau \rightarrow$P(X) be an operation defined by γ (A) = A. Then, we see that the set A = {a} is a γ -sαg*-open set but not a γ -open set.

Remark 3.6: The union and intersection of γ -sαg*-open sets are not γ -sαg*-open.

In Example 3.2, the sets {b} and {c} are sαg*-open sets but their union {b, c} is not γ-sαg*-open. Also, the sets {a, b} and {a, c} are γ-sαg*-open but their intersection {a} is not not a sαg*-open set.

Definition 3.7: A subset B of (X, τ) is said be γ -sαg*- closed in (X, τ), if X – B is
γ-sαg*-open in (X, τ).

Definition 3.8: A topological space (X, τ) is said to be γ -sαg*-regular where γ is an operation on τ, if for each x $\in$ X and for every open set U of x, there exists a sαg*- open set W of x such that $W^{\gamma} \subseteq U$.

Proposition 3.9: Every γ - regular space is γ -sαg*-regular space.

Proof: Let (X, τ) be a γ -regular space. Then for each x$\in$X and for every open neighbourhood U of x, there exists an open neighbourhood W of x such that $W^{\gamma} \subseteq U$. But every open set is sαg*-open and therefore for each x $\in$X and for every open set U of x, there exits a sαg*-open set W of x such that $W^{\gamma} \subseteq U$. Hence (X, τ) is γ-sαg*- regular space.

The converse of the above theorem is not true always. The topological space in the Example 3.5 is a γ -sαg* - regular space, but not a γ-regular space.

Proposition 3.10: Let γ: $\tau \rightarrow$P(X) be an operation on a ${}_{s*}T_c$ space (X, τ). Then (X, τ) is γ -sαg* -regular if and only if $\tau_{\gamma} = \tau_{\gamma s*}$

Proof: Necessity: Since $\tau_{\gamma} \subseteq \tau_{\gamma s*}$, it is enough to prove that $\tau_{\gamma s*} \subseteq \tau_{\gamma}$. Let A be an open set. For any x $\in$A, there exists an open set U of x such that U $\subseteq$ A. By the γ -sαg* - regularity of X, there exists an sαg* -open set W of x such that $W^{\gamma} \subseteq U$. Since (X, τ) is a ${}_{s*}T_c$ space, W is open. Thus, for each x $\in$ A, we have an open set W and hence an open neighbourhood such that x$\in$W and $W^{\gamma} \subseteq A$. Then A is γ -open. Therefore $\tau_{\gamma s*} \subseteq \tau_{\gamma.}$

Sufficiency: Let x $\in$ X and V be an open set of x. Since V $\in \tau_{\gamma} = \tau_{\gamma s*}$, there exists an s$\alpha$g* -open set W of x such that $W^{\gamma} \subseteq V$. This implies that (X, τ) is γ -sαg* - regular.

Definition 3.11: Let (X, τ) be a topological space. An operation γ is said to be sαg* -regular if for every pair of open sets U and V of each x $\in$X, there exists an sαg* -open set W of x such that $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma}$.

Proposition 3.12: Every regular operation is sαg* -regular operation.

Proposition 3.13: On any ${}_{s*}T_c$ space (X, τ), let γ:$\tau \rightarrow$ P(X) be a regular operation on τ.

(i) If A and B are γ -sαg* -open then A $\cap$ B is γ -sαg* -open.

(ii) $\tau_{\gamma s*}$ is a topology on X.

Proof: (i) Let x $\in$ A$\cap$B. Then, x $\in$A and x $\in$B. So, there exists an sαg*-open set U such that x$\in$U, $U^{\gamma} \subseteq A$ and a sαg*-open set V such that x$\in$V, $V^{\gamma} \subseteq B$. Since (X, τ) is a ${}_{s*}T_c$ Space, U and V are open sets. Since γ is regular there exists an open neighbourhood W of x such that $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma}$ and hence $W^{\gamma} \subseteq A \cap B$. Since every open set is sαg* -open, for each x in A$\cap$B, there exists an sαg* -open set W containing x such that $W^{\gamma} \subseteq A \cap B$. Hence A $\cap$ B is γ - sαg* -open.

(ii): Clearly, $\phi \in \tau_{\gamma s*}$. Let x $\in$ X, then X is a sαg*-open set containing x such that $X^{\gamma} \subseteq X$. Hence $X \in \tau_{\gamma s*.}$ By (i), $\tau_{\gamma s*}$ is closed under finite intersections. Let {A_i }, i $\in$ I be any arbitrary collection of γ -sαg* open sets. Let $x \in \cup_{i \in I} A_i$. Then, x $\in A_i$ for some i. Since A_i is γ-sαg* open , there is a sαg*-open set U_i such that x$\in U_i$ and $U^{\gamma}_i \subseteq A_i \subseteq \cup_{i \in I} A_i$. Hence $\cup_{i \in I} A_i$ is a γ -sαg* -open set. Thus $\tau_{\gamma s*}$ is a topology on X.

Remark 3.14: If γ is not regular then the above theorem is not true, that is $\tau_{\gamma s*}$ is not a topology in general. For example, consider the space and the operation γ of Example 3.2. We note that γ is not regular. Also $\tau_{\gamma s*}$ = {ϕ, X, {b}, {c}, {a, b}, {a, c}} which is not a topology on X.

Definition 3.15: (i) The point $x \in X$ is in the γ_{s*}–closure of a set $A \subseteq X$ if $U^{\gamma} \cap A \neq \phi$ for each sαg*-open set U of x. The γ_{s*} -closure of A is denoted by $cl_{\gamma s*}(A)$.
(ii) For a family $\tau_{\gamma s*}$, we define a set of $\tau_{\gamma s*}$- cl(A) as follows:

$$\tau_{\gamma s*}\text{-cl}(A) = \cap\{F: F \supseteq A, X - F \in \tau_{\gamma s*}\}$$

Proposition 3.16: For a point $x \in X$, $x\in\tau_{\gamma s*}$-cl(A) if and only if $V\cap A \neq \phi$ for every γ-sαg*-open set V containing x.

Proof: Assume that $x \in \in\tau_{\gamma s*}$-cl(A). Let V be any γ-sαg*-open set containing x. We have to show that $V\cap A \neq \phi$. Suppose, $V\cap A = \phi$. Then $V^c \supseteq A$, where V^c is a γ -sαg* - closed set containing A. Since $x \in \tau_{\gamma s*}$-cl(A), $x \in V^c$ which contradicts the fact that V contains x. Hence $V\cap A \neq \phi$.

Conversely, let F be any γ -sαg* - closed set containing A. We have to show that $x \in F$. If possible suppose that $x \notin F$. Then, $x \in F^c$. Now, F^c is a γ -sαg* -open set containing x. But F^c and A are disjoint. This contradicts the hypothesis. Therefore, $x \in F$. This implies $x \in\tau_{\gamma s*}$ - cl(A).

Proposition 3.17: For any subset A of (X, τ), we have

(i) $cl_{\gamma s*}(A) \subseteq cl_{\gamma}(A)$
(ii) $cl(A) \supseteq cl_{\gamma s*}(A)$
(iii) $cl_{\gamma s*}(A) \subseteq \tau_{\gamma s*}$ - cl(A)

Proof: (i) Let $x \notin cl_{\gamma}(A)$. Then there exists an open set U of x such that $U^{\gamma} \cap A = \phi$[21]. Since every open set is sαg*-open we have $U^{\gamma} \cap A = \phi$ for a sαg* -open set U. Thus, $x \notin cl_{\gamma s*}(A)$. Therefore, $cl_{\gamma s*}(A) \subseteq cl_{\gamma}(A)$.

(ii) Let $x \notin cl(A)$. Then there is an open set U such that $U \cap A=\phi$. Since every open set is sαg* -open, $x \notin cl_{\gamma s*}(A)$. Therefore, $cl_{\gamma s*}(A) \subseteq cl(A)$.

(iii) Let $x \notin \tau_{\gamma s*}$-cl(A). Then by Proposition 3.16, there is a γ -sαg* -open set U containing x such that $U\cap A = \phi$. Since U is a γ -sαg* -open set containing x, there is an sαg* -open set W such that $x \in W$ and $W^{\gamma} \subseteq U$. Hence $W^{\gamma} \cap A = \phi$. Therefore, $x \notin cl_{\gamma s*}(A)$. Thus $cl_{\gamma s*}(A) \subseteq \tau_{\gamma s*}$ - cl(A)

Proposition 3.18: Let $\gamma: \tau \to P(X)$ be an operation on τ and A be a subset of X.
(i) The subset $cl_{\gamma s*}(A)$ is closed in (X, τ).
(ii) If (X, τ) is γ -sαg* -regular, $cl_{\gamma s*}(A) = cl(A)$.
(iii) If γ is open and (X, τ) is a ${}_{s*}T_c$ space, then $cl_{\gamma s*}(A) = \tau_{\gamma s*}$ - cl(A) and $cl_{\gamma s*}(cl_{\gamma s*}(A)) = cl_{\gamma s*}(A)$.

Proof:(i) Let $y \in cl(cl_{\gamma s*}(A))$. We have to prove that $y \in cl_{\gamma s*}(A)$. Let G be a sαg*-open set of y. Therefore, we have $G \cap cl_{\gamma s*}(A) \neq \phi$. So, there exists a point z such that $z \in G$ and $z \in cl_{\gamma s*}(A)$. Since $z \in cl_{\gamma s*}(A)$ and G is sαg* -open set of z, $G^{\gamma} \cap A \neq \phi$. Thus, for each sαg* -open set G of y, we have $G^{\gamma} \cap A \neq \phi$. Hence, $y \in cl_{\gamma s*}(A)$. Therefore, $cl(cl_{\gamma s*}(A)) \subseteq cl_{\gamma s*}(A)$. This implies that $cl_{\gamma s*}(A)$ is closed in (X, τ).

(ii) By Proposition 3.17, it is sufficient to prove that the inclusion $cl(A) \subseteq cl_{\gamma s*}(A)$. Let $x \in cl(A)$. Then for every open set U of x we have $U \cap A \neq \phi$. Since γ is γ -sαg*-regular, we have for every open neighbourhood U of x, there exists a open neighbourhood V of x such that $V^{\gamma} \subseteq U$. Since every open set is sαg*-open, we have $x \in cl_{\gamma s*}(A)$. Hence, the proof of (ii).

(iii) Suppose $x \notin cl_{\gamma s*}(A)$. Then there exists a sαg* -open set U such that $x \in U$ and $U^{\gamma} \cap A = \phi$. Since (X, τ) is a ${}_{s*}T_c$ space U is an open set. Since γ - is open, there is a open set S such that $x \in S \subseteq U^{\gamma}$. ie. A sαg*-open set such that $x \in S \subseteq U^{\gamma}$. We have $S\cap A = \phi$. By Proposition 3.16, $x \notin \tau_{\gamma s*}$ - cl(A). Hence $\tau_{\gamma s*}$ - cl(A) $\subseteq cl_{\gamma s*}(A)$. By Proposition 3.17, we have $cl_{\gamma s*}(A) \subseteq \tau_{\gamma s*}$ cl(A). Hence $cl_{\gamma s*}(A) = \tau_{\gamma s*}$ - cl(A).

Lemma 3.19: For any ${}_{s*}T_c$ space (X, τ) if γ is γ-sαg*-regular then $cl_{\gamma s*}(A\cup B) = cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B)$ for any subsets A and B of X.

Proof: Let $x \notin cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B)$. Therefore, $x \notin cl_{\gamma s*}(A)$ and $x \notin cl_{\gamma s*}(B)$. So there exists an sαg* -open set U of x such that $U^{\gamma} \cap A = \phi$ and a sαg* open set V of x such that $V^{\gamma} \cap B= \phi$. Since (X, τ) is a ${}_{s*}T_c$ space U and V are open in (X, τ). Since γ is γ -sαg*-regular, there exists a sαg*-open set W of x such that $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma}$. Thus $W^{\gamma} \subseteq U^{\gamma}$ and $W^{\gamma} \subseteq V^{\gamma}$. So $W^{\gamma} \cap A = \phi$. $W^{\gamma} \cap B = \phi$. Hence $W^{\gamma} \cap (A\cup B) = \phi$. This implies $x \notin cl_{\gamma s*}(A\cup B)$ and hence $cl_{\gamma s*}(A\cup B) \subseteq cl_{\gamma s*}(A)\cup cl_{\gamma s*}(B)$.

To prove the reverse inclusion, let $x \notin cl_{\gamma s*}(A\cup B)$. Then there exists a sαg* -open set U of x such that $U^{\gamma} \cap (AUB) = \phi$. This implies $U^{\gamma} \cap A = \phi$ and $U^{\gamma} \cap B = \phi$ and so $x \notin cl_{\gamma s*}(A)$ and $x \notin cl_{\gamma s*}(B)$. Therefore, $x \notin cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B)$. Hence, $cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B) \subseteq cl_{\gamma s*}(A\cup B)$. Thus, $cl_{\gamma s*}(A\cup B) = cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B)$

Remark 3.20: Even if γ is not γ -sαg *-regular and (X, τ) is not ${}_{s*}T_c$ space,
from the above Lemma 3.19, we observe that for any subsets A and B of X, $cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B) \subseteq cl_{\gamma s*}(A\cup B)$ always.

Corollary 3.21: For any ${}_{s*}T_c$ space (X, τ), if γ is γ -sαg* -regular on (X, τ), then the operation $cl_{\gamma s*}$ satisfies the Kurotowski closure axioms

Proof: We have to prove that

(i) $cl_{\gamma s*}(\phi) = \phi$.

(ii) A $\subseteq$ $cl_{\gamma s*}(A)$
(iii) $cl_{\gamma s*}(cl_{\gamma s*}(A)) = cl_{\gamma s*}(A)$
(iv) $cl_{\gamma s*}(A\cup B) = cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B)$ for any subsets A and B of X.

From the definition of γ_{s*} - closure of a set, it follows that $cl_{\gamma s*}(\phi)= (\phi)$. Hence (i). From the Definition 3.15, A $\subseteq cl_{\gamma s*}(A)$, for any subset A of X. By Proposition 3.18, $cl_{\gamma s*}[cl_{\gamma s*}(A)] = cl_{\gamma s*}(A)$ for any subset A of X. Hence (iii). Also, from the Lemma 3.19, we have $cl_{\gamma s*}(A\cup B) = cl_{\gamma s*}(A) \cup cl_{\gamma s*}(B)$ for any two subsets A and B of X. Hence (iv). Thus the operation $cl_{\gamma s*}$ satisfies the Kurotowski closure axioms.

Proposition 3.22: Every γ -sαg* -open set is open on a ${}_{s*}T_c$ space.

Proof: Let A be a γ -sαg* -open set. Let $x\in A$. Then there exists an sαg* -open set U such that $x \in U$ and $U^{\gamma} \subseteq$ A. But $U \subseteq U^{\gamma}$. Therefore, $U \subseteq A$. Since every sαg* -open set U open in ${}_{s*}T_c$ space, for every $x \in A$, we get an open set U such that $x \in U \subseteq A$. Hence A is open.

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