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Paley Wiener Theorem

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Abstract: *In this work we study how to apply PALEY WIENER theorem to the Fourier transforms of functions with compact support.*

I. Introduction

One of the classical theorems of Paley and wiener characterizes the entire functions of exponential type, whose restriction to the real axis is in L^2 as being exactly the Fourier transformation of L^2 -functions with compact support. We shall give two analogues of this (in several variables), one for $C^\infty(R) = \{f / f \text{ is differential at everywhere}\}$ with compact support, one for distributions with compact support

Note:

In the following two theorems,Support of rB= $\{x \in R^n / |x| \le r\}$.

Theorem:

If $\Phi \in (R^n)$ has its support in rB and if $f(z) = \int_{R^n} \Phi(t) e^{-izt}\ dm(t) \qquad (z \in C^n) \qquad (1)$ then is entire, and there are constants $\gamma_N < \infty$ such that $|f(z)| \le \gamma_N (1+|z|)^{-N} e^{r|im\, z|}$ (2) $(z \in C^n,\ N = 0,1,2\ldots)$. Conversely, if f is an entire function in C^n which satisfies(2) for some N then there exists $u \in D'(R^n)$, with support in rB, such that (1) holds.

Proof

rB= $\{x \in R^n / |x| \le r\}$

If $t \in rB$ then $|t| \le r$, consider $|e^{-izt}| = e^{y.t} \le e^{|y||t|} \le e^{|im\, z|\, r}$

let K= support of $\Phi \in rB$.

Claim: $\Phi(t) e^{-itz} \in L'(R)$

Since $\Phi \in D(R^n)$, Φ is differentiable $\Rightarrow \Phi$ is continuous complex function

$\Rightarrow \Phi$ is measurable

Also e^{-itz} continuous complex function

Hence $\Phi(t) e^{-itz}$ is complex measurable function

consider $\int_R |\Phi(t) e^{-itz} dm(t)| \le \int_R |\Phi(t)| e^{r|im\, z|}\ dm(t)$

$$= e^{r|im\, z|} \int_R \Phi(t)\ dm(t) \qquad (3)$$

since Φ is continuous and support of Φ is compact, Φ is continuous function defined on a compact set

Hence Φ is bounded, there exists a real number M such that $|\Phi(t)| \le M \quad \forall\ t$

(3) becomes $\int_R |\Phi(t) e - itz|\ dmt \le e^{r|im\, z|} \int_R M\ dm(t)$

$$\le M\, e^{r|im\, z|}\ m(k)$$

$$\le \infty$$

Hence $\Phi(t)e^{-itz} \in L'(R)$, therefore for every $z \in C$, $f(z) = \int \Phi(t)\, e^{-itz} dm(t)$ exists on C.

Now to prove f is an entire function, for that it is enough to prove that f is analytic, for proving f is analytic , we have to use morera's theorem(statement: If f is continuous and $\int_\Gamma f(z)dz = 0$, then f is analytic.)

so first we have to prove that f is continuous

if $z_n \to z$, then $|f(z_n) - f(z)| = \left|\int_R \Phi(t)e^{-iz_n t} dm(t) - \int \Phi(t)e^{-izt} dm(t)\right|$

$$= \left|\int_k \Phi(t)\left(e^{-iz_n t} - e^{-izt}\right)dm(t)\right| \text{ since,outsideK, } \Phi(t) = 0$$

$$\leq \int_k |\Phi(t)|\left|e^{-iz_n t} - e^{-izt}\right| dm(t)$$

$$= \int_k M \left|e^{-iz_n t} - e^{-izt}\right| dm(t) \qquad (4)$$

consider $\left|e^{-iz_n t} - e^{-izt}\right| \leq \left|e^{-iz_n t}\right| + \left|e^{-izt}\right|$

$$\leq e^{ty_n} + e^{ty}$$

$$\leq e^{|t||y_n|} + e^{|t||y|}$$

$$\leq e^{|t|(1+|y|)} + e^{|t||y|} \quad [\text{since } y_n \to y,\ |y_n - y| \leq 1, |y_n| \leq 1 + |y|]$$

$$= e^{|t||y|}\left(e^{|t|} + 1\right)$$

$$\leq e^{r|im\, z|}\left(1 + e^r\right) = g(z) \text{ (say)}$$

now consider $\int_R |g(x)|\, dm(x) = \int_k \left|e^{r|y|}\left(1 + e^r\right)\right| dm(x)$

$$= \left|e^{r|y|}\left(1 + e^r\right)\right| \int_k dm(x)$$

$$= \left|e^{r|y|}\left(1 + e^r\right)\right| m(k) \text{ which is finite}$$

also $e^{r|im\, z|}\left(1 + e^r\right)$ is continuous, therefore measurable

hence $g(z) \in L'(R)$

also since $z_n \to z \Rightarrow e^{-iz_n t} \to e^{-izt} \quad as \quad n \to \infty$

$$\Rightarrow \left(e^{-iz_n t} - e^{-izt}\right) \to 0 \quad as \quad n \to \infty$$

hence by dominated convergent theorem , $\int_k \left|e^{-iz_n t} - e^{-izt}\right| dm(t) \to 0$

therefore (4) becomes $|f(z_n) - f(z)| \leq \int_k M \left|e^{-iz_n t} - e^{-izt}\right| dm(t)$

$$\to 0 \qquad as \quad n \to \infty$$

$$f(z_n) \to f(z) \quad as \quad n \to \infty$$

hence f is continuous

Claim: $\int_{\Gamma_\alpha} f(z)\, dz = 0$

let $z = \omega(s)\ a \le s \le b\ ,\ dz = \omega'(s)\ d\omega$

consider $\int_a^b f(\omega)\omega'(s)\ d\omega = \int_a^b \int_R \Phi(t)\ e^{-it\omega}\ \frac{dt}{\sqrt{2\pi}}\ \omega'(s)\ d\omega$

since $\Phi(t) \in D(R^n) \Rightarrow \Phi(t)$ *is* differentiable

$\Rightarrow \Phi$ is continuous

$\Rightarrow \Phi$ is measurable

$e^{-it\omega}$ is continuous, therefore measurable

hence $\Phi(t)e^{-it\omega}$ is measurable

consider $\int_a^b \omega'(s) \int_k \left|\Phi(t)e^{-it\omega}\right| \frac{dt}{\sqrt{2\pi}}\ d\omega \le \int_a^b \omega'(s) \int_k M\ e^{r|im\,\omega|}\ \frac{dt}{\sqrt{2\pi}}\ d\omega$

$\le \int_a^b \omega'(s)\ M\ e^{r|im\,z|}\ \frac{m(k)}{\sqrt{2\pi}}\ d\omega$ which is finite

Conversely assume that f is an entire function and $|f(z)| \le \gamma_N (1+|z|)^{-N}\ e^{r|img\,z|}$.

Define $\Phi(t) = \int f(x)e^{itx}\ dm(x) \qquad (t \in R)$ (5)

Claim: $(1+|x|)^{-N}\ f(x) \in L'(R)$

Since $f(x)$ is analytic, f is continuous.

Therefore f is measurable.

consider $\int_R |1+|x||^{-N}\ |f(x)|\ dm(x) \le \int_{-\infty}^{\infty} |1+|x||^{-N}\gamma_N\ |1+|x||^{N-2}\ e^{r|img\,x|}\ \frac{dx}{\sqrt{2\pi}}$

$= \int_{-\infty}^{\infty} |1+|x||^{-2}\gamma_N\ \ e^{r(0)}\ \frac{dx}{\sqrt{2\pi}}$

$= \frac{\gamma_N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |1+|x||^{-2}\ dx$

$\le \frac{\gamma_N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |1+|x||^{-2}\ dx$ [since $\frac{1}{(1+x)^2} \le \frac{1}{(1+x^2)}$]

$< \infty$

$|1+|x||^{-N}\ f(x) \in L'(R)$.

Claim: $\Phi \in C^{\infty}(R)$

Let $f_n(x) = \frac{f(x)\left(e^{it_n x} - e^{isx}\right)}{t_n - s}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f(x)\left(e^{it_n x} - e^{isx}\right)}{t_n - s}$

$= f(x)\ \lim_{n\to\infty} \frac{\left(e^{it_n x} - e^{isx}\right)}{t_n - s}$

$= \quad f(x)\ e^{isx}\ \lim_{n\to\infty} \frac{\left(e^{i(t_n - s)x} - 1\right)}{t_n - s}$

$$= f(x)\, e^{isx} \lim_{n\to\infty} \frac{\left(e^{ih_n x} - 1\right)}{h_n} \qquad \text{where } h_n = t_n - s$$

$$\lim_{n\to\infty} f_n(x) = f(x)\, e^{isx}\; ix \left(\lim_{n\to\infty} \frac{1 + ih_n x + \frac{(ih_n x)^2}{2!} + \dots}{h_n} \right)$$

$$\lim_{n\to\infty} f_n(x) = f(x)\, e^{isx}\; ix \tag{6}$$

$f_n(x) = f(x) \dfrac{\left(e^{it_n x} - e^{isx}\right)}{t_n - s}$ is measurable.

Consider $\left| f_n(x) \right| = \left| f(x) \dfrac{\left(e^{it_n x} - e^{isx}\right)}{t_n - s} \right|$

$$= \left| f(x)\, e^{isx} \frac{\left(e^{i(t_n - s)x} - 1\right)}{t_n - s} \right|$$

$$= \left| f(x)\, e^{isx} \frac{\left(e^{ih_n x} - 1\right)}{h_n} \right|$$

$$= \left| f(x)\, e^{isx}\, e^{i\frac{h_n}{2}x} \frac{\left(e^{i\frac{h_n}{2}x} - e^{-i\frac{h_n}{2}x} \right)}{h_n} \right|$$

$$= |f(x)| \left| \frac{\sin \frac{h_n}{2} x}{h_n} \right|$$

$$= |f(x)| \left| \frac{\sin \frac{h_n}{2} x}{h_n \cdot \frac{x}{2}} \frac{x}{2} \right|$$

$$= |f(x)| \left| \frac{x}{2} \right| \left| \frac{\sin \frac{h_n}{2} x}{\frac{x h_n}{2}} \right|$$

$$\le |f(x)| \left| \frac{x}{2} \right| \qquad [\text{since } \left| \frac{\sin \frac{h_n}{2} x}{h_n} \right| \le 1]$$

let $g(x) = |f(x)| \left| \dfrac{x}{2} \right|$

Claim: $g \in L'(R)$

$$\int_R |g(x)|\ dm(x) = \int_{-\infty}^{\infty} |f(x)|\left|\frac{x}{2}\right|\ dm(x)$$

$$\leq \int_{-\infty}^{\infty} \gamma_N (1+|x|)^{-N} e^{r|img\ x|}\left|\frac{x}{2}\right|\ dm(x)$$

$$\leq\ \gamma_N \int_{-\infty}^{\infty} (1+|x|)^{-N}\left|\frac{x}{2}\right|\ dm(x)$$

Choose N=2 $\quad = \ \gamma_2 \int_{-\infty}^{\infty} (1+|x|)^{-2}\left|\frac{x}{2}\right|\ dm(x)$

$$\leq\ \gamma_2 \int_{-\infty}^{\infty} \frac{1}{(1+|x|^2)}\left|\frac{x}{2}\right| \frac{dx}{\sqrt{2\pi}}$$

$$< \infty$$

therefore $f(x)\left|\frac{x}{2}\right| \in L'(R)$.

by dominated convergent theorem,

$$\lim_{n\to\infty} \int_R \frac{f(x)\left(e^{it_n x} - e^{isx}\right)}{t_n - s} dm(x) = \int_R f(x)e^{isx}\ ix\, dm(x)$$

$$\lim_{n\to\infty} \frac{\Phi(t_n) - \Phi(s)}{t_n - s} = \int_R f(x)e^{isx}\ ix\, dm(x)$$

$$\Phi'(t) = \int_R f(x)e^{isx}\ ix\, dm(x)$$

$\Phi'(t)$ exists

Φ is differential at everywhere

$\Phi \in C^\infty(R)$

Claim: $\int f(\xi + i\eta)e^{it(\xi+i\eta)}d\xi$ is independent of η for arbitrary t .

Let Γ be a rectangular path in $(\xi + i\eta)$ plane with one edge on the real axis, one on the line $\eta = \eta_1$ whose vertical edges move off to infinity

Since f *and* $e^{it(\xi+i\eta)}$ are analytic

therefore $f\ \ e^{it(\xi+i\eta)}$ is analytic and Γ be the closed path ,

therefore by cauchy's theorem $\int_\Gamma f(\xi + i\eta)e^{it(\xi+i\eta)}\ d\xi = 0$

$$\int_{\xi_1}^{\xi_2} f(\xi)e^{it\xi}d\xi + \int_0^{\eta_1} f(\xi_2 + i\eta)e^{it(\xi_2+i\eta)}id\eta + \int_{\xi_2}^{\xi_1} f(\xi + i\eta_1)e^{it(\xi+i\eta_1)}d\xi + \int_{\eta_1}^{0} f(\xi_1 + i\eta)e^{it(\xi_1+i\eta)}id\eta = 0$$

(7)

Consider $\left|\int_0^{\eta_1} f(\xi_2 + i\eta)e^{it(\xi_2+i\eta)}id\eta\right| \leq \int_0^{\eta_1} |f(\xi_2 + i\eta)|\left|e^{it(\xi_2+i\eta)}\right|d\eta$

$$= \int_0^{\eta_1} |f(\xi_2 + i\eta)|e^{-t\eta}d\eta$$

$$\leq \int_0^{\eta_1} \left(1+\left|\xi_2+i\eta\right|\right)^{-N} \gamma_N\, e^{r\eta} e^{-t\eta} d\eta$$

$$= \gamma_N \int_0^{\eta_1} \frac{e^{(r-t)\eta}}{\left(1+\left|\xi_2+i\eta\right|\right)^N} d\eta$$

$$\leq \gamma_N \int_0^{\eta_1} \frac{e^{(r-t)\eta}}{1+\left|\xi_2\right|^N} d\eta \text{ since} \left|\xi_2+i\eta\right|^N \geq \left|\xi_2\right|^N$$

$$\leq \gamma_N \int_0^{\eta_1} \frac{e^{(r-t)\eta}}{\left|\xi_2\right|^N} d\eta$$

$$= \frac{\gamma_N}{\left|\xi_2\right|^N} \; \frac{e^{(r-t)\eta_1}-1}{r-t}$$

$$= \frac{\gamma_N}{\left|\xi_2\right|^N} \; \frac{1-e^{(r-t)\eta_1}}{t-r}$$

$$\leq \frac{\gamma_N}{\left|\xi_2\right|^N}$$

$$\to 0 \qquad as\ \xi_2 \to \infty \text{(8)}$$

Similarly $\int_{\eta_1}^{0} f\left(\xi_1+i\eta\right)e^{it(\xi_1+i\eta)} i d\eta = 0$ as $\xi_1 \to \infty$

Taking $\xi_1 \to -\infty$ in (7)

$$\underset{\xi_1\to-\infty}{Lim} \int_{\xi_1}^{\xi_2} f(\xi)e^{it\xi}d\xi + \underset{\xi_1\to-\infty}{Lim} \int_0^{\eta_1} f\left(\xi_2+i\eta\right)e^{it(\xi_2+i\eta)} i d\eta + \underset{\xi_1\to-\infty}{Lim} \int_{\xi_2}^{\xi_1} f\left(\xi+i\eta_1\right)e^{it(\xi+i\eta_1)}d\xi + \underset{\xi_1\to-\infty}{Lim} \int_{\eta_1}^{0} f\left(\xi_1+i\eta\right)e^{it(\xi_1+i\eta)} i d\eta = 0$$

$$\underset{\xi_1\to-\infty}{Lim} \int_{\xi_1}^{\xi_2} f(\xi)e^{it\xi}d\xi + \underset{\xi_1\to-\infty}{Lim} \int_{\xi_2}^{\xi_1} f\left(\xi+i\eta_1\right)e^{it(\xi+i\eta_1)}d\xi + \underset{\xi_1\to-\infty}{Lim} \int_{\eta_1}^{0} f\left(\xi_1+i\eta\right)e^{it(\xi_1+i\eta)} i d\eta = 0 \text{ by(8)}$$

Taking $\xi_2 \to \infty$

$$\int_{-\infty}^{\infty} f(\xi)e^{it\xi}d\xi + \int_{\infty}^{-\infty} f\left(\xi+i\eta_1\right)e^{it(\xi+i\eta_1)}d\xi = 0$$

$$\int_R f(\xi)e^{it\xi}d\xi - \int_R f\left(\xi+i\eta_1\right)e^{it(\xi+i\eta_1)}d\xi = 0 \text{ (9)}$$

From (9) , (5) becomes $\Phi(t) = \int f(x+iy)e^{it(+iyx)}\ dx \qquad (t \in R)$

Given $t \in R^n, \quad t \neq 0$

Choose $y = \frac{\lambda t}{|t|} \qquad where \quad \lambda > 0$

Then $t \cdot y = t \cdot \frac{\lambda t}{|t|}$

If $t < 0$ then $t \cdot y = t \cdot \frac{\lambda t}{-t} = -t\lambda$

If $t > 0$ then $t \cdot y = t \cdot \frac{\lambda t}{t} = t\lambda$

Therefore $t \cdot y = \lambda |t|$

$|y| = |\lambda| = \lambda$ $\quad [\sin ce\, \lambda > 0]$

Now consider $\left| f(x+iy)e^{it(x+iy)} \right| = \left| f(x+iy) \right| \left| e^{it(x+iy)} \right|$

$$\leq \gamma_N \left(1+|x+iy|\right)^{-N} e^{r|img(x+iy)|} \left| e^{-ty} \right|$$

$$= \gamma_N \left(1+|x+iy|\right)^{-N} e^{r|y|} \left| e^{-ty} \right| \qquad [\text{since } |x+iy| > |x|]$$

$$\leq \gamma_N \left(1+|x|\right)^{-N} e^{r|y|} e^{-|t||y|}$$

$$\leq \gamma_N \left(1+|x|\right)^{-N} e^{r|y|} e^{-|t||y|}$$

$$= \gamma_N \left(1+|x|\right)^{-N} e^{(r-|t|)|y|}$$

$$= \gamma_N \left(1+|x|\right)^{-N} e^{(r-|t|)\lambda}$$

Now consider $\Phi(t) = \int_R f(x+iy)e^{it(+iyx)}\, dmx$

$$|\Phi(t)| \leq \int_R \left| f(x+iy)e^{it(+iyx)} \right| dmx$$

$$\leq \int_R \gamma_N e^{|r-|t||\lambda} \left(1+|x|\right)^{-N} dmx$$

$$= \gamma_N e^{|r-|t||\lambda} \int_R \left(1+|x|\right)^{-N} dmx$$

Where N is chosen so large, choose N=2

$$|\Phi(t)| = \gamma_2 e^{|r-|t||\lambda} \int_R \left(1+|x|\right)^{-2} dmx$$

$$\leq \frac{\gamma_2 e^{|r-|t||\lambda}}{\sqrt{2\pi}} \int_R \left(1+|x|^{-2}\right) dx$$

$$< \infty$$

Now to prove support of Φ in rB

If $|t| > r$

Then $\Phi(t) \leq \gamma_N e^{-(|t|-r)\lambda} \int_R \left(!+|x|\right)^{-N} dmx$

As $\lambda \to \infty$ $|\Phi(t)| = 0$

$\Phi(t) = 0$ if $|t| > r$

$\Phi(t) \neq 0$ if $|t| \leq r$

Therefore support of Φ in rB

Apply inversion theorem to (A) we get, $f(x) = \int_R \Phi(t)e^{-itx} dm(t) \quad \text{for} \quad \text{real} \quad z$

This completes the proof.

II. Conclusion:

I have tried a brief note on PaleyWiener in C. This is a very useful result as it enables one pass to the Fourier transform of a function in the Hardy space and perform calculations in the easily understood space.

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