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Weakly quasi-conformally symmetries of generalized (k,μ) space forms

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Abstract: *In this paper we study symmetries and weak symmetries of generalized (k,μ) space forms with respect to Quasi-conformal curvature tensor. We establish relations regarding the associated 1-forms of weakly symmetric manifolds.*

Keywords: *generalized (k,μ) space forms, Quasi-conformal curvature tensor,Weakly Quasi-conformally-symmetric,Weakly Quasi-conformally ϕ - Ricci symmetric, η -Einstein manifolds.*

I. Introduction

A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \qquad (2)$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$
$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z$$
$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \qquad (3)$$

for any vector fields X,Y,Z on M. In [2] , the authors defined a generalized (k,μ) space form as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \quad (4)$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on M and R_1,R_2,R_3 are tensors defined above and

$$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y$$
$$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX$$
$$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,$$

for any vector fields $X,Y,Z,$ where $2h = L_\xi\phi$ and L is the usual Lie derivative.This manifold was denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

Natural examples of generalized (k,μ) space forms are (k,μ) space forms and generalized Sasakian space forms. The authors in[1] proved that contact metric generalized (k,μ) space forms are generalized (k,μ) spaces and if dimension is greater than or equal to 5, then they are (k,μ) spaces with constant $\phi-$sectional curvature $2f_6 - 1.$ They gave a method of constructing examples of generalized (k,μ) space forms and proved that generalized (k,μ) space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [3], it is proved that under D_a-homothetic deformation generalized (k,μ) space form structure is preserved for dimension 3, but not in general. Another interesting and important class of manifolds is a class of manifolds of constant curvature. As the generalization of this class of manifolds the notion of symmetric Riemannian manifolds was introduced. The notion of symmetric manifolds has been weakened by many authors in several ways such as pseudo-symmetric manifolds introduced by Chaki [4] and their generalization of weakly symmetric manifolds and weakly projectively symmetric Riemannian manifolds introduced by Tamassy and Binh [5]. In analogy the authors Jaiswal and Ojha [6] introduced weakly pseudo-projectively symmetric manifolds.The notion of the Quasi-conformal curvature tensor was given by Yano and Sawaki[7] and also studied by Amur and Maralabhavi[8]. Motivated by the above studies, in this paper we study symmetries and weak symmetries of generalized (k,μ) space forms with respect to Quasi-conformal curvature tensor.

II. Preliminaries

A (2n+1)-dimensional Riemannian manifold (M,g) is said to be an almost contact metric manifold if it admits a tensor field ϕ of type (1,1), a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0, \quad (5)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (6)$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \phi X) = 0, g(X, \xi) = \eta(X). \quad (7)$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$,

where $\Phi(X,Y) = g(X,\phi Y)$ is the fundamental 2-form of M.

It is well known that on a contact metric manifold (M,ϕ,ξ,η,g), the tensor h is defined by $2h = L_\xi \phi$ which is symmetric and satisfies the following relations.

$$h\xi = 0, h\phi = -\phi h, trh = 0, \eta \circ h = 0, \quad (8)$$

$$\nabla_X \xi = -\phi X - \phi h X, (\nabla_X \eta)Y = g(X + hX, \phi Y). \quad (9)$$

In a $(2n+1)$-dimensional (k,μ)-contact metric manifold, we have [9]

$$h^2 = (k-1)\phi^2, k \le 1, \quad (10)$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (11)$$

$$\begin{aligned}(\nabla_X h)(Y) = \; & [(1-k)g(X,\phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi h X) \\ & -\mu\eta(X)\phi h Y.\end{aligned} \quad (12)$$

Definition 1: *A contact metric manifold M is said to be*

(i) Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and S is the Ricci tensor,

(ii) η-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions on M.

In a $(2n+1)$-dimensional generalized (k,μ) space-form, the following relations hold.

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \quad (13)$$

$$\begin{aligned}QX = \; & [2nf_1 + 3f_2 - f_3]X + [(2n-1)f_4 - f_6]hX \\ & -[3f_2 + (2n-1)f_3]\eta(X)\xi,\end{aligned} \quad (14)$$

$$\begin{aligned}S(X,Y) = \; & [2nf_1 + 3f_2 - f_3]g(X,Y) + [(2n-1)f_4 - f_6]g(hX,Y) \\ & -[3f_2 + (2n-1)f_3]\eta(X)\eta(Y),\end{aligned} \quad (15)$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \quad (16)$$

$$r = 2n[(2n+1)f_1 + 3f_2 - 2f_3], \quad (17)$$

for any vector fields X,Y,Z where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M(f_1,...,f_6)$.

The relation between the associated functions $f_i, i=1,...,6$ of $M(f_1,...,f_6)$ was recently discussed by Carriazo et al. [2].

III. Quasi-conformal curvature tensor in Generalized (k,μ)-space forms

For a (2n+1) dimensional Riemannian manifold , the Quasi-conformal curvature tensor field $\tilde{C}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z = \ & aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ & -\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \quad (18)$$

where a and b are constants such that $a,b \neq 0$ and r is the scalar curvature.

Taking $Z=\xi$ in (18), then making use of (7), (13) and(16) we have

$$\begin{aligned}\tilde{C}(X,Y)\xi = \ & a[(f_1-f_3)(\eta(Y)X-\eta(X)Y)+(f_4-f_6)(\eta(Y)hX-\eta(X)hY)] \\ & +b[2n(f_1-f_3)(\eta(Y)X-\eta(X)Y)+QX\eta(Y)-QY\eta(X)] \\ & -\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)[\eta(Y)X-\eta(X)Y].\end{aligned} \quad (19)$$

Contracting the above with respect to ξ and by using (7), we deduce that

$$\eta(\tilde{C}(X,Y)\xi)=0. \quad (20)$$

Let $e_i, i=1,2,3,...2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then from (18), we have the following

$$\tilde{C}(Y,Z)=[a+(2n-1)b]S(Y,Z)+r\left[\frac{(1-2n)b-a}{2n+1}\right]g(Y,Z), \quad (21)$$

where

$$\tilde{C}(Y,Z)=\sum_{i=1}^{i=2n+1}\tilde{C}(e_iY,Z,e_i).$$

From (21), it follows that

$$Q_C Y=[a+(2n-1)b]QY+r\left[\frac{(1-2n)b-a}{2n+1}\right]Y, \quad (22)$$

where Q_C and Q, respectively are called Quasi-conformal Ricci and Ricci operators.

3.1 Weakly Quasi-conformally-symmetric Generalized (k,μ)-space forms

A space M is said to be weakly Quasi-conformally-symmetric if the Quasi-conformal curvature tensor $\tilde{C}$ of type (0,4) is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_X\tilde{C})(Y,Z,U,V) = \ & A(X)\tilde{C}(Y,Z,U,V)+B(Y)\tilde{C}(X,Z,U,V)+C(Z)\tilde{C}(Y,X,U,V) \\ & +D(U)\tilde{C}(Y,Z,X,V)+E(V)\tilde{C}(Y,Z,U,X),\end{aligned} \quad (23)$$

for all vector fields X,Y,Z,U,V on M. Such a manifold will be denoted by $(WQCS)_n$.

It is shown that in a $(WQCS)_n$ the associated 1-forms B=C and D=E, and hence the defining condition of $(WQCS)_n$ reduces to the following form,

$$\begin{aligned}(\nabla_X \tilde{C})(Y,Z,U,V) = \; & A(X)\tilde{C}(Y,Z,U,V)+B(Y)\tilde{C}(X,Z,U,V)+B(Z)\tilde{C}(Y,X,U,V) \\ & +D(U)\tilde{C}(Y,Z,X,V)+D(V)\tilde{C}(Y,Z,U,X),\end{aligned} \tag{24}$$

where A,B and D are one forms.

ie.,

$$\begin{aligned}(\nabla_X \tilde{C})(Y,Z)U = \; & A(X)\tilde{C}(Y,Z)U+B(Y)\tilde{C}(X,Z)U+B(Z)\tilde{C}(Y,X)U \\ & +D(U)\tilde{C}(Y,Z)X+g(\tilde{C}(Y,Z)U,X)\rho,\end{aligned} \tag{25}$$

where ρ is the vector field such that $g(X,\rho)=D(X)$.

Contracting with respect V and setting $Y=V=e_i$ in (24) and taking summation over i, $1\le i\le 2n+1$, we get

$$\begin{aligned}& [a+(2n-1)b](\nabla_X S)(Z,U)+dr(X)\left(\frac{(1-2n)b-a}{2n+1}\right)g(Z,U) \\ & = A(X)\left[(a+(2n-1)b)S(Z,U)+r\left(\frac{(1-2n)b-a}{2n+1}\right)g(Z,U)\right] \\ & +B(Z)\left[(a+(2n-1)b)S(X,U)+r\left(\frac{(1-2n)b-a}{2n+1}\right)g(X,U)\right] \\ & +D(U)\left[(a+(2n-1)b)S(Z,X)+r\left(\frac{(1-2n)b-a}{2n+1}\right)g(Z,X)\right] \\ & +B(\tilde{C}(X,Z)U)-g(\tilde{C}(U,X)Z)\rho.\end{aligned} \tag{26}$$

Taking $X=Z=U=\xi$ in (26) and then using (5),(16) and (18),we obtain

$$[A(\xi)+B(\xi)+D(\xi)]\left[(a+(2n-1)b)2n(f_1-f_3)+r\left(\frac{(1-2n)b-a}{2n+1}\right)\right]=0 \tag{27}$$

and hence

$$A(\xi)+B(\xi)+D(\xi)=0, \tag{28}$$

provided $r=\left(\frac{(a+(2n-1)b)2n(f_1-f_3)(2n+1)}{a-(1-2n)b}\right)=0$.

So we state the following

Theorem 1: *In a weakly Quasi-conformally-symmetric Generalized (k,μ)-space form is of constant curvature ,the associated 1-forms are related by the relation(28) provided* $r=\left(\frac{(a+(2n-1)b)2n(f_1-f_3)(2n+1)}{a-(1-2n)b}\right)=0$.

3.2 Weakly Quasi-conformally ϕ-symmetric Generalized (k,μ)-space forms

A space M is said to be weakly Quasi-conformally ϕ-symmetric if the Quasi-conformal curvature tensor $\tilde{C}$ satisfies

$$\begin{aligned}(\phi^2(\nabla_W \tilde{C})(X,Y)Z) = \; & A(W)\phi^2(\tilde{C}(X,Y)Z)+B(X)\phi^2(\tilde{C}(W,Y)Z)+B(Y)\phi^2(\tilde{C}(X,W)Z) \\ & +D(Z)\phi^2(\tilde{C}(X,Y)W)+g(\tilde{C}(X,Y)Z,W)\phi^2\rho,\end{aligned} \tag{29}$$

where ρ is the vector field associated to the 1-forms D such that $D(Z)=g(Z,\rho)$ and A,B,D are 1-forms(not simultaneously zero). If in particular A=B=D=0, then the manifold is said to be Quasi-conformally ϕ-symmetric.

Let M be a weakly Quasi-conformally ϕ-symmetric Generalized (k,μ)-space form.

By virtue of (5),it follows from (29) that

$$\begin{aligned} &-((\nabla_W\tilde{C})(X,Y)Z)+\eta((\nabla_W\tilde{C})(X,Y)Z)\xi \\ =\ & A(W)[-\tilde{C}(X,Y)Z+\eta(\tilde{C}(X,Y)Z)\xi]+B(X)[-\tilde{C}(W,Y)Z+\eta(\tilde{C}(W,Y)Z)\xi] \\ &+B(Y)[-\tilde{C}(X,W)Z+\eta(\tilde{C}(X,W)Z)\xi]+D(Z)[-\tilde{C}(X,Y)W+\eta(\tilde{C}(X,Y)W)\xi] \\ &+g(\tilde{C}(X,Y)Z,W)\phi^2\rho. \end{aligned} \tag{30}$$

Replacing Z by ξ in (30), we get

$$\begin{aligned} &-((\nabla_W\tilde{C})(X,Y)\xi)+\eta((\nabla_W\tilde{C})(X,Y)\xi)\xi \\ =\ & A(W)[-\tilde{C}(X,Y)\xi+\eta(\tilde{C}(X,Y)\xi)\xi]+B(X)[-\tilde{C}(W,Y)\xi+\eta(\tilde{C}(W,Y)\xi)\xi] \\ &+B(Y)[-\tilde{C}(X,W)\xi+\eta(\tilde{C}(X,W)\xi)\xi]+D(\xi)[-\tilde{C}(X,Y)W+\eta(\tilde{C}(X,Y)W)\xi] \\ &+g(\tilde{C}(X,Y)\xi,W)\phi^2\rho. \end{aligned} \tag{31}$$

Contracting (31) with respect to ξ, we have

$$D(\xi)g(\tilde{C}(X,Y)W,\xi)=g(\tilde{C}(X,Y)\xi,W)g(\phi^2\rho,\xi). \tag{32}$$

From (19) and (32) , we obtain

$$\begin{aligned} D(\xi)\tilde{C}(X,Y)W = & \Bigg[\left((a+2bn)(f_1-f_3)-\frac{r(a+4nb)}{2n(2n+1)}\right) \\ &+a(f_4-f_6)[g(hX,W)\eta(Y)-g(hY,W)\eta(X)] \\ &+b[S(X,W)\eta(Y)-S(Y,W)\eta(X)]\Bigg]\phi^2\rho. \end{aligned} \tag{33}$$

Thus we have

Theorem 2: *In a weakly Quasi-conformally ϕ-symmetric Generalized (k,μ)-space form, the Quasi-conformal curvature tensor is in the form (33).*

Contracting (33) with respect to U and setting $X=U=e_i$ and summing up with respect to i, by using (5) and (21) in (33), we have

$$\begin{aligned} &D(\xi)\left[(a+(2n-1)b)S(Y,W)+r\left(\frac{(1-2n)b-a}{2n+1}\right)g(Y,W)\right] \\ &=\left[(a+2bn)(f_1-f_3)-\frac{r(a+4nb)}{2n(2n+1)}\right][D(W)\eta(Y)-\eta(Y)\eta(W)\eta(\rho)] \\ &+b[S(W,\rho)\eta(Y)-2n(f_1-f_3)\eta(Y)\eta(W)\eta(\rho)]. \end{aligned} \tag{34}$$

Changing Y by ϕY and W by ϕW and using (5) in (34), we have

$$D(\xi)[a+(2n-1)b]S(\phi Y,\phi W)=r\left(\frac{a+(2n-1)b}{2n+1}\right)g(\phi Y,\phi W). \tag{35}$$

The above equation yields

$$S(\phi Y,\phi W)=r\left(\frac{a+(2n-1)b}{2n+1}\right)g(\phi Y,\phi W), \tag{36}$$

provided $D(\xi)[a+(2n-1)b]\neq 0$.

Again changing Y by ϕY and W by ϕW in (36) and using (5) and (7), we obtain

$$S(Y,W)=r\left(\frac{a+(2n-1)b}{2n+1}\right)g(Y,W)+[2n(f_1-f_3)-1]r\left(\frac{a+(2n-1)b}{2n+1}\right). \tag{37}$$

Setting $Y=W=e_i$ in (37) and taking summation over $i=1,2,...,2n+1$, we get

$$r = \frac{2n(f_1 - f_3)(2n+1)}{[(2n+1)-(a+(2n-1)b)2n]}. \qquad (38)$$

In view of (37) and (38), we have

$$\begin{aligned} S(Y,W) = {} & \left(\frac{2n(f_1-f_3)(a+(2n-1)b)}{[(2n+1)-(a+(2n-1)b)2n]}\right)g(Y,W) \\ & + \left(\frac{2n(f_1-f_3)(2n(f_1-f_3)-1)(a+(2n-1)b)}{[(2n+1)-(a+(2n-1)b)2n]}\right)\eta(Y)\eta(W). \end{aligned} \qquad (39)$$

We state the following

Theorem 3: *A weakly Quasi-conformally ϕ-symmetric Generalized (k,μ)-space form is an η-Einstein manifold provided* $D(\xi)[a+(2n-1)b] \neq 0$.

3.3 Weakly Quasi-conformally ϕ- Ricci symmetric Generalized (k,μ)-space forms

A Generalized (k,μ)-space form M is said to be weakly Quasi-conformally ϕ- Ricci symmetric if the Quasi-conformal- Ricci operator Q_C satisfies

$$\phi^2((\nabla_X Q_C)Y) = A(X)\phi^2(Q_C Y) + B(Y)\phi^2(Q_C X) + \tilde{C}(Y,X)\phi^2\rho, \qquad (40)$$

where $\tilde{C}(X,Y) = g(Q_C X, Y)$.

By virtue of (5),(40) takes the form

$$\begin{aligned} -(\nabla_X Q_C)Y + \eta((\nabla_X Q_C)Y)\xi = {} & A(X)[-Q_C Y + \eta(Q_C Y)\xi] \\ & + B(Y)[-Q_C X + \eta(Q_C X)\xi] + \tilde{C}(Y,X)[-\rho + \eta(\rho)\xi], \end{aligned} \qquad (41)$$

from which it follows that,

$$\begin{aligned} & -g(\nabla_X Q_C Y, Z) + \tilde{C}(\nabla_X Y), Z) + \eta((\nabla_X Q_C)Y)\eta(Z) \\ & = A(X)[-\tilde{C}(Y,Z) + \eta(Q_C Y)\eta(Z)] + B(Y)[-\tilde{C}(X,Z) \\ & + \eta(Q_C X)\eta(Z)] + \tilde{C}(Y,X)[-D(Z) + \eta(\rho)\eta(Z)]. \end{aligned} \qquad (42)$$

Taking $Y = \xi$ in (42) and using (5),(9),(14),(21) and (22) we have

$$\begin{aligned} & (a+(2n-1)b)\left[2n(f_1-f_3)(g(\phi X,Z)+g(\phi hX,Z)) - S(\phi X,Z) - S(\phi hX,Z)\right] \\ & = B(\xi)\Bigg[-(a+(2n-1)b)S(X,Z) - r\left(\frac{(1-2n)b-a}{2n+1}\right)g(X,Z) \\ & + \left((a+(2n-1)b)2n(f_1-f_3) + r\left(\frac{(1-2n)b-a}{2n+1}\right)\right)\eta(X)\eta(Z)\Bigg] \\ & + \left[(a+(2n-1)b)2n(f_1-f_3) + r\left(\frac{(1-2n)b-a}{2n+1}\right)\right]\eta(X) \\ & [-D(Z) + \eta(\rho)\eta(Z)]. \end{aligned} \qquad (43)$$

Replace X by ϕX in (43) and using (5), (10) and (15) we have

$$S(X,Z) = M_1 g(X,Z) + M_2\eta(X)\eta(Z) + M_3 g(hX,Z) + M_4 g(\phi X,Z) + M_5 g(\phi X,hZ), \qquad (44)$$

where

$$\begin{aligned} & M_1 = 2n(f_1 - f_3) - M_2,\ M_2 = (k-1)[(2n-1)f_4 - f_6] \\ & M_3 = 3f_2 + (2n-1)f_3,\ M_4 = B(\xi)\left[\frac{r}{2n+1} - (2nf_1 + 3f_2 - f_3)\right] \\ & M_5 = B(\xi)[f_6 - (2n-1)f_4]. \end{aligned}$$

Replace X by ϕX and Z by hZ in (44) and use (15) to obtain

$$\begin{aligned} S(X,Z) = & \left[M_1 + \frac{M_5{}^2(k-1)}{M_2+M_3}\right] g(X,Z) + \left[M_2 - \frac{M_5{}^2(k-1)}{M_2+M_3}\right]\eta(X)\eta(Z) \\ & + \left[M_4 + \frac{M_5[(1-k)M_3 + (2n-1) f_4 - f_6]}{M_2+M_3}\right] g(\phi X, Z) \\ & + \left[M_3 - \frac{M_4 M_5}{M_2+M_3}\right] g(hX,Z). \end{aligned} \tag{45}$$

Again replace X by hX in (45) and use (15) to get

$$S(X,Z) = M_6 g(X,Z) + M_7 \eta(X)\eta(Z) + M_8 g(\phi X, Z), \tag{46}$$

where

$$\begin{aligned} M_6 = \; & M_1 + \frac{M_5{}^2(k-1)}{M_2+M_3} + \left(\frac{M_3}{M_{10}} - \frac{M_4 M_5}{(M_2+M_3)M_{10}}\right) \\ & \left[M_2 + (1-k)\left(M_3 + \frac{(M_9 - M_4)M_5}{M_2+M_3}\right)\right] \\ M_7 = \; & M_2 - \frac{M_5{}^2(k-1)}{M_2+M_3} + \left(\frac{M_3}{M_{10}} - \frac{M_4 M_5}{(M_2+M_3)M_{10}}\right) \\ & \left[(k-1)\left(M_3 + \frac{(M_9 - M_4)M_5}{M_2+M_3}\right) - M_2\right] \\ M_8 = \; & M_9 - \left(\frac{M_3}{M_{10}} - \frac{M_4 M_5}{(M_2+M_3)M_{10}}\right) \\ & \left(\frac{M_9((1-k)M_3 + (2n-1) f_4 - f_6)}{M_2+M_3}\right) \\ M_9 = \; & M_4 + \left(\frac{M_5((1-k)M_3 + (2n-1) f_4 - f_6)}{M_2+M_3}\right) \\ M_{10} = \; & \frac{(M_2+M_3)^2 - M_5{}^2(k-1) - M_4 M_9}{M_2+M_3}. \end{aligned}$$

Now replace X by ϕX in (46) and use (15) to get

$$S(X,Z) = \mu_1 g(X,Z) + \mu_2 \eta(X)\eta(Z), \tag{47}$$

where

$$\mu_1 = M_6 + \frac{M_8}{L_2}\left[\frac{(-M_2)M_5}{M_2+M_3} - M_8 + \left(\frac{((2n-1)f_4 - f_6)M_4}{(M_2+M_3)M_{10}}\right)\left(M_2 + (1-k)\left(M_3 + \frac{(M_9-M_4)M_5}{M_2+M_3}\right)\right)\right]$$

$$\mu_2 = M_7 + \frac{M_8}{L_2}\left[\frac{M_2M_5}{M_2+M_3} + M_8 + \left(\frac{((2n-1)f_4 - f_6)M_4}{M_2+M_3}\right)\left((k-1)\left(\frac{(M_9-M_4)M_5}{M_2+M_3}\right) - M_2\right)\right]$$

$$L_1 = 2nf_1 + 3f_2 - f_3 - M_6 + \left(\frac{(2n-1)f_4 - f_6}{M_2+M_3}\right)[(1-k)M_3 + (2n-1)f_4 - f_6]$$

$$L_2 = L_1 + \left(\frac{((2n-1)f_4 - f_6)M_4M_9}{(M_2+M_3)^2M_{10}}\right)[(1-k)M_3 + (2n-1)f_4 - f_6].$$

Thus we can state that

Theorem 4: *A weakly Quasi-conformally ϕ - Ricci symmetric Generalized (k,μ) -space form is an η - Einstein manifold.*

3.4 ϕ -Quasi-conformally flat Generalized (k,μ) -space form

An Generalized (k,μ) -space form M is said to be ϕ -Quasi-conformally flat if

$$\phi^2(\tilde{C}(\phi X,\phi Y)\phi Z) = 0 \quad (48)$$

holds for all vector fields X,Y,Z on M. We now prove the following theorem.

Theorem 5 : *A ϕ -Quasi-conformally flat Generalized (k,μ) -space form M is an η* -Einstein manifold.

Proof: Suppose that M is a ϕ -Quasi-conformally flat Generalized (k,μ) -space form. Then it is easy to see that $\phi^2(\tilde{C}(\phi X,\phi Y)\phi Z) = 0$ holds if and only if $g(\tilde{C}(\phi X,\phi Y)\phi Z,\phi W) = 0$ for any vector fields X,Y,Z on M. From (18), we have

$$\begin{aligned} -aR(\phi X,\phi Y,\phi Z,\phi W) &= b\big[S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W) \\ &+ g(\phi Y,\phi Z)g(Q\phi X,\phi W) - g(\phi X,\phi Z)g(Q\phi Y,\phi W)\big] \\ &- \left(\frac{r(a+4nb)}{2n(2n+1)}\right)[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \quad (49)$$

Let $e_1,....,e_{2n},\xi$ be an orthonormal basis of the vector field on M. Then $\phi e_1,....,\phi e_{2n},\xi$ is also local orthonormal basis. We have the following

$$\sum_{i=1}^{i=2n} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) = S(\phi Y,\phi Z) - [(f_1 - f_3)g(\phi Y,\phi Z) + (f_4 - f_6)g(h\phi Y,\phi Z)], \quad (50)$$

$$\sum_{i=1}^{i=2n} S(e_i,e_i) = \sum_{i=1}^{i=2n} S(\phi e_i,\phi e_i) = r - 2n(f_1 - f_3), \quad (51)$$

$$\sum_{i=1}^{i=2n} g(\phi e_i,\phi Z)S(\phi Y,\phi e_i) = S(\phi Y,\phi Z), \quad (52)$$

$$\sum_{i=1}^{i=2n} g(e_i,e_i) = \sum_{i=1}^{i=2n} g(\phi e_i,\phi e_i) = 2n, \qquad (53)$$

$$\sum_{i=1}^{i=2n} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = g(\phi Y,\phi Z). \quad (54)$$

If we put $X = W = e_i$ in (49) and sum up with respect to i , then from (50)-(54) and (15), we have

$$\begin{aligned} -[a+(2n-2)b]S(\phi Y,\phi Z) &= a(f_4 - f_6)g(hY,Z) \\ &+\left[a(f_3 - f_1)+b(r-2n(f_1 - f_3))-\frac{r(a+4nb)(2n-1)}{2n(2n+1)}\right]g(\phi Y,\phi Z). \end{aligned} \qquad (55)$$

Replace Y by ϕY and Z by ϕZ in (55) and use (5), (17) to get

$$S(Y,Z) = A_1 g(Y,Z) + A_2\eta(Y)\eta(Z) + A_3 g(hY,Z), \qquad (56)$$

where

$$\begin{aligned} A_1 &= \frac{a(f_1 - f_3)-b(4n^2 f_1+6nf_2-2nf_3)}{a+(2n-2)b}+\frac{(a+4nb)(2n-1)[(2n+1)f_1+3f_2-2f_3]}{(2n+1)(a+(2n-2)b)} \\ A_2 &= \frac{a(f_3 - f_1)+b(4n^2 f_1+6nf_2-2nf_3)}{a+(2n-2)b}-\frac{(a+4nb)(2n-1)[(2n+1)f_1+3f_2-2f_3]}{(2n+1)(a+(2n-2)b)} \\ A_3 &= \frac{a(f_4 - f_6)}{a+(2n-2)b}. \end{aligned}$$

Taking $Y = hY$ in (56) and using (15), we obtain

$$S(Y,Z) = \mu_3 g(Y,Z) + \mu_4\eta(Y)\eta(Z), \quad (57)$$

where

$$\begin{aligned} \mu_3 &= A_1 + A_3B_1,\ B_1 = \frac{[A_1+(k-1)((2n-1)f_4 - f_6)](a+(2n-2)b)}{(2nf_1+3f_2 - f_3)-a(f_4 - f_6)} \\ \mu_4 &= A_2 + A_3B_2,\ B_2 = \frac{[A_2+(1-k)((2n-1)f_4 - f_6)]}{(2nf_1+3f_2 - f_3)-a(f_4 - f_6)}. \end{aligned}$$

Hence the proof.

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