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Fixed Point Theorems for Two Weakly Increasing Mappings by Using Delbosco's Set in Ordered G-Metric Spaces

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Abstract: *We give some fixed point theorems for two weakly increasing self-mappings T and S satisfying contractive type conditions by using Delbosco's set in ordered G-metric spaces.*
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I. Introduction

In [1], to give a unified approach for contraction mappings D. Delbosco's considered the set $\mathcal{F}$ of all continuous function $g: [0, +\infty)^3 \to [0, +\infty)$ satisfying the following conditions:

(g-1) : $g(1,1,1) = h < 1$,

(g-2) : If $u, v \in [0, +\infty)$ are such that

$$u \leq g(u, v, v) \text{or} u \leq g(v, u, v) \text{ or } u \leq g(v, v, u)$$

then $u \leq hv$.

And proved the following.

Theorem: 1.1 (see [1]) Let(X, d) be a complete metric space. If S and T are two mappings from X into itself, satisfying the following conditions:

(1.1) $$d(Sx, Ty) \leq g\big(d(x, y), d(x, Sx), d(y, Ty)\big)$$

for all $x, y \in X$. where, $g \in \mathcal{F}$. Then S and T have a unique common fixed point in X.

Some authors proved many kinds of fixed point theorems for contractive type mappings by using Delbosco's set. (see [2-4]).The basic topological properties of ordered sets were discussed by Wolk [5] and Manjardet [6]. The existence of fixed point in partially ordered metric spaces was considered by Ran and Reurings [7]. The notion of G-metric space was introduced by Mustafa and Sims [8] as a generalization of the notion of metric spaces. Mustafa et al. studied many fixed point results in G-metric space [9-13].

II. Basic Concepts

In this section, we present the necessary definitions and theorems in G-metric spaces.

Throughout this paper, we will adopt the following notations: $\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}^+$ is the set of all non-negative real numbers. Consistent with Mustafa and Sims [8], the following definitions and results will be needed in the sequel.

Definition 2.1 (see [8])let X is a nonempty set and $G: X \times X \times X \to \mathbb{R}^+$be a function satisfying the following properties:

[G1] $G(x. y, z) = 0$ if $x = y = z$.

[G2] $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$.

[G3] $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$.

[G4] $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ ... ...Symmetry in all three variables.

[G5] $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$for all $x, y, z, a \in X$ (Rectangle inequality)

Then the function G is called a generalized metric or more specifically a G-metric on X and pair (X, G) is called a G-metric space.

Definition 2.2(see [8])Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to+\infty} G(x_n, x_m, x_m) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x.Thus $x_n \to x$ in a G-metric space (X,G) if for any $\varepsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$.

Proposition: 2.1(see [8]) Let(X, G) be a G-metric space. Then the following are equivalent:

(1). $\{x_n\}$ is G-convergent to;

(2). $G(x_n, x_n, x) \to 0$ as$n \to \infty$;

(3). $G(x_n, x, x) \to 0$ as $n \to \infty$;
(4). $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.
Definition: 2.3 (see [8]) Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq k$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.
Proposition: 2.2(see [8]) Let (X, G) be a G-metric space. Then the following are equivalent:
(1) The sequence $\{x_n\}$ is G-Cauchy;
(2) For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq k$.
Definition: 2.4(see [8])Let (X, G) and (X', G')be G-metric spaces, and let $T: (X, G) \to (X', G')$ be a function. Then T is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(T(a), T(x), T(y)) < \varepsilon$. A function T is G-continuous at X if and only if it is G-continuous at all $a \in X$.
Proposition: 2.3(see [8])Let (X, G) and (X', G') are G-metric spaces. Then $T: X \to X'$ is G-continious at $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(T(x_n))$ is G-convergent to $T(x)$.
Proposition: 2.4(see [8]) let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variable.
Definition: 2.5(see [8]) A G-metric space (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).
Definition: 2.6(see [8]) A G-metric space on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.
Definition: 2.7 Let $(X, \leq)$ be a partially ordered set and $T : X \to X$ be say that non-decreasing mapping if for $x, y \in X$, $x \leq y \Rightarrow Tx \leq Ty$.
The notion of weakly increasing mappings was introduced in by Altun and Simsek [14].
Definition 2.8(see [14]) Let $(X, \leq)$ be a partially ordered set. Two mappings $T, S: \to X$ are said to be weakly increasing if $Tx \leq STx$ and $Sx \leq TSx$ for all $x \in X$.Two weakly increasing mappings need not be non-decreasing.
Example: 2.1(see [14]) Let $X = \mathbb{R}^+$, endowed with the usual ordering. Let $T, S: \to X$ defined by

$$Tx = \begin{cases} x, & 0 \leq x \leq 1, \\ 0, & 1 < x < +\infty, \end{cases}$$
$$Sx = \begin{cases} \sqrt{x}, & 0 \leq x \leq 1, \\ 0, & 1 < x < +\infty. \end{cases}$$

Then T and S are weakly increasing mappings. Note that T and S are not non-decreasing.

III. Main Results

We will prove the following result:
Theorem: 3.1 Let $(X, \leq)$ be a partially ordered set and suppose that there exists G-metric in X such that (X, G) is G-complete. Let $T, S: X \to X$ be two weakly increasing mappings with respect to $\leq$, satisfying the following conditions:
(3.1) $G(Tx, Sx, Sx) \leq g\big(G(x, y, y), G(x, Tx, Tx), G(y, Sy, Sy)\big)$
(3.2) $G(Sx, Ty, Ty) \leq g\big(G(x, y, y), G(x, Sx, Sx), G(y, Ty, Ty)\big)$
for all comparative $x, y \in X$. where, $g \in \mathcal{F}$. If T or S is G-continuous, then T and S have a common fixed point u in X.
Proof: Let x_0 be an arbitrary point in X. choose $x_1 \in X$ such that $x_1 = Tx_0$. Again choose $x_2 \in X$ such that $Sx_1 = x_2$. Also choose $x_3 \in X$ such that $x_3 = Tx_2$. Continuing this fashion, we can construct a sequence in $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$, $\forall\, n \in N \cup \{0\}$ and $x_{2n+2} = Sx_{2n+1}$, $\forall\, n \in N \cup \{0\}$. Since T and S are weakly increasing with respect to $\leq$, we get:
(3.3) $$x_1 = Tx_0 \leq S(Tx_0) = Sx_1 = x_2 \leq T(Sx_1) = Tx_2 = x_3 \\ \leq S(Tx_2) = Sx_3 = x_4 \leq \cdots \ldots \ldots \ldots$$
Form (3.1), we have
$$\begin{aligned} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &= G(Tx_{2n}, Sx_{2n+1}, Sx_{2n+1}) \\ &\leq g\begin{pmatrix} G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n}, Tx_{2n}, Tx_{2n}), \\ G(x_{2n+1}, Sx_{2n+1}, Sx_{2n+1}) \end{pmatrix} \\ &= g\begin{pmatrix} G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n}, x_{2n+1}, x_{2n+1}), \\ G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \end{pmatrix} \end{aligned}$$
Thus, by (g-2), we have
(3.4) $$G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq hG(x_{2n}, x_{2n+1}, x_{2n+1})$$
Similarly, by (3.2), we have
$$G(x_{2n}, x_{2n+1}, x_{2n+1}) = G(Sx_{2n-1}, Tx_{2n}, Tx_{2n})$$

$$\leq g_1\begin{pmatrix} G(x_{2n-1}, x_{2n}, x_{2n}), G(x_{2n-1}, Sx_{2n-1}, Sx_{2n-1}), \\ G(x_{2n}, Tx_{2n}, Tx_{2n}) \end{pmatrix}$$
$$= g_1\begin{pmatrix} G(x_{2n-1}, x_{2n}, x_{2n}), G(x_{2n-1}, x_{2n}, x_{2n}), \\ G(x_{2n}, x_{2n+1}, x_{2n+1}) \end{pmatrix}$$

Thus, from (g-2), we obtain:

(3.5) $$G(x_{2n}, x_{2n+1}, x_{2n+1}) \leq hG(x_{2n-1}, x_{2n}, x_{2n})$$

Therefore, by (1.4) and (1.5),

(3.6) $$G(x_n, x_{n+1}, x_{n+1}) \leq hG(x_{n-1}, x_n, x_n) \forall n \in N.$$

If $x_0 = x_1$, we get $G(x_n, x_{n+1}, x_{n+1}) = 0$ for each $n \in N$. Hence $x_n = x_0$ for each $n \in N$. Therefore $\{x_n\}$ is G-Cauchy sequence in X. So without loss of generality, we assume that $x_0 \neq x_1$. Let $m, n \in N with m > n$. By axiom[G5] of the definition of G-metric space, we get:

(3.7) $$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \ldots + G(x_{m-1}, x_m, x_m)$$

Using (3.6), we have

(3.8) $$G(x_n, x_m, x_m) \leq [h^n + h^{n+1} + \cdots \ldots \ldots + h^{m-1}]G(x_0, x_1, x_1)$$
$$\leq \frac{h^n}{1-h} G(x_0, x_1, x_1)$$

On making limit $m, n \to \infty$ in (3.8), we get

(3.9) $$\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 0$$

This implies that $\{x_n\}$ is G-Cauchy sequence in (X, G) and so, since (X, G) is G-complete; it converges to a point u in X. Also the sub-sequences $(x_{2n+1}) = (Tx_{2n})$ and $(x_{2n+2}) = (Sx_{2n+1})$ converge to u.

Further, the G-continuity of T implies

(3.10) $$Tu = T(\lim_{n\to\infty} x_{2n}) = \lim_{n\to\infty} Tx_{2n}$$
$$= \lim_{n\to\infty} x_{2n+1} = u$$

And this proves that u is a fixed point of T. Now, we claim that$Tu = u$. Since $u \leq u$, by inequality (3.1), we have

$$G(u, Su, Su) = G(Tu, Su, Su)$$
$$\leq g_1\big(G(u, u, u), G(u, Tu, Tu), G(u, Su, Su)\big)$$
$$= g_1\big(0,0, G(u, Su, Su)\big)$$
$$= 0 \text{ ,by property (g-2)}$$

That is, $Su = u$, which means that the point $u \in X$ is a common fixed point of T and S. If S is G-continuous. By similar argument as above we shows that S and T have a common fixed point. This finishes the proof.

In what fallows, we prove that Theorem 3.1 is still valid for T and S, not necessarily continuous, assuming the following hypothesis in X:

(3.11) If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then$x = \sup\{x_n\}$, for all $n \in N$.

Theorem: 3.2 Let$(X, \leq)$ be a partially ordered set and suppose that there exists G-metric in X such that (X, G) is G-complete. Let T and S be two weakly increasing mappings with respect to $\leq$, satisfying the conditions (3.1) and (3.2). Assume that X satisfies (3.11). Then T and S have a common fixed point u in X.

Proof: Following the proof of Theorem 1, we only have to check $Tu = Su = u$.

As $\{x_n\}$ is an increasing sequence in X and$x_n \to u$. Thus $(x_{2n}), (x_{2n+1}), (Tx_{2n})$ and (Sx_{2n+1}) converge to u. since X satisfies property (3.11), we get that$u = \mathrm{Sup}\{x_n\}$, particularly, $x_n \leq u$ for all $n \in N$. Thus x_{2n} and u are comparative. By (3.1), we have

(3.12) $$G(Tx_{2n}, Su, Su) \leq g\big(G(x_{2n}, u, u), G(x_{2n}, Tx_{2n}, Tx_{2n}), G(u, Su, Su)\big)$$

On making limit $n \to +\infty$ in (3.12) and using the fact that g and Gare continuous, by property (g-2), we obtain:

$$G(u, Su, Su) \leq g\big(0, 0, G(u, Su, Su)\big)$$
$$= hG(u, Su, Su)$$

which means that $G(u, Su, Su) = 0$ that is,$u = Su$.

By similar argument, we may show that$u = fu$. This finishes the proof.

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