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System of Variational-like Inequalities

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Abstract: *In this paper we consider variational-like inequality problem over product of sets, which is equivalent to the system of variational-like inequalities. New concept of η-relative monotonicity is introduced for solving variational-like inequality problem over product of sets. As an application of our results, we prove the existence of a coincidence point of two families of nonlinear operators.*
Keywords: *coincidence point, monotonicity, operator, system of variational-like inequalities, solution.*

I. Introduction

Variational inequalities proved to be a very useful tool for investigation and solutions of various equilibrium-type problems arising in Operations Research, Economics, Mathematical Physics and other fields, see for example, [1,2, 3,4,5,6,7,8, 9].It is well known that most of such problems arising in game theory, transportation and network economics have a decomposable structure, i.e. , they can be formulated asvariational inequalities over Cartesian product sets; see e.g. Nagurney [8] and Ferris and Pang [4]. In 1980, Aubin [2] has pointed out that the Nash equilibrium problem [10] for differentiable functions can be formulated in the form of a variational inequality problem defined over the product of sets (for short, VIPPS). Further Pang [9] showed that not only Nash equilibrium problem but also various equilibrium-type problems, like, traffic equilibrium, spatial equilibrium, and general equilibrium programming problems from Operations Research, Economics, Game theory, Mathematical Physics, and other areas, can also be uniformly modeled as a VIPPS. Later, it is found that VIPPS is equivalent to the problem of system of variational inequalities (for short, SVI),see for example, [11] and references therein. In 1999, Ansari and Yao [1] used a fixed point theorem for a family of multivalued maps to prove the existence of a solution of SVI. Since then several authors, see for instance, [1, 11, 12], studied the existence theory of various classes of systems of variational-like inequalities by exploiting fixed point theorems and maximal element theorems for a family of multivalued maps. In the recent past, system of variational-like inequalities emerged as tools to prove the existence of a solution of Nash equilibrium problem [10] for differentiable and non-differentiable functions, respectively. See for example, [1, 12] and references therein.

Inspired by the work of Luc [13], we introduce the concept of η-relatively quasimonotonicity and η-densely relatively pseudomonotonicity which are much weaker than the relatively pseudomonotonicity considered by Konnov [11]. We also define the η-relatively B-pseudomonotonicity and η-relatively demimonotonicity which extend in a natural way the well knownpseudomonotonicity in the sense of Brezis [14] (see also [15]). We establish some existence results for a solution of variational-like inequality problem over product of sets under these monotonicities. As an application of our results, we derive the existence of a coincidence point of two families of nonlinear operators.

II. Formulations And Preliminaries

Let I be a finite index set, that is, $I = \{1,2,\dots,n\}$. For each $i \in I$, let X_i be a Hausdroff topological vector space with its dual X_i^*, K_i a nonempty convex subset of X_i , $K = \prod_{i\in I} K_i$, $X = \prod_{i\in I} X_i$, and $X^* = \prod_{i\in I} X_i^*$. For each $i \in I$, when X_i is a normed space, its norm is denoted by $\|.\|_i$ and the product norm on Xwill be denoted by $\|.\|$. We denote by $\langle .,. \rangle$ the pairing between X_i^* and X_i . For each $x \in X$, we write $x = (x_i)_{i\in I}$, where $x_i \in X_i$, that is, for each $x \in X$, $x_i \in X_i$ denotes the ith component of x. For each $i \in I$, let $f_i : K \to X_i^*$ be a nonlinear operator and $\eta_i : K_i \times K_i \to X_i$ be a bifunction.We consider the following problem of system of variational-like inequalities (SVLI) which is the model of various equilibrium-type problems from operations research, economics, game theory, mathematical physics and other areas, see for example [1, 4, 8, 9] and references therein:

(SVLI) Find $\bar{x} \in K$ such that for each $i \in I, \langle f_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \geq 0$ for all $y_i \in K_i$

It is easy to that (SVLI) is equivalent to the following variational-like inequality problem overproduct of sets (VLIPPS):

(VLIPPS) Find $\bar{x} \in K$ such that $\sum_{i\in I}\langle f_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \geq 0$ for all $y_i \in K_i$, $i \in I(1)$

If $\eta_i(y_i, \bar{x}_i) = y_i - \bar{x}_i$, then (SVLI) and (VLIPPS) reduce to the following problems of (SVI) and (VIPPS) respectively, introduced and studied by Ansari and Zubair [16]:

(SVI) Find $\bar{x} \in K$ such that for each $i \in I, \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0$ for all $y_i \in K_i$
(VIPPS) Find $\bar{x} \in K$ such that $\sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0$ for all $y_i \in K_i$, $i \in I$

Of course, if we define the mapping $f : K \to X^*$ and $\eta : K \times K \to X$ respectively by:
$f(x) = \left(f_i(x)\right)_{i \in I}$ and $\eta(y, x) = \left(\eta_i(y_i, x_i)\right)_{i \in I}$ (2)
then (VLIPPS) can be equivalently rewritten as the usual variational-like inequality problem of finding $\bar{x} \in K$ such that:
$\langle f(\bar{x}), \eta(y, \bar{x}) \rangle \geq 0$ for all $y \in K$ (3)

Konnov [11] introduced the concept of relatively pseudomonotonicity and strongly relatively pseudomonotonicity to prove some existence results for a solution of (VIPPS) in the setting of Banach spaces. Konnov [17] also studied combined relaxation method for solving (VIPPS). He essentially exploited the decomposable structure of (3) and simplified their implementation. He also noted that the method cannot be extended directly due to its two-level structure and a binding condition in its line search procedure.

For every nonempty set A, we denote by 2^A (respectively, $\boldsymbol{F}(A)$) the family of all subsets (respectively, finite subsets) of A. If A is a nonempty subset of a vector space, then coA denotes the convex hull of A.

We shall use the following particular form of Fan-KKM lemma (see [18])

Theorem 2.1 Let K be a compact and convex subset of a Hausdroff topological vector space X and $K^0 \subseteq K$ be nonempty. Assume that $G: K^0 \to 2^K \setminus \{\emptyset\}$ be a multivalued satisfying the following conditions:

(i) For each $x \in K^0$, $G(x)$ is closed;
(ii) For every finite set $\{x^1, \ldots, x^m\}$ of K^0 one has $co\{x^1, \ldots, x^m\} \subseteq \bigcup_{k=1}^{m} G(x^k)$
Then $\bigcap_{x \in K^0} G(x) \neq \emptyset$.

Now we mention the following generalization of Browder fixed point theorem.

Theorem 2.2 Let K be a nonempty and convex subset of a topological vector space (not necessarily Hausdroff) X and $T : K \to 2^K$ a multivalued map. Assume that the following conditions hold:

(i) For all $x \in K$, $T(x)$ is convex
(ii) For each $A \in \boldsymbol{F}(K)$ and for all $y \in coA$, $T^{-1}(y) \cap coA$ is open in coA.
(iii) For each $A \in \boldsymbol{F}(K)$ and all $x, y \in coA$ and every net $\{x^\alpha\}_{\alpha \in \Gamma}$ in K converging to x such that $ty + (1-t)x \notin T(x_\alpha)$ for all $\alpha \in \Gamma$ and for all $t \in [0,1]$, we have $y \notin T(x)$.
(iv) There exists a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that $\tilde{y} \in T(x)$ for all $x \in K \setminus D$.
(v) For all $x \in D$, $T(x)$ is nonempty.

Then there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

III. Existence Results

Definition 3.1 Let $\eta_i : K_i \times K_i \to X_i$ be a bifunction. The map $f : K \to X^*$ defined by (2) is said to be:

(i) *η-relative pseudomonotone at* $y \in K$ [11] if for all $x \in K$ we have:

$$\sum_{i \in I} \langle f_i(x), \eta_i(y_i, x_i) \rangle \geq 0 \Rightarrow \sum_{i \in I} \langle f_i(y), \eta_i(y_i, x_i) \rangle \geq 0$$

and *η-relative strictly pseudomonotone at* $y \in K$ if the second inequality is strict for all $x \neq y$;

(ii) *η-relative quasimonotone at* $y \in K$ if for all $x \in K$ we have:

$$\sum_{i \in I} \langle f_i(x), \eta_i(y_i, x_i) \rangle > 0 \Rightarrow \sum_{i \in I} \langle f_i(y), \eta_i(y_i, x_i) \rangle \geq 0$$

If f is η-relative pseudomonotone (respectively, η-relative strictly pseudomonotone and η-relative quasimonotone) at each $y \in K$, then we say that it is η-relative pseudomonotone (respectively, η-relative strictly pseudomonotone and η-relative quasimonotone) on K.

Definition 3.2 The map $f : K \to X^*$ defined by (2) is said to be *η-hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \mapsto \langle f\left(x + \lambda(y - x)\right), \eta(y, x) \rangle$ is continuous.

Lemma 3.1 Let f , defined by (2), be η-hemicontinuous and η-relative quasimonotone on K. Then for every $x, y \in K$ with $\sum_{i \in I} \langle f_i(x), \eta_i(y_i, x_i) \rangle \geq 0$, we have either:

$\sum_{i \in I} \langle f_i(y), \eta_i(y_i, x_i) \rangle \geq 0$ or $\sum_{i \in I} \langle f_i(x), \eta_i(z_i, x_i) \rangle \leq 0$ for all $z_i \in K_i$, $i \in I$.

Proof It is sufficient to show that if for all $z_i \in K_i, i \in I$:

$$\sum_{i \in I} \langle f_i(x), \eta_i(z_i, x_i) \rangle > 0$$

then we have:

$$\sum_{i \in I} \langle f_i(y), \eta_i(y_i, x_i) \rangle \geq 0$$

Let us set $y^t = tz + (1-t)y$ for $0 < t \leq 1$. Then obviously, $y^t \in K$ and:

$$\sum_{i\in I}\langle f_i(x),\eta_i(y_i^t,x_i)\rangle > 0$$

By η-relative quasimonotonicity of f, we get:

$$\sum_{i\in I}\langle f_i(y^t),\eta_i(y_i^t,x_i)\rangle \geq 0$$

Now let $t\to 0$. Since $y^t\to y$ along a line segment, and by η-hemicontinuity of f, we have:

$$\sum_{i\in I}\langle f_i(y),\eta_i(y_i,x_i)\rangle \geq 0$$

This completes the proof.

Definition 3.3 [13] A subset K^0 of K is said to be *segment-dense* in K if for all $x\in K$, there can be found $x^0\in K^0$ such that x is a cluster point of the set $[x,x^0]\cap K^0$, where $[x,x^0]$ denotes the line segment joining x and x^0 including end points .

Definition3.4 [13] For each $i\in I$ let K_i be a nonempty convex subset of X_i. For each $i\in I$, we set:

$K_i^{\perp}:=\{\xi_i\in X_i^*:\ \langle\xi_i,\eta_i(y_i,x_i)\rangle=0\}$for all $x_i,y_i\in K_i$

and call it the *η-orthogonal complement* of K_i. Then:

$K^{\perp}:=\prod_{i\in I}K_i^{\perp}=\prod_{i\in I}\{\xi_i\in X_i^*:\ \langle\xi_i,\eta_i(y_i,x_i)\rangle=0\}$for all $x_i,y_i\in K_i$

$=\{\xi:=(\xi_i)_{i\in I}\in X^*: for each\, i\in I, \langle\xi_i,\eta_i(y_i,x_i)\rangle=0\,\}$for all $x_i,y_i\in K_i$

Remark 3.1 For a given $\xi_i\in X_i^*$ the following two statements are equivalent:

(a) For each $i\in I$, $\langle\xi_i,\eta_i(y_i,x_i)\rangle=0$ for all $x_i,y_i\in K_i$;

(b) $\sum_{i\in I}\langle\xi_i,\eta_i(y_i,x_i)\rangle=0$for all $x_i,y_i\in K_i$, $i\in I$.

Indeed, (a) implies (b) is obvious. For (b) implies (a), let $y_j=x_j$for $j\neq i$, in (b) then we obtain (a).

In view of above remark, we have:

$K^{\perp}=\{\xi:=(\xi_i)_{i\in I}\in X^*: \sum_{i\in I}\langle\xi_i,\eta_i(y_i,x_i)\rangle=0\}$for all $x_i,y_i\in K_i$, $i\in I$.

and we call it the *η-orthogonal complement* of K.

Definition 3.5 Let f be a map from $K$to X^*, defined by (2). We say that $x^0\in K$ is a positive point of f on K if for all $x\in K$ one has either $f(x)\in K^{\perp}$, that is, for each $i\in I$, $f_i(x)\in K_i^{\perp}$ or there exists $y\in K$ such that:

$$\sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i^0)\rangle > 0$$

The set of all positive points of f on K is denoted by K_f.

We denote by $f(K)$ the image of Kunder f, that is, $f(K)=\{f(x):x\in K\}$.

Proposition 3.1 Let f, defined by (2), be η-hemicontinuous and η-relative quasimonotone on K such that $f(K)\cap K^{\perp}=\emptyset$, that is, for each $i\in I$, $f_i(K)\cap K_i^{\perp}=\emptyset$. Then f is η-relative pseudomonotone at every positive point.

Proof Let $y\in K_f$ and $x\in K$ be any point such that$\sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle\geq 0$. Then by Lemma 3.1, we have either:

$\sum_{i\in I}\langle f_i(y),\eta_i(y_i,x_i)\rangle\geq 0$or $\sum_{i\in I}\langle f_i(x),\eta_i(z_i,x_i)\rangle\leq 0$ for all $z_i\in K_i$, $i\in I$(4)

To complete the proof, it is sufficient to show that the second inequality in (4) is impossible. Indeed, since $y\in K_f$ and for each $i\in I$, $f_i(x)\notin K_i^{\perp}$, then there exists $z\in K$ such that:

$$\sum_{i\in I}\langle f_i(x),\eta_i(z_i,y_i)\rangle > 0$$

then:

$$\sum_{i\in I}\langle f_i(x),\eta_i(z_i,x_i)\rangle=\sum_{i\in I}\langle f_i(x),\eta_i(z_i,y_i)\rangle+\sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle > 0$$

which shows that the second inequality in (4) is impossible, and the proof is completed.

Proposition 3.2 Let K be closed and convex subset of X and K^0 a segment-dense subset of K. If f, defined by (2) is η-relativequasimonotone at every point of K^0 and η-hemicontinuous on K, then it is η-relativequasimonotone on K.

Proof Let $x,y\in K$ with:

$\sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle > 0$ (5)

Since K^0 is a segment-dense subset of K, we can find $y^0\in K^0$ and $y^m\in[y,y^0]\cap K^0$ for all $m\in\mathbb{N}$ such that $\lim y^m=y$. Then from (5), we obtain:

$$\sum_{i\in I}\langle f_i(x),\eta_i(y_i^m,x_i)\rangle > 0 \text{ for all } m\in\mathbb{N}.$$

Since $f$is η-relativequasimonotone at y^m , we get:

$$\sum_{i\in I}\langle f_i(y^m),\eta_i(y_i^m,x_i)\rangle \geq 0$$

Since $\lim y^m = y$ and by η-hemicontinuity of f, we have

$$\sum_{i\in I}\langle f_i(y), \eta_i(y_i, x_i)\rangle \geq 0$$

Hence $f$is η-relativequasimonotone on K.

Now we are ready to define a new concept of η-densely relative pseudomonotonicity, which generalize the notion of densely pseudomonotonicity considered by Luc [13].

Definition 3.6 The map $f : K \to X^*$ defined by (2) is said to be *η-densely relativepseudomonotone (respectively, η-densely relative strictly pseudomonotone) on K* if there exists a segment-dense subset $K^0 \subseteq K$ such that f is η-relativepseudomonotone (respectively, η-relative strictly pseudomonotone) on K^0.

Next, we define the η-relative B-pseudomonotonicity and η-relativedemimonotonicity which extend the definition of an η-pseudomonotone map, introduced by Brezis [14].

Definition 3.7 The map $f : K \to X^*$ defined by (2) is said to be *η-relative B-pseudomonotone* (respectively, *η-relativedemimonotone*) if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha\in\Gamma}$ in K converging to x(respectively, weakly to x) with:

$$\liminf_\alpha \left[\sum_{i\in I}\langle f_i(x^\alpha), \eta_i(x_i, x_i^\alpha)\rangle\right] \geq 0$$

we have:

$$\sum_{i\in I}\langle f_i(x), \eta_i(y_i, x_i)\rangle \geq \limsup_\alpha \left[\sum_{i\in I}\langle f_i(x^\alpha), \eta_i(y_i, x_i^\alpha)\rangle\right]$$

for all $y \in K$.

The following Lemma can be treated as a generalization of Minty lemma (see for example, [19] Chapter 3, Lemma 1.5) to (VLIPPS).

Lemma 3.2 Let K be a nonempty convex subset of X and K^0 be the same as in the definition of η-densely relativepseudomonotone map. Iff, defined by (2), is η-hemicontinuous and η-densely relativepseudomonotone, then the following problem is equivalent to (VLIPPS):

$$(\text{MVLIPPS})^0 \begin{cases} \text{find } \bar{x} \in K \text{ such that} \\ \sum_{i\in I}\langle f_i(y), \eta_i(y_i, \bar{x}_i)\rangle \geq 0 \end{cases}$$

for all $y_i \in K_i^0, i \in I$. The solution sets of (VLIPPS) and $(\text{MVLIPPS})^0$are denoted by $K_s$and K_{sm}^0 , respectively.

Proof By the η-densely relativepseudomonotonicityof f, we have $K_s \subseteq K_{sm}^0$.

Conversely, let $\bar{x} \in K$ be a solution of $(\text{MVLIPPS})^0$. Then:

$\sum_{i\in I}\langle f_i(y), \eta_i(y_i, \bar{x}_i)\rangle \geq 0$for all $y_i \in K_i^0, i \in I$ (6)

Since K^0 is segment-dense, for all $z \in K$, we can find $z^0 \in K^0$ and $z^m \in [z, z^0] \cap K^0$ for all $m \in \mathbb{N}$ such that $\lim z^m = z$. Then from (6), we get:

$$\sum_{i\in I}\langle f_i(z^m), \eta_i(z_i^m, \bar{x}_i)\rangle \geq 0 \text{ for all } m \in \mathbb{N}.$$

Since $\lim z^m = z$ and f is η-hemicontinuous, we obtain:

$$\sum_{i\in I}\langle f_i(z), \eta_i(z_i, \bar{x}_i)\rangle \geq 0 \text{for all } z_i \in K_i, i \in I.$$

Again by η-hemicontinuity of f(see the proof of lemma 2 in [11]), we have:

$$\sum_{i\in I}\langle f_i(\bar{x}), \eta_i(z_i, \bar{x}_i)\rangle \geq 0 \text{for all } z_i \in K_i, i \in I.$$

Hence $\bar{x} \in K_s$ and thus $K_s = K_{sm}^0$.

Theorem 3.1 For each $i \in I$, let K_i be a nonempty , compact and convex subset of X_i, and f, defined by (2), be η-hemicontinuous andη-densely relativepseudomonotone on K. Then (VLIPPS) has a solution and hence (SVLI) has a solution.

Proof Let K^0 be the same as in the definition of a η-densely relativepseudomonotone map. For each $y \in K^0$, define two multivalued maps $S, T : K^0 \to 2^K$ by:

$$S(y) = \left\{x \in K : \sum_{i\in I}\langle f_i(x), \eta_i(y_i, x_i)\rangle \geq 0 \right\}$$

and:

$$T(y) = \left\{x \in K : \sum_{i\in I}\langle f_i(y), \eta_i(y_i, x_i)\rangle \geq 0 \right\}$$

then for each $y \in K^0$, $T(y)$ is closed, and also by η-relativepseudomonotonicity of f on K^0, we have $S(y) \subseteq T(y)$. By using the standard argument, it is easy to see that for every finite set $\{y^1, \ldots, y^m\}$of K^0, one has $co\{y^1, \ldots, y^m\} \subseteq \bigcup_{k=1}^m S(y^k)$(see for example, the proof of Theorem1 in [11]). Since for all $\in K^0$, $S(y) \subseteq T(y)$, we also have $co\{y^1, \ldots, y^m\} \subseteq \bigcup_{k=1}^m T(y^k)$. By applying Theorem 2.1, we have $\bigcap_{y\in K^0} T(y) \neq \emptyset$, that is, there exists $\bar{x} \in K$ such that:

$$\sum_{i\in I}\langle f_i(y),\eta_i(y_i,x_i)\rangle \geq 0$$

for all $y_i \in K_i^0, i \in I$. By lemma 3.1, $\bar{x} \in K$is a solution of (VLIPPS).

Corollary 3.1 For each$i \in I$, let $K_i$be a nonempty, compact and convex subset ofX_i, and$f$, defined by (2), be η-hemicontinuous and η-relativequasimonotone on$K$such that$K_f$is segment-dense in$K$ .Then (VLIPPS) has a solution and hence (SVLI) has a solution.

ProofLet $\bar{x} \in K$ such that $f(\bar{x}) \in K^{\perp}$, then $\sum_{i\in I}\langle f_i(x),\eta_i(y_i,\bar{x}_i)\rangle = 0$ for all $y_i \in K_i, i \in I$. Hence$\bar{x} \in K$ is a solution of (VLIPPS). Therefore, we may assume that $f(K) \cap K^{\perp} = \emptyset$. Then by proposition 3.1, f is η-relatively pseudomonotone at every point of K_f . Since K_f is segment-dense in K, f is η-densely relativepseudomonotone on K. Thus by Theorem 3.1, (VLIPPS) has a solution.

Corollary 3.2 For each$\in I$, let$K_i$be a nonempty , compact and convex subset ofX_i, and$f$, defined by (2) beη-hemicontinuous andη-densely relative strictly pseudomonotone on$K$. Then (VLIPPS) has a solution $\bar{x} \in K$and it is unique if$\bar{x} \in K^0$. Further,$\bar{x} \in K$is a solution of (SVLI), and it is unique if$\bar{x} \in K^0$, where $K^0$is the same as in the definition of η-densely relativepseudomonotone map.

ProofIn view of Theorem 3.1, it is sufficient to show that (VLIPPS) has at most one solution. Assume to the contrary that $x', x'' \in K^0$ are two solutions of (VLIPPS) such that $x' \neq x''$. Then:

$$\sum_{i\in I}\langle f_i(x'),\eta_i(x_i'',x')\rangle \geq 0$$

By η-densely relative strictly pseudomonotonicity of $f$onK^0. We have:

$\sum_{i\in I}\langle f_i(x''),\eta_i(x_i'',x')\rangle > 0$i.e.$\sum_{i\in I}\langle f_i(x''),\eta_i(x_i',x'')\rangle < 0$

Thus x'' is not a solution of (VLIPPS), which is a contradiction of our assumption. This completes the proof.

Theorem 3.2 For each$i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdroff) X_i. Let f, defined by (2), be η-relative B-pseudomonotone such that for each $A \in \boldsymbol{F}(K)$, $x \mapsto \sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle$ is upper semicontinuous on coA. Assume that there exists a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K\backslash D$, $\sum_{i\in I}\langle f_i(x),\eta_i(\tilde{y}_i,x_i)\rangle < 0$. Then (VLIPPS) has a solution and hence (SVLI) has a solution.

Proof For each $x \in K$, define a multivalued map $T : K \to 2^K$ by:

$$T(x) = \left\{y \in K\colon \sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle < 0\right\}$$

then for all $x \in K, T(x)$is convex. Let $\in \boldsymbol{F}(K)$, then for all $y \in coA$,

$$[T^{-1}(y)]^c \cap coA = \left\{x \in coA\colon \sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle \geq 0\right\}$$

is closed in coA by upper semicontinuity of the map $x \mapsto \sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle$ on coA. Hence $[T^{-1}(y)] \cap coA$ is open in coA.

Suppose that $x, y \in coA$ and $\{x^\alpha\}_{\alpha\in\Gamma}$ is a net in Kconverging to $x$such that:

$\sum_{i\in I}\langle f_i(x^\alpha),\eta_i(ty_i + (1-t)x_i, x_i^\alpha)\rangle \geq 0$for all $\alpha \in \Gamma$ and $t \in [0,1]$.

For $t = 0$, we have:

$\sum_{i\in I}\langle f_i(x^\alpha),\eta_i(x_i,x_i^\alpha)\rangle \geq 0$for all $\alpha \in \Gamma$,

and therefore:

$$liminf_\alpha\left[\sum_{i\in I}\langle f_i(x^\alpha),\eta_i(x_i,x_i^\alpha)\rangle\right] \geq 0$$

By the η-relative B-pseudomonotonicity of f, we have:

$\sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle \geq limsup_\alpha\,[\sum_{i\in I}\langle f_i(x^\alpha),\eta_i(y_i,x_i^\alpha)\rangle]$ (7)

For $t = 1$, we have:

$\sum_{i\in I}\langle f_i(x^\alpha),\eta_i(y_i,x_i^\alpha)\rangle \geq 0$for all $\alpha \in \Gamma$

and therefore:

$\liminf_{\alpha\in\Gamma}[\sum_{i\in I}\langle f_i(x^\alpha),\eta_i(y_i,x_i^\alpha)\rangle] \geq 0$ (8)

From (7) and (8), we obtain:

$$\sum_{i\in I}\langle f_i(x),\eta_i(y_i,x_i)\rangle \geq 0$$

and thus $y \notin T(x)$.

Assume that for all $x \in D$, $T(x)$ is nonempty. Then all the conditions of Theorem 2.2 are satisfied. Hence there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$, that is,

$$0 = \sum_{i\in I} \langle f_i(\hat{x}), \eta_i(\hat{x}_i, \hat{x}_i)\rangle < 0$$

a contradiction. Thus there exists $\bar{x} \in K$ such that $T(\bar{x}) = \emptyset$, that is:

$$\sum_{i\in I}\langle f_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \geq 0 \text{for all } \ y_i \in K_i\ ,\ i \in I..$$

Hence $\bar{x}$ is a solution of (VLIPPS).

Corollary 3.3 For each $i \in I$, let K_i be a nonempty, closed and convex subset of a reflexive Banach spaceX_iLet$f$, defined by (2), be η-relativedemimonotone such that for each $A \in \boldsymbol{F}(K)$, $x \longmapsto \sum_{i\in I}\langle f_i(x), \eta_i(y_i, x_i)\rangle$ is upper semicontinuous on coA. Assume that there exists $\tilde{y} \in K$ such that $\lim_{\|x\|\to\infty, x\in K} \sum_{i\in I}\langle f_i(x), \eta_i(\tilde{y}_i, x_i)\rangle < 0$ (9)

Then (VLIPPS) has a solution and hence (SVLI) has a solution.

Proof Let $\alpha = \lim_{\|x\|\to\infty, x\in K} \sum_{i\in I}\langle f_i(x), \eta_i(\tilde{y}_i, x_i)\rangle < 0$ .Then by (9), $\alpha < 0$. Let $r > 0$be such that $\|\tilde{y}\| \leq r$and$\sum_{i\in I}\langle f_i(x), \eta_i(\tilde{y}_i, x_i)\rangle < \frac{\alpha}{2}$for all $x \in K$with$\|x\| > r$. For each $i \in I$, let$B_i^r = \{x_i \in K_i : \|x_i\|_i \leq r\}$, and we denote by $B^r = \prod_{i\in I} B_i^r$. Then B^r is a nonempty and weakly compact subset of K. Note that for any $x \in K \backslash B^r$,$\sum_{i\in I}\langle f_i(x), \eta_i(\tilde{y}_i, x_i)\rangle < \frac{\alpha}{2} < 0$, and the conclusion follows from Theorem 3.2

IV. Coincidence Theorem

As an application of corollary 3.3, we establish some existence results for a coincidence point of two families of nonlinear operators.

Theorem 4.1 For each $i \in I$, let X_i be a real reflexive Banach space. Let $f, g : X \to X_i^*$be defined as $f(x) = \left(f_i(x)\right)_{i\in I}$ and $g(x) = \left(g_i(x)\right)_{i\in I}$, respectively, for all $x \in X$, , where for each $i \in I$, $g_i : X \to X_i^*$, is a nonlinear operator. Assume that$(f - g)$ is η-relativedemimonotone and for each $A \in \boldsymbol{F}(X)$, , $x \longmapsto \sum_{i\in I}\langle f_i(x), \eta_i(y_i, x_i)\rangle$ is upper semicontinuous on coA. Further, assume that there exists $\tilde{y} \in X$ such that:

$$\lim_{\|x\|\to\infty, x\in X} \sum_{i\in I} \langle (f_i - g_i)\ \ (x), \eta_i(\tilde{y}_i, x_i)\rangle < 0$$

Then there exists $\bar{x} \in X$ such that $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$.

Proof From the Corollary 3.3, there exists $\bar{x} \in X$such that for each $i \in I$,

$$\langle f_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \geq \langle g_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle$$

for all$y_i \in X_i$. Therefore we have, $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$.

Finally, we give another application of Corollary 3.3 in the setting of Hilbert spaces.

Theorem4.2 For each $i \in I$, let $(X_i, \langle .,. \rangle)$ be a real Hilbert space and K_i a nonempty, closed and convex subset of X_i. . Let f, defined by (2), be η-relative demimonotone such that for each $A \in \boldsymbol{F}(K)$,$x \longmapsto \sum_{i\in I}\langle f_i(x), \eta_i(y_i, x_i)\rangle$ is lower semicontinuous on coA. Assume that there exists $\tilde{y} \in K$ such that:

$$\lim_{\|x\|\to\infty, x\in K} \sum_{i\in I} \langle x_i - f_i(x), \eta_i(\tilde{y}_i, x_i)\rangle < 0$$

Then there exists $\bar{x} \in K$ such that $f_i(\bar{x}) = \bar{x}_i$ for each $i \in I$..

Proof For each $i \in I$, define a nonlinear operator $S_i : K \to X_i$ by $S_i(x) = x_i - f_i(x)$ for all $x \in K$. Then obviously, for each $i \in I, S_i$satisfies all the conditions of Corollary 3.3. Hence there exists $\bar{x} \in K$ such that for each $i \in I, \langle S_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \geq 0$for all $y_i \in K_i$. . For each $i \in I$, , let $y_i = f_i(\bar{x})$, we have $\|\bar{x}_i - f_i(\bar{x})\| \leq 0$. Therefore for each $i \in I$, $f_i(\bar{x}) = \bar{x}_i$.

V. Conclusion

By using a particular form of Fan-KKM Lemma and generalization of Browder Fixed Point Theorem, we have proved some existence results for a solution of (VLIPPS) in the setting of Hausdroff topological vector spaces.

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