IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 1 Ver. IV. (Feb. 2014), PP 116-122
www.iosrjournals.org

Some Characterization of Normal Multi-fuzzy and Normal Multi-anti fuzzy Subgroup

R.Muthuraj[1], S.Balamurugan[2]
[1](*PG and Research Department of Mathematics, H.H. The Rajah's College,Pudukkottai,Tamilnadu, India.*
[2](*Department of Mathematics, Velammal College of Engineering and Technology,Madurai,Tamilnadu,India.*

Abstract : *This paper is mainly concerned with a generalization of the concept of a normal multi-fuzzy subgroup and a normal multi-anti fuzzy subgroup. We introduce the concept of a normal multi-fuzzy subgroup of a multi-fuzzy subgroup and examine its basic properties. Also we introduce the concept of a normal multi-anti fuzzy subgroup of a multi-anti fuzzy subgroup and examine its basic properties. We use them to develop results concerning multi-fuzzy subgroups of a multi-fuzzy subgroup and multi-anti fuzzy subgroups of a multi-anti fuzzy subgroup.*

Mathematics Subject Classification: MSC: 20N25; 03E72; 08A72
Keywords: *Fuzzy set, multi-fuzzy set, fuzzy subgroup , multi-fuzzy subgroup, anti fuzzy subgroup, multi-anti fuzzy subgroup, normal multi-fuzzy subgroup, normal multi-anti fuzzy subgroup, multi-fuzzy coset.*

I. INTRODUCTION

S.Sabu and T.V.Ramakrishnan [9] proposed the theory of multi-fuzzy sets in terms of multi-dimensional membership functions and investigated some properties of multi-level fuzziness. L.A.Zadeh [11] introduced the theory of multi-fuzzy set is an extension of theories of fuzzy sets. N.Palaniappan and R.Muthuraj [7] introduced the inter-relationship between the Anti fuzzy group and its Lower level subgroups. P.S.Das [1] studied the inter-relationship between the fuzzy subgroup and its level subsets. Liu, Mukharjee and Bhattacharya [2,3] proposed the concept of normal fuzzy sets. R.Muthuraj and S.Balamurugan [6] proposed the inter-relationship between the multi-fuzzy group and its level subgroups. R.Muthuraj and S.Balamurugan [4] also proposed the inter-relationship between the multi-anti fuzzy group and its lower level subgroups. In this paper we define a new algebraic structures of a normal multi-fuzzy subgroup and a normal multi-anti fuzzy subgroup.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel.

2.1 Definition Let X be any non-empty set. A fuzzy subset μ of X is $\mu : X \to [0,1]$.

2.2 Definition Let X be a non-empty set. A multi-fuzzy set A in X is defined as a set of ordered sequences:

$$A = \{ (x, \mu_1(x), \mu_2(x), \ldots , \mu_i(x), \ldots) : x \in X \}, \text{ where } \mu_i : X \to [0,1] \text{ for all } i.$$

Remark

i. If the sequences of the membership functions have only k-terms (finite number of terms), then k is called the dimension of A.

ii. The set of all multi-fuzzy sets in X of dimension k is denoted by M^kFS(X).

iii. The multi-fuzzy membership function μ_A is a function from X to $[0,1]^k$ such that for all x in X,

$$\mu_A(x) = (\mu_1(x), \mu_2(x), \ldots ,\mu_k(x)).$$

iv. For the sake of simplicity, we denote the multi-fuzzy set

$$A = \{ (x, \mu_1(x), \mu_2(x), \ldots , \mu_k(x)) : x \in X \} \text{ as } A=(\mu_1, \mu_2, \ldots , \mu_k).$$

2.3 Definition Let k be a positive integer and let A and B in M^kFS(X), where $A=(\mu_1, \mu_2, \ldots ,\mu_k)$ and $B=(\nu_1, \nu_2, \ldots ,\nu_k)$, then we have the following relations and operations:

i. $A \subseteq B$ if and only if $\mu_i \le \nu_i$, for all i = 1, 2, ... , k;

ii. A = B if and only if $\mu_i = \nu_i$, for all i = 1, 2, ... , k;

iii. $A \cup B = (\mu_1 \cup \nu_1, \ldots , \mu_k \cup \nu_k) = \{(x, \max(\mu_1(x), \nu_1(x)), \ldots , \max(\mu_k(x), \nu_k(x))) : x \in X\}$;

iv. $A \cap B = (\mu_1 \cap \nu_1, \ldots , \mu_k \cap \nu_k) = \{(x, \min(\mu_1(x), \nu_1(x)), \ldots , \min(\mu_k(x), \nu_k(x))) : x \in X\}$;

v. $A+B = (\mu_1+\nu_1, \dots, \mu_k+\nu_k) = \{(x, \mu_1(x)+\nu_1(x)-\mu_1(x)\nu_1(x), \dots, \mu_k(x)+\nu_k(x)-\mu_k(x)\nu_k(x)) : x\in X\}$.

2.4 Definition Let $A=(\mu_1, \mu_2, \dots, \mu_k)$ be a multi-fuzzy set of dimension k and let μ_i' be the fuzzy complement of the ordinary fuzzy set μ_i for i=1, 2, ... ,k. The Multi-fuzzy Complement of the multi-fuzzy set A is a multi-fuzzy set $(\mu_1', \dots, \mu_k')$ and it is denoted by C(A) or A' or A^C.
That is, $C(A) = \{(x, c(\mu_1(x)), \dots, c(\mu_k(x))) : x\in X\} = \{(x, 1-\mu_1(x), \dots, 1-\mu_k(x)) : x\in X\}$, where c is the fuzzy complement operation.

2.5 Definition Let μ be a fuzzy set on a group G. Then μ is said to be a fuzzy subgroup of G if for all $x,y\in G$,

i. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

ii. $\mu(x^{-1}) = \mu(x)$

2.6 Definition A multi-fuzzy set A of a group G is called a multi-fuzzy subgroup of G if for all $x,y\in G$,

i. $A(xy) \geq \min\{A(x), A(y)\}$

ii. $A(x^{-1}) = A(x)$

2.7 Definition Let μ be a fuzzy set on a group G. Then μ is called an anti fuzzy subgroup of G if for all $x,y\in G$,

i. $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$

ii. $\mu(x^{-1}) = \mu(x)$

2.8 Definition A multi-fuzzy set A of a group G is called a multi-anti fuzzy subgroup of G if for all $x,y\in G$,

i. $A(xy) \leq \max\{A(x), A(y)\}$

ii. $A(x^{-1}) = A(x)$

2.9 Definition Let A and B be any two multi-fuzzy sets of a non-empty set X. Then for all $x\in X$,

i. $A \subseteq B$ iff $A(x) \leq B(x)$

ii. $A = B$ iff $A(x) = B(x)$

iii. $(A\cup B)(x) = \max\{A(x), B(x)\}$

iv. $(A\cap B)(x) = \min\{A(x), B(x)\}$

2.10 Definition Let A and B be any two multi-fuzzy sets of a non-empty set X. Then

i. $A\cup A = A$, $A\cap A = A$

ii. $A \subseteq A\cup B$, $B \subseteq A\cup B$, $A\cap B \subseteq A$ and $A\cap B \subseteq B$

iii. $A \subseteq B$ iff $A\cup B = B$

iv. $A \subseteq B$ iff $A\cap B = A$

2.11 Definition We define the binary operation '∘' on MFP(G), the multi-fuzzy power set of a group G and the unary operation $^{-1}$ on MFP(G) as follows:
$\forall A,B\in MFP(G)$ and $\forall x\in G$, $(A\circ B)(x) = \max\{\min\{A(y), B(z) / y,z\in G, yz = x\}\}$ and $A^{-1}(x) = A(x^{-1})$. We call $A\circ B$ as the product of A and B and A^{-1} as the inverse of A. The binary operation '∘' is associative.

III. Normal multi-fuzzy subgroup of the multi-fuzzy subgroup

In this section, we introduce the concept of normal multi-fuzzy subgroup of a multi-fuzzy subgroup and examine its basic properties.

3.1 Definition A multi-fuzzy subgroup *A* of a group *G* is called normal if for each $x \in G$,

$$A(x) \leq \min_{g\in G} \{ A(gxg^{-1}) \}$$

3.2 Definition Let $A,B\in MF(G)$ and $A \subseteq B$. Then A is called a normal multi-fuzzy subgroup of the multi-fuzzy subgroup B, written as $A \lhd B$, if $A(xyx^{-1}) \geq \min\{A(y), B(x)\}$, $\forall x,y\in G$.

Remarks: The following statements are immediate from the above definition 3.2:

1. Every multi-fuzzy subgroup is a normal multi-fuzzy subgroup of itself.

2. $A \in MFP(G)$ is a normal multi-fuzzy subgroup of G $\Leftrightarrow$ A is a normal multi-fuzzy subgroup of the multi-fuzzy subgroup 1_G

Notations: The following notations to be used in this paper with the following meaning:
1. MF(G) is the set of all multi-fuzzy subgroups of a group G.
2. NMF(G) is the set of all normal multi-fuzzy subgroups of a group G.

3.3 Definition Let μ be a fuzzy subgroup of a group G and $x \in G$. The fuzzy subsets $\mu(e)_{\{x\}} \circ \mu$ and $\mu \circ \mu(e)_{\{x\}}$ are referred to as the left fuzzy coset and right fuzzy coset of μ with respect to x and written as $x\mu$ and μx, respectively.

3.4 Definition Let $A \in MF(G)$ be a multi-fuzzy subgroup of a group G and $x \in G$. The multi-fuzzy sets $A(e)_{\{x\}} \circ A$ and $A \circ A(e)_{\{x\}}$ are referred to as the left multi-fuzzy coset and right multi-fuzzy coset of A with respect to x and written as xA and Ax respectively.

Remark: If $A \in NMF(G)$ is a normal multi-fuzzy subgroup of a group G, then the left multi-fuzzy coset xA is just the right multi-fuzzy coset Ax. Thus, in this case, we call xA as a multi-fuzzy coset for short.

3.1 Theorem: Let $A,B \in MF(G)$ and $A \subseteq B$. Then the following assertions are equivalent:
1. A is a normal multi-fuzzy subgroup of B
2. $A(yx) \geq \min\{A(xy), B(y)\}, \forall x,y \in G$
3. $A(e)_{\{x\}} \circ A \supseteq (A \circ A(e)_{\{x\}}) \cap B, \forall x \in G$

Proof: (1) $\Rightarrow$ (2)
Since A is a normal multi-fuzzy subgroup of B,

$$\begin{aligned} A(yx) &= A(yxyy^{-1}) \\ &= A(y(xy)y^{-1}) \\ &\geq \min\{A(xy), B(y)\}, \forall x,y \in G \text{ ,since by the definition 3.2.} \end{aligned}$$

Proof: (2) $\Rightarrow$ (1)

$$\begin{aligned} A(xyx^{-1}) &= A(x(yx^{-1})) \\ &\geq \min\{A((yx^{-1})x), B(x)\}, \text{ since by the hypothesis(2).} \\ &= \min\{A(y), B(x)\} \end{aligned}$$

That is, A is normal multi-fuzzy subgroup of B.
That is, $A \triangleleft B$.
Proof: (2) $\Rightarrow$ (3)

$$\begin{aligned} \forall z \in G, (A(e)_{\{x\}} \circ A)(z) &= \max\{\min\{A(e)_{\{x\}}(p), A(q)\} / pq = z \text{ where } p,q \in G\}, \text{since by the definition 2.11.} \\ &= \max\{\min\{A(e)_{\{x\}}(x), A(q)\} / xq = z\}, \text{ since take } p = x. \\ &= \max\{\min\{A(e), A(x^{-1}z)\}\}, \text{ since } A(xq) = A(z) \Rightarrow A(q) = A(x^{-1}z). \\ &= \max\{A(x^{-1}z)\} \\ &= \max\{A((x^{-1}z)^{-1})\} \\ &= \max\{A(z^{-1}x)\} \\ &\geq \max\{\min\{A(xz^{-1}), B(z^{-1})\}\}, \text{ since by the hypothesis (2)} \\ &= \max\{\min\{A((xz^{-1})^{-1}), B(z^{-1})\}\} \\ &= \max\{\min\{A(zx^{-1}), B(z^{-1})\}\} \\ &= \min\{(A \circ A(e)_{\{x\}})(z), B(z)\}, \text{ since } (A \circ A(e)_{\{x\}})(z) = A(zx^{-1}) \\ &= [(A \circ A(e)_{\{x\}}) \cap B](z) \text{ , since by the definition of '}\cap\text{'} \end{aligned}$$

Hence the proof (2) $\Rightarrow$ (3).
Proof: (3) $\Rightarrow$ (2)

$$\begin{aligned} \forall x,y \in G, \quad A(yx) &= A((yx)^{-1}) \\ &= A(x^{-1}y^{-1}) \\ &= (A(e)_{\{x\}} \circ A)(y^{-1}) \\ &\geq ((A \circ A(e)_{\{x\}}) \cap B)(y^{-1}) \text{ , since by the hypothesis(3)} \\ &= \min\{(A \circ A(e)_{\{x\}})(y^{-1}), B(y^{-1})\}, \text{ since by the definition of '}\cap\text{'} \\ &= \min\{A(y^{-1}x^{-1}), B(y^{-1})\}, \text{ since by the definition 2.11} \\ &= \min\{A((xy)^{-1}), B(y)\} \\ &= \min\{A(xy), B(y)\} \end{aligned}$$

Hence the proof (3) $\Rightarrow$ (2) and hence the Theorem also.

3.2 Theorem: Let $A,B \in MF(G)$. Then A is a normal multi-fuzzy subgroup of B $\Leftrightarrow$ A_t is a normal subgroup of B_t, $\forall t \in \{ b / b = (b_1, b_2, \dots, b_i, \dots), b_i \in [0,1], \forall i$ such that $b \le A(e) \}$.

Proof: ($\Rightarrow$ part)

Suppose A is a normal multi-fuzzy subgroup of B.

Let $t \in \{ b / b = (b_1, b_2, \dots, b_i, \dots), b_i \in [0,1], \forall i$ such that $b \le A(e) \}$.

Then A_t is a subgroup of B_t.

Let $y \in A_t$ and $x \in B_t$.

Then $A(y) \ge t$ and $B(x) \ge t$ …………………(1)

By the hypothesis, $A(xyx^{-1}) \ge \min\{A(y), B(x)\}$

$\ge \min\{t, t\}$, since by (1)

$= t$

That is, $A(xyx^{-1}) \ge t$

Hence $xyx^{-1} \in A_t$

That is, A_t is a normal subgroup of B_t

Conversely, Suppose A_t is a normal subgroup of B_t, $\forall t \in \{ b / b = (b_1, b_2, \dots, b_i, \dots), b_i \in [0,1], \forall i$ such that $b \le A(e)\}$

Let $A(y) = t$; $B(x) = b$ and suppose that $b \ge t$ …………..(2)

Then this implies that $B(x) \ge t$

$\Rightarrow x \in B_t$

$\Rightarrow xyx^{-1} \in A_t$ since by the hypothesis.

Thus, $\Rightarrow A(xyx^{-1}) \ge t = \min\{t, b\}$, since by (2)

$\Rightarrow A(xyx^{-1}) \ge \min\{A(y), B(x)\}$, since by (2)

$\Rightarrow$ A is a normal multi-fuzzy subgroup of B, since by the definition 3.2 …………………………(I)

Suppose $b < t \Rightarrow b < A(y)$,since by (2)

$\Rightarrow A(y) > b$

$\Rightarrow y \in A_b$

$\Rightarrow xyx^{-1} \in A_b$, A_b is a normal subgroup of B_b, by the hypothesis.

$\Rightarrow A(xyx^{-1}) \ge b = \min\{b,t\}$

$\Rightarrow A(xyx^{-1}) \ge \min\{B(x), A(y)\}$, since by (2)

$\Rightarrow$ A is a normal multi-fuzzy subgroup of B ……………..(II) ,since by the definition 3.2.

Therefore, from I and II, we get the proof and hence the Theorem.

3.3 Theorem: Let $A,B \in MF(G)$ and A be a normal multi-fuzzy subgroup of B. Then A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*.

Proof: If $x \in A_*$, $y \in B_*$ and A be a normal multi-fuzzy subgroup of B, then this implies that $A(y^{-1}xy) \ge \min\{A(x), B(y)\}$

$\Rightarrow A(y^{-1}xy) = \min\{A(e), B(e)\}$

$\Rightarrow A(y^{-1}xy) = A(e)$

$\Rightarrow A_*$ is a normal subgroup of B_*

Similarly, If $x \in A^*$, $y \in B^*$ and A be a normal multi-fuzzy subgroup of B, then this implies that

$A(y^{-1}xy) \ge \min\{A(x), B(y)\}$

$\Rightarrow A(y^{-1}xy) > 0$, since $A(x), B(y) > 0$

$\Rightarrow A^*$ is a normal subgroup of B^* and hence the Theorem.

3.4 Theorem: If $A \in NMF(G)$ and $B \in MF(G)$, then $(A \cap B)$ is a normal multi-fuzzy subgroup of B.

Proof: Clearly, $A \cap B \in MF(G)$ and $A \cap B \subseteq B$.

Now, $\forall x,y \in G$, $(A \cap B)(xyx^{-1}) = \min\{ A(xyx^{-1}), B(xyx^{-1}) \}$

$= \min\{ A(y), B(xyx^{-1}) \}$, since $A \in NMF(G)$

$\ge \min\{ A(y), \min\{ B(x), B(y), B(x^{-1}) \} \}$

$= \min\{ A(y), \min\{ B(x), B(y) \} \}$

$= \min\{ \min\{A(y), B(y)\}, B(x) \}$

$= \min\{ (A \cap B)(y), B(x) \}$

This implies that $(A \cap B)$ is a normal multi-fuzzy subgroup of B, since by the definition of 3.2. and hence the Theorem.

3.5 Theorem: Let $A,B,C \in MF(G)$ be such that A and B are normal multi-fuzzy subgroups of C. Then $(A \cap B)$ is a normal multi-fuzzy subgroup of C.

Proof: Observe that $(A\cap B)\in MF(G)$ and $(A\cap B) \subseteq C$. Now,

$(A\cap B)(xyx^{-1}) = \min\{ A(xyx^{-1}), B(xyx^{-1}) \}$

$\geq \min\{\min\{A(y),C(x)\}, \min\{B(y),C(x)\}\}$,since A and B are normal multi-fuzzy subgroups of C.

$\geq \min\{ \min\{A(y), B(y)\}, C(x) \}$

$= \min\{ (A\cap B)(y), C(x) \}$

Therefore, $(A\cap B)$ is a normal multi-fuzzy subgroup of C ,since by the definition 3.2 and hence the Theorem.

IV. Normal multi-anti fuzzy subgroup of the multi-anti fuzzy subgroup

In this section, we introduce the concept of normal multi-anti fuzzy subgroup of a multi-anti fuzzy subgroup and examine its basic properties.

4.1 Definition A multi-anti fuzzy subgroup A of a group G is called normal if for each $x\in G$,

$$A(x) \geq \max_{g\in G} \{ A(gxg^{-1}) \}$$

4.2 Definition Let $A,B\in MAF(G)$ and $A\subseteq B$. Then A is called a normal multi-anti fuzzy subgroup of the multi-anti fuzzy subgroup B, written as $A \lhd B$, if $A(xyx^{-1}) \leq \max\{A(y), B(x)\}$, $\forall x,y\in G$.

Remarks: The following statements are immediate from the above definition :

1. Every multi-anti fuzzy subgroup is a normal multi-anti fuzzy subgroup of itself.
2. $A\in MAFP(G)$ is a normal multi-anti fuzzy subgroup of G $\Leftrightarrow$ A is a normal multi-anti fuzzy subgroup of the multi-anti fuzzy subgroup 1_G

Notations: The following notations to be used in this paper with the following meaning:

1. MAF(G) is the set of all multi-anti fuzzy subgroups of a group G
2. NMAF(G) is the set of all normal multi-anti fuzzy subgroups of a group G

4.3 Definition Let μ be any anti-fuzzy subgroup of a group G and $x\in G$. The fuzzy subsets $\mu(e)_{\{x\}}\circ\mu$ and $\mu\circ\mu(e)_{\{x\}}$ are referred to as the left fuzzy coset and right fuzzy coset of μ with respect to x and written as $x\mu$ and μx , respectively.

4.4 Definition Let A be a multi-anti fuzzy subgroup of a group G and $x\in G$. The multi-fuzzy sets $A(e)_{\{x\}}\circ A$ and $A\circ A(e)_{\{x\}}$ are referred to as the left multi-fuzzy coset and right multi-fuzzy coset of A with respect to x and written as xA and Ax respectively.

Remark: If $A\in NMAF(G)$ is a normal multi-anti fuzzy subgroup of a group G, then the left multi-fuzzy coset xA is just the right multi-fuzzy coset Ax. Thus, in this case, we call xA as a multi-fuzzy coset for short.

4.1 Theorem: Let $A,B\in MAF(G)$ and $A\subseteq B$. Then the following assertions are equivalent:

1. A is a normal multi-anti fuzzy subgroup of B
2. $A(yx) \leq \max\{A(xy), B(y)\}$, $\forall x,y\in G$
3. $A(e)_{\{x\}}\circ A \subseteq (A\circ A(e)_{\{x\}}) \cup B$, $\forall x\in G$.

Proof: (1) $\Rightarrow$ (2)

Since A is a normal multi-anti fuzzy subgroup of B,

$A(yx) = A(yxyy^{-1})$

$= A(y(xy)y^{-1})$

$\leq \max\{A(xy), B(y)\}$, $\forall x,y\in G$, since by the definition 4.2.

Proof: (2) $\Rightarrow$ (1)

$A(xyx^{-1}) = A(x(yx^{-1}))$

$\leq \max\{A((yx^{-1})x), B(x)\}$, since by the hypothesis(2).

$= \max\{A(y), B(x)\}$

That is, A is normal multi-anti fuzzy subgroup of B

That is, $A \lhd B$.

Proof: (2) $\Rightarrow$ (3) $\forall z\in G$,

$(A(e)_{\{x\}}\circ A)(z) = \max\{ \min\{A(e)_{\{x\}}(p), A(q)\} / pq = z$ where $p,q\in G\}$,since by the definition 2.11.

$= \max\{\min\{A(e)_{\{x\}}(x), A(q)\} / xq = z \}$, since take $p = x$

$= \max\{ \min\{A(e), A(x^{-1}z)\} \}$, since $A(xq) = A(z) \Rightarrow A(q) = A(x^{-1}z)$
$= \min\{ \max\{A(e), A(x^{-1}z)\} \}$
$= \min\{A(x^{-1}z)\}$
$= \min\{A((x^{-1}z)^{-1})\}$
$= \min\{A(z^{-1}x)\}$
$\leq \min\{ \max\{ A(xz^{-1}), B(z^{-1}) \} \}$, since by the hypothesis (2)
$= \min \{ \max\{A((xz^{-1})^{-1}), B(z^{-1})\} \}$
$= \min \{ \max\{A(zx^{-1}), B(z^{-1})\} \}$
$= \max \{ \min\{ A(zx^{-1}), B(z^{-1})\}\}$
$= \max\{(A\circ A(e)_{\{x\}})(z), B(z)\}$, since $(A\circ A(e)_{\{x\}})(z)=A(zx^{-1})$
$= [(A\circ A(e)_{\{x\}}) \cup B](z)$, since by the definition of '$\cup$'

Hence the proof (2) $\Rightarrow$ (3).

Proof: (3) $\Rightarrow$ (2)

$\forall x,y\in G$, $A(yx) = A((yx)^{-1})$
$= A(x^{-1}y^{-1})$
$= (A(e)_{\{x\}}\circ A)(y^{-1})$
$\leq ((A\circ A(e)_{\{x\}}) \cup B)(y^{-1})$, since by the hypothesis (3)
$= \max\{(A\circ A(e)_{\{x\}})(y^{-1}), B(y^{-1})\}$, since by the definition of '$\cup$'
$= \max\{A(y^{-1}x^{-1}), B(y^{-1})\}$, since by the definition 2.11
$= \max\{A((xy)^{-1}), B(y)\}$
$= \max\{A(xy), B(y)\}$

Hence the proof (3) $\Rightarrow$ (2) and hence the Theorem also.

4.2 Theorem: Let $A,B\in MAF(G)$. Then A is a normal multi-anti fuzzy subgroup of B $\Leftrightarrow$ A_t is a normal subgroup of B_t ,$\forall t\in\{b / b = (b_1 ,b_2 , \ldots ,b_i , \ldots), b_i\in[0,1], \forall i$ such that $b \geq A(e)$ }

Proof: ($\Rightarrow$ part)

Suppose A is a normal multi-anti fuzzy subgroup of B.

Let $t\in\{ b / b = (b_1 ,b_2 , \ldots ,b_i , \ldots), b_i\in[0,1], \forall i$ such that $b \geq A(e)$ }.

Then A_t is a subgroup of B_t .

Let $y\in A_t$ and $x\in B_t$

Then $A(y) \leq t$ and $B(x) \leq t$ …………………(1)

By the hypothesis, $A(xyx^{-1}) \leq \max\{A(y), B(x)\}$
$\leq \max\{t, t\}$, since by (1)
$= t$

That is, $A(xyx^{-1}) \leq t$

Hence $xyx^{-1}\in A_t$

That is, A_t is a normal subgroup of B_t

Conversely, Suppose A_t is a normal subgroup of B_t , $\forall t\in\{ b / b = (b_1 ,b_2 , \ldots, b_i , \ldots), b_i\in[0,1], \forall i$ such that $b\geq A(e)\}$

Let $A(y) = t$; $B(x) = b$ and suppose that $b \leq t$ …………..(2)

Then this implies that $B(x) \leq t$
$\Rightarrow x\in B_t$
$\Rightarrow xyx^{-1}\in A_t$ since by the hypothesis.

Thus, $\Rightarrow A(xyx^{-1}) \leq t = \max\{t, b\}$,since by (2)
$\Rightarrow A(xyx^{-1}) \leq \max\{A(y), B(x)\}$,since by (2)
$\Rightarrow$ A is a normal multi-anti fuzzy subgroup of B ,since by the definition 4.2 ………………………….(I)

Suppose $b > t$ $\Rightarrow b > A(y)$,since by (2)
$\Rightarrow A(y) < b$
$\Rightarrow y\in A_b$
$\Rightarrow xyx^{-1}\in A_b$,since A_b is a normal subgroup of B_b ,by the hypothesis.
$\Rightarrow A(xyx^{-1}) \leq b = \max\{b,t\}$
$\Rightarrow A(xyx^{-1}) \leq \max\{B(x), A(y)\}$,since by (2)
$\Rightarrow$ A is a normal multi-anti fuzzy subgroup of B ,since by the definition 4.2.……………..(II)

Therefore, from I and II, we get the proof and hence the Theorem .

4.3 Theorem: Let $A,B\in MAF(G)$ and A be a normal multi-anti fuzzy subgroup of B. Then A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^* .

Proof: If $x \in A_*$, $y \in B_*$ and A be a normal multi-anti fuzzy subgroup of B, then this implies that $A(y^{-1}xy) \le \max\{A(x), B(y)\}$

$\Rightarrow A(y^{-1}xy) = \max\{A(e), B(e)\}$

$\Rightarrow A(y^{-1}xy) = B(e)$

$\Rightarrow A_*$ is a normal subgroup of B_*

Similarly, If $x \in A^*$, $y \in B^*$ and A be a normal multi-anti fuzzy subgroup of B, then this implies that $A(y^{-1}xy) \le \max\{A(x), B(y)\}$

$\Rightarrow A(y^{-1}xy) > 0$,since $A(x), B(y) > 0$

$\Rightarrow A^*$ is a normal subgroup of B^* and hence the Theorem.

4.4 Theorem: If $A \in NMAF(G)$ and $B \in MAF(G)$, then $(A \cup B)$ is a normal multi-anti fuzzy subgroup of B.

Proof: Clearly, $A \cup B \in MAF(G)$ and $B \subseteq A \cup B$.

Now, $\forall x,y \in G$,

$$\begin{aligned}(A\cup B)(xyx^{-1}) &= \max\{A(xyx^{-1}), B(xyx^{-1})\}\\ &= \max\{A(y), B(xyx^{-1})\} \text{ ,since } A \in NMAF(G)\\ &\le \max\{A(y), \max\{B(x), B(y), B(x^{-1})\}\}\\ &= \max\{A(y), \max\{B(x), B(y)\}\}\\ &= \max\{\max\{A(y), B(y)\}, B(x)\}\\ &= \max\{(A\cup B)(y), B(x)\}\end{aligned}$$

This implies that $(A \cup B)$ is a normal multi-anti fuzzy subgroup of B ,since by the definition 4.2 and hence the Theorem.

4.5 Theorem: Let $A,B,C \in MAF(G)$ be such that A and B are normal multi-anti fuzzy subgroups of C. Then $(A \cup B)$ is a normal multi-anti fuzzy subgroup of C.

Proof: Observe that $(A \cup B) \in MAF(G)$ and $(A \cup B) \subseteq C$. Now,

$(A\cup B)(xyx^{-1}) = \max\{A(xyx^{-1}), B(xyx^{-1})\}$

$\le \max\{\max\{A(y),C(x)\}, \max\{B(y),C(x)\}\}$,since A and B are normal multi-anti fuzzy subgroups of C.

$\le \max\{\max\{A(y), B(y)\}, C(x)\}$

$= \max\{(A\cup B)(y), C(x)\}$

Therefore,$(A \cup B)$ is a normal multi-anti fuzzy subgroup of C ,since by the definition 4.2 and hence the Theorem.

V. Conclusion

In this paper we discussed normal multi-fuzzy subgroup of multi-fuzzy subgroup and normal multi-anti fuzzy subgroup of multi-anti fuzzy subgroup. Multi-fuzzy cosets are very useful for the theory of multi-fuzzy set, multi-fuzzy subgroup and multi-anti fuzzy subgroups.

REFERENCES

Journal Papers:

[1] Das. P.S., Fuzzy groups and level subgroups, Journal of Mathematical Analysis and Applications, 84 (1981), 264-269.
[2] Liu W.J., Fuzzy invariant subgroup and fuzzy ideal, Fuzzy Sets and Systems, 8(1981),133-139.
[3] Mukharjee N.P. and Bhattacharya P., Fuzzy normal subgroups and fuzzy cosets, Information Sciences, 34 (1984), 225-239.
[4] Muthuraj.R and Balamurugan.S, Multi-anti fuzzy group and its Lower level subgroups, Int. Journal of Engineering Research and Applications, Vol. 3, Issue 6, Nov-Dec 2013, pp.1498-1501.
[5] Muthuraj.R, Sitharselvam.P.M and Muthuraman.M.S, Anti Q-fuzzy group and its Lower level subgroups, International Journal of Computer Applications(0975-8887), Volume 3-No. 3,June2010,16-20.
[6] Muthuraj.R and Balamurugan.S, Multi-fuzzy group and its level subgroups, Gen. Math. Notes, Vol. 17,No. 1, July, 2013, pp. 74-81.
[7] Palaniappan.N, Muthuraj.R, Anti fuzzy group and its Lower level subgroups, Antartica J.Math., 1 (1)(2004) , 71-76.
[8] Rosenfeld.A, fuzzy groups, J. math. Anal.Appl. 35 (1971), 512-517.
[9] Sabu.S and Ramakrishnan.T.V, Multi-fuzzy sets, International Mathematical Forum, 50 (2010), 2471-2476.
[10] Sabu.S and Ramakrishnan.T.V, Multi-fuzzy subgroups, Int. J. Contemp. Math. Sciences,Vol.6, 8 (2011), 365-372.
[11] Zadeh.L.A, Fuzzy sets, Information and Control, 8 (1965), 338-353.