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A Note on Generalized Weighted Arithmetic Mean Summability Factors via Quasi-B-Power Increasing Sequence

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Abstract: *In this paper we have established a theorem on generalized summability factors via quasi-β-power increasing sequence, which gives some new results and generalizes some previous known results.*
Keywords: *Weighted arithmetic mean summability, summability, summability factors and quasi-β-power increasing sequence.*
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I. Introduction:

Let Σa_n be a given infinite series with partial sums $\{s_n\}$. We denote by u_n^α and t_n^α the *n*-th Cesaro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n) respectively such that

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v \tag{1.2}$$

where $A_n^\alpha = O(n^\alpha), \alpha > -1, A_0^\alpha = 1$ and $A_{-n}^\alpha = 0$ for $n > 0.$
A series Σa_n is said to be summable $|C,\alpha|_k, k \geq 1$. If (FLETT [7]).

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{|t_n^\alpha|^k}{n} < \infty \tag{1.3}$$

and Σa_n is said to be summable $|C,\alpha,\delta|_k, k \geq 1$ and $\delta \geq 0$ if (FLETT [7]).

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_n^\alpha|^k < \infty \tag{1.4}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \; as \; n \to \infty (P_{-i} = p_{-i} = 0, \; i \geq 1) \tag{1.5}$$

The sequence to sequence transformation

$$\sigma_n = \frac{1}{P_n} \Sigma p_v s_v \tag{1.6}$$

defines the sequence (σ_n) of the Weighted arithmetic mean or simply $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (HARDY [8]). The series Σa_n is said to be summable $|\bar{N}, p_n|_k, k \geq 1$ if (BOR [7])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty \tag{1.7}$$

and it is said to be summable $|\bar{N}, p_n, \delta|_k; k \geq 1$ and $\delta \geq 0$ if (BOR [3])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\Delta\sigma_{n-1}|^k < \infty \quad (1.8)$$

where $\Delta\sigma_{n-1} = \sigma_n - \sigma_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} p_{v-1} a_v ; n \geq 1.$

In the special case $P_n = 1, k = 1$ and $\delta = 0$ for all values of $n, |\bar{N}, p_n, \delta|_k$ summability is reduces to $|\bar{N}, p_n|$-summability.

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants *A* and *B* such that –

$$Ac_n \leq b_n \leq \beta c_n \quad (1.9)$$

A positive sequence (γ_n) is said to be quasi-β-power increasing sequence if there exist a constant $k = k(\beta, \gamma) \leq 1$ such that-

$$kn^{\beta}\gamma_n = m^{\beta}\gamma_m \quad (1.10)$$

Hold for all $n \geq m+1$. It should be noted the every almost increasing sequence is a quasi-β-power increasing sequence for any non-negative β and converse is not true.

II. Known Results:

BOR [2] has proved the following theorem for $|\bar{N}, p_n|_k$-summability factors.

Theorem 2.1: Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n \quad (2.1)$$

$$\beta_n \to 0 \text{ as } n \to \infty \quad (2.2)$$

$$\sum_{n=1}^{\infty} n |\Delta\beta_n| X_n < \infty \quad (2.3)$$

$$|\lambda_n| X_n = O(1) \quad (2.4)$$

If

$$\sum_{v=1}^{n} \frac{|t_v|^k}{v} = O(X_n) \text{ as } n \to \infty \quad (2.5)$$

where (t_n) is the *n*-th $(C,1)$ mean of the sequence (na_n) and (p_n) is the sequence such that

$$P_n = O(np_n) \quad (2.6)$$

$$P_n \Delta p_n = O(p_n p_{n+1}) \quad (2.7)$$

then the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k, k \geq 1.$

Later BOR [4] has generalized Theorem 2.1 for the $|\bar{N}, p_n, \delta|_k$ summability factors.

Theorem 2.2: Let (X_n) be a positive non-decreasing sequence and the sequence (β_n) and (λ_n) are such that the condition (2.1) – (2.7) of Theorem (2.1) are satisfied with condition (2.5) replaced by

$$\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|t_v|^k}{v} = O(X_n) \text{ as } n \to \infty. \quad (2.8)$$

If

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left(\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{1}{P_v}\right) \text{ as } m\to\infty. \qquad (2.9)$$

Then the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$ is summable $|\bar{N},p_n,\delta|_k$ $k\geq 1$ and $0\leq\delta\leq\frac{1}{k}$.

Recently BOR [5] has proved the following theorem.

Theorem 2.3: Let (X_n) be an almost increasing sequence. If the condition (2.1)-(2.4) and (2.6)-(2.9) are satisfied then the series $\sum_{n=1}^{\infty}a_n\frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N},p_n,\delta|_k, k\geq 1$ and $0\leq\delta<\frac{1}{k}$.

III. Main Results:

The aim of this paper is to prove the Theorem 2.3 under more weaker conditions for this we use the concepts of quasi-β-power increasing sequence. Now we shall prove the following theorem.

Theorem 3.1: Let (X_n) be quasi-β-power increasing sequence if the condition (2.1)-(2.4) and (2.6)-(2.9) are satisfied, then the series $\sum_{n=1}^{\infty}a_n\frac{P_nA_n}{np_n}$ is summable $|\bar{N},p_n,\delta|_k, k\geq 1$ and $0\leq\delta<\frac{1}{k}$.

IV. Lemma:

We need the following lemma for the proof of Theorem 3.1.

Lemma 4.1: (LEINDER [9]) Under the condition on $(X_n),(\lambda_n)$ and (β_n) where (X_n) is quasi-β-power increasing sequence as taken in the statement of the theorem the following condition holds

$$n\beta_nX_n=O(1) \text{ as } n\to\infty. \qquad (4.1)$$

and

$$\sum_{n=1}^{\infty}\beta_nX_n<\infty. \qquad (4.2)$$

Lemma 4.2: (BOR [2]) If condition (2.6) and (2.7) are satisfied then we have

$$\Delta\left(\frac{P_n}{n^2p_n}\right)=O\left(\frac{1}{n^2}\right) \qquad (4.3)$$

Lemma 4.3: (BOR [2]) If the condition (2.1)-(2.4) are satisfied, then we have

$$\lambda_n=O(1) \qquad (4.4)$$

$$\Delta\lambda_n=O\left(\frac{1}{n}\right) \qquad (4.5)$$

2.5 PROOF OF THE THEOREM:

Let (T_n) be the sequence of $(\bar{N},p_n)$ means of the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$. Then we have

$$T_n=\frac{1}{P_n}\sum_{v=1}^{n}p_v\sum_{r=1}^{v}\frac{a_rP_r\lambda_r}{rp_r}$$
$$=\frac{1}{P_n}\sum_{v=1}^{n}(P_v-P_{v-1})\frac{a_vP_v\lambda_v}{vp_v} \qquad (5.1)$$

then, for $n\geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v p_v}$$
$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta\left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{n} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v$$
$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_v \Delta\lambda_v (v+1) \frac{t_v}{v^2 p_v}$$
$$- \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_{v+1} (v+1) t_v \left(\frac{P_v}{v^2 p_v}\right) + \frac{\lambda_n t_n (n+1)}{n^2}$$
$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \quad \text{(say)}$$

To complete the proof of the theorem by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,r}|^k < \infty, \text{ for } r = 1, 2, 3, 4 \qquad (5.2)$$

Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,1}|^k$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v |t_v| |\lambda_v| \frac{1}{v} \right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n=1} \left(\frac{P_v}{p_v}\right)^k p_v |t_v|^k |\lambda_v|^k \frac{1}{v^k} \left[\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right]^{k-1}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v |t_v|^k |\lambda_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right) |\lambda_v|^{k-1} |\lambda_v| p_v |t_v|^k \frac{1}{v^k} \frac{1}{P_v} \left(\frac{P_v}{p_v}\right)^{\delta k}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} |\lambda_v| |t_v|^k \frac{1}{v^k} \left(\frac{P_v}{p_v}\right)^{\delta k}$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} v^{k-1} \frac{1}{v^k} |\lambda_v| |t_v|^k$$
$$= O(1) \sum_{v=1}^{m} |\lambda_v| \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|t_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{r}+O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)\frac{|t_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v|+O(1)|\lambda_m|X_m$$

$$= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m$$

$$= O(1) \text{ as } m\to\infty$$

Next

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}|\Delta\lambda_v||t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k|\Delta\lambda_v|^k|t_v|^k\ p_v\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v|t_v|^k|\Delta\lambda_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\lambda_v|^k\ p_v|t_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\lambda_v|^k|t_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^{k-1}|\Delta\lambda_v|^{k-1}|\Delta\lambda_v||t_v|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}v^{k-1}\frac{1}{v^{k-1}}|\Delta\lambda_v||t_v|^k$$

$$= O(1)\sum_{v=1}^{m}\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m}v\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_v}{p_r}\right)^{\delta k}\frac{|t_r|^k}{r}+O(1)m\beta_m\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=12}^{m-1}\beta_v X_v+O(1)m\beta_m X_m$$

$$= O(1) \text{ as } m \to \infty$$

Next,

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,3} |^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v | \lambda_{v+1} || t_v | \frac{1}{v} \left(\frac{v+1}{v} \right) \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v | \lambda_{v+1} | \frac{1}{v} | t_v | \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k p_v \frac{1}{v^k} | \lambda_{v+1} |^k | t_v |^k \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k p_v \frac{1}{v^k} | \lambda_{v+1} |^{k-1} | \lambda_{v+1} | | \mathrm{t}_v |^k \left\{ \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \right\}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{k-1} \frac{1}{v^k} | \lambda_{v+1} || t_v |^k \left(\frac{P_v}{p_v} \right)^{\delta k}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k} v^{k-1} \frac{1}{v^k} | \lambda_{v+1} || t_v |^k$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k} | \lambda_{v+1} | \frac{| t_v |^k}{v}$$

$$= O(1) \sum_{v=1}^{m-1} \Delta | \lambda_{v+1} | \sum_{r=1}^{v} \left(\frac{P_r}{p_r} \right)^{\delta k} \frac{| t_r |^k}{r} + O(1) | \lambda_{m+1} | \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k} \frac{| t_v |^k}{v}$$

$$= O(1) \sum_{v=1}^{m-1} | \Delta\lambda_{v+1} | X_v + O(1) | \lambda_{m+1} | X_m$$

$$= O(1) \sum_{v=1}^{m-1} | \Delta\lambda_{v+1} | X_v + O(1) | \lambda_{m+1} | X_m$$

$$= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) | \lambda_{m+1} | X_{m+1}$$

$$= O(1) \text{ as } m \to \infty$$

Finally

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,4} |^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\frac{n+1}{n} \right)^k \frac{1}{n^k} | \lambda_n |^k | t_n |^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} n^{k-1}\frac{1}{n^k}|\lambda_n|^{k-1}|t_n|^k|\lambda_n|$$

$$= O(1)\sum_{n=1}^{m}|\lambda_m|\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|t_n|^k}{n}$$

$$= O(1) \text{ as } m \to \infty.$$

Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,r}|^k = O(1) \text{ as } m\to\infty \text{ for } r = 1,2,3,4$$

This completes the proof of the theorem.

V. Corollary:

Our theorem has following results as a corollary.

Corollary 6.1: Taking $\delta = 0$ in theorem 4.1, we get Theorem 2.1 as a corollary. Since for $\delta = 0, |\bar{N}, p_n, \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ -summability.

VI. Conclusion:

Our theorem have the more general result rather than any previous known results. So our theorem enrich the literature of summability theory.

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