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Generalized Metrizable Spaces, the D-Property And Mappings

a] P.Rajendran b] R.Ragavendran c] Dr.K.Vijayalakshmi
[a]Saveetha School of Engineering, Saveetha University, Chennai, India [b] Saveetha School of Engineering, Saveetha University, Chennai, India [c]Saveetha School of Engineering, Saveetha University, Chennai, India

Abstract: *The basic properties of D-spaces are discussed and a generalized left separated space is introduced with D-space. Generalized metric spaces that are D-spaces and the behavior of the D-property with respect to the mappings is discussed in this paper.*

I. Introduction

E.K.Van Douwen and W.F.Pfeffer has introduced the concepts of D-spaces in 1979. In this paper we show that spaces with additional structure such as a base property or generalized metric property are D-spaces.

Definition 1.1.[1]A neighbourhood assignment for a topological space (X,τ) is a function $N: X \to \tau$such that $x \in N(x)$ for each $x \in X$. $X$is said to be a D-space if for every neighbourhood assignment N, there is a closed discrete subset D of X such that $N(x)\backslash x \in D$ covers X.

Definition 1.2.[2]We say that the positive cardinality of a family γ of sets in Xdoes not exceed a cardinal number τ , if for each x the cardinality of the collection of all members of γcontaining x, does not exceed τ. If the pointwise cardinality of a family does not exceed $\aleph_0$, the family is said to be point countable.

Theorem 1.3. Every space X with a point-countable base$B$ is a D −space.

Proof: Let φ be any arbitrary neighbourhood assignment on . Since B is a base for X, for each $x \in X$we can fix $\psi(x) \in B$such that $x \in \psi(x) \subset \varphi(x)$. Then $\psi$is also a neighbourhood assignment on X. The family $\psi(x)$is point countable, since $\psi(x)$is contained in B. By lemma(2.3), there exists a locally finite subset$A$ of Xsuch that $\psi(A)$covers X. Hence $\varphi(A)$also covers X. Therefore$X$ is a D −space.

Recall that for topological spaces $X, Y f : X \to Y$is said to be an$s$ −mapping if fibers of $f$are seperable.

Corollary1.4. Open continuous s −image of a metrizable space is a D −space.

Proof: Since open continuous s −image of a metrizable space is a space with a point countable base from theorem(2.4) we have the statement proved.

Corollary1.5. Metrizable spaces are D −spaces.

Proof: The Nagata-Smirnov metrization theorem states that "A space $X$is a metrizable if and only if $X$is regular and has a countably locally finite basis". Any countably locally finite basis is point countable. Hence by theorem (2.4) $X$is a D −space.\

Definition 1.6.[2] Let X be a topological space. A sequence of open covers $\{G_n\}_{n<\omega}$ is a development for X if G_{i+1} is a refinement for G_i and if x is a point in X , U is an open set in X containing x, then there is a $k < \omega$such that $st(x, G_k) \subset U$ where $st(x, G_k) = \cup\{G \in G_k / x \in G\}$. A topological space X with a development is called a developable space.

Definition 1.7.[4] A Moore space is a regular developable space.

Definition 1.8.[4] A family neighbourhoodsη of a point $x \in X$ is said to be a neighbourhood base at x if for each neighbourhood $U$of x, there is a Vin η such that $V \subset U$.

Definition 1.9.[5] Let X be a topological space. A distance function $\rho : X \times X \to [0,\infty)$ is said to be a semimetric if whenever $x, y \in X$

1. $\rho(x,y) \geq 0$
2. $\rho(x,y) = 0$ if and only if $x = y$
3. $\rho(x,y) = \rho(y,x)$.

If ρ is a semimertic on X such that $\{S_\epsilon(x)/\epsilon > 0\}$, where $S_\epsilon(x) = \{y \in X/\rho(x,y) < \epsilon\}$ is a neighbourhood base at each $x \in X$ then X is called a semimetrizable space.

Theorem 1.10. Every developable space is semimetrizable.

Proof:X be a developable space with a development $\{U_n\}_{n<\omega}$. Then define $\rho$by $\rho(x,y) = inf\{1/n/y \in st(x, U_n)\}$. By definition of $st(x, U_n)$, if $y \in st(x, U_n)$, then $x \in st(y, U_n)$. Hence $\rho(x,y) = \rho(y,x)$. Also $\rho(x,y) \geq 0$ whenever $x \neq y. \rho(x,x) = inf\{1/n/x \in st(x, U_n)\} \to 0$ as$x \in U_n$ for each$n$. Soρ is a semimetric on .For any$x \in X, s_\epsilon(x) = \{y \in X/\rho(x,y) < \epsilon\} = \{y \in X/inf\{1/n/y \in st(x, U_n)\} < \epsilon\}$. i.e.,$s_\epsilon(x)$consists precisely those$y \in st(x, U_n)$with$\{inf\{1/n/y \in st(x, U_n)\} < \epsilon\}$. Hence for any neighbourhood $U$of$x$ there exists$\epsilon > 0$, such that$s_\epsilon(x) \subset U$. Hence$\{s_\epsilon(x)/\epsilon > 0\}$ is a neighbourhood base at each$x \in X$. Hence $X$is a semimetrizable space.

Definition 1.11.[5] A topological space$X$ is said to be semistttratifiable if for each open set$U \subset X$ one can assign a countable sequence$\{U_n\}_{n<\omega}$ of closed subsets of$X$ such that

1. $\bigcup_{n<\omega} U_n = U$.
2. $U_n \subset V_n$ whenever $U \subset V$, where $\{V_n\}_{n<\omega}$ is the sequence assigned to V.

Equivalently another definition of a semistratifiable space can be stated as follows: A topological space X is said to be semistratifiable if for each closed set $H \subset X$ One can assign a countable sequence $\{U(n,H\}_{n<\omega}$ of open subsets of X such that

1. $\bigcap_{n<\omega} U(n,H) = H$
2. If K is closed with $H \subseteq K$ then $U(n,H) \subseteq U(n,K)$ for all $n < \omega$.

Theorem 1.12. Every semimetrizable space is semistratifiable.

Proof. Let X be a semimetrizable space with semimetric ρ. Let H be a closed subset of $X$and $U(n,H) = \bigcup\{S_{1/2^n}(x)^\circ/x \in H\}$. Then$\{U(n,H)\}_{n<\omega}$ is a countable sequence of open sets in X.For any$y \in H, y \in S_{1/2^n}(y)^\circ$ for each$n < \omega$. Hence$y \in U(n,H)$ for each$n < \omega$.Hence$y \in \bigcap\{U(n,H)/n < \omega\}$.Therefore$H \subseteq \bigcap\{U(n,H)/n < \omega\}$.Similarly$\bigcap\{U(n,H)/n < \omega\} \subseteq H$.Also if K is any closed set with$\subseteq K$, then$U(n,H) \subseteq U(n,K)$ for all$n < \omega$. Hence $X$is a semistratifiable space.

Theorem 1.13. Semistratifiable spaces are D-spaces.

Proof. Let(X,τ) be a semistratifiable space. To each$W \in \tau$ we can assign a sequence$\{F(W,n)\}_{n<\omega}$ of closed subsets of$X$ such that$W = \bigcup_{n<\omega} F(W,n)$ and$F(W,n) \subset F(V,n)$whenever$\subset V$. Let$U: X \to \tau$ be a neighbourhood assignment in X with range space $\{U_x/x \in X\}$. Let $U_n = \{U_x/x \in F(U_x, n\}$and $X_n = \{x \in X/U_x \in U_n\}$. Note that $X = \bigcup_{j<\omega} X_j$. Let$j_0$ be the smallest element$< \omega$ such thatU_{j_0} and$X_{j_0} \neq \emptyset$. By transfinite induction pick$U_{x_\alpha} \in U_{j_0}, \alpha < \gamma_0$ such that

1. $\alpha < \beta < \gamma_0$implies $x_\beta \notin U_{x_\alpha}$.
2. $X_{j_0} \subset \bigcup_{\alpha<\gamma_0} U_{x_\alpha}$.

Let $D_0 = \{x_\alpha/\alpha < \gamma_0\}$. $D_0$is a closed discrete subset of X. Pick $z \in \overline{D_0}$ note that$z \in \bigcup_{\alpha<\gamma_0} U_{x_\alpha}$because$D_0 \subset F(\bigcup_{\alpha<\gamma_0} U_{x_\alpha}, j_0) \subset \bigcup_{\alpha<\gamma_0} U_{x_\alpha}$. Let$\alpha_0$ be the smallest element$<\gamma_0$ such that$z \in U_{x_{\alpha_0}}$. Then$V = U_{x_{\alpha_0}} \setminus F(\bigcup_{\alpha<\gamma_0} U_{x_\alpha}, j_0)$ is a neighbourhood of$z$ such that$V \bigcap D_0 = \{x_{\alpha_0}\}$. This shows that$D_0$ is closed and discrete. Let$X_0' = \bigcup_{\alpha<\gamma_0} U_{x_\alpha}$ andj_1 be the smallest element$< \omega$such that$X_{j_1} \setminus X_0' \neq \emptyset$. Then let$U_{j_1}' = \{U_x \in U_{j_1}/x \in X_{j_1} \setminus X_0'\}$. Again by transfinite induction, pick$U_{x_\alpha} \in U_{j_1}', \alpha < \gamma_1$ such that

1. $\alpha < \beta < \gamma_1$implies $x_\beta \notin U_{x_\alpha}$
2. $X_{j_1} \setminus X_0' \subset \bigcup_{\alpha<\gamma_1} U_{x_\alpha}$

Again we get $D_1 = \{x_\alpha/\alpha < \gamma_1\}$ is a closed discrete subset of X and letting $X_1' = \bigcup_{\alpha<\gamma_1} U_{x_\alpha}$ we get that $X_{j_1} \subset X_0' \bigcup X_1'$. Hence by ordinary induction we can find integers $j < 1, j < 2, \ldots$ and closed discrete subsets $D_i \subset X_{j_i}, i < \omega$ such that

1. Each $X_{j_i} \subset \bigcup\{U_x/x \in \bigcup_{k=0}^{'} D_k\}$.
2. Each $D_{i+1} \bigcap (\bigcup\{U_x/x \in \bigcup_{k=0}^{'} D_k\}) = \emptyset$.

Now letting $D = \bigcup_{i<\omega} D_i$. we see that$\{U_x/x \in D\}$ covers X due to (1). To show that D is closed discrete subset of . Pick$z \in X$ and let n be the smallest integer such that z is in some$U_{x_\alpha}$with$x_\alpha \in D_n$. Then$V = (U_{x_\alpha}/\bigcup_{k=0}^{n} D_k) \bigcup \{x_\alpha\}$ is a neighbourhood of$z$ such that$V \bigcap D = \{x_\alpha\}$ which shows that$D$ is closed and discrete. Since$D$ is closed and discrete and$\{U_x/x \in D\}$covers$X$,X is aD-space.

Definition 1.14.[5] Suppose X is a topological and $d: X \times X \to [0,\infty)$ such that for all $(x,y) \in X \times X$,

- $d(x,y) = d(y,x)$and
- $d(x,y) = 0$ if and only if $x = y$.

The function d is said to be symmetric for X provided for all non-empty $A \subseteq X$, A is closed in X if and only if $\inf\{d(x,z)/z \in A\} > 0$ for every $x \in X \setminus A$ and X is said to be symmetrizable with symmetric d.

Notation 1.15. For $x \in X$and $n \in N$, let$B(x, 1/n) = \{z \in X \;/\; d(x,z) < 1/n\}$.

Claim 1.$D \subseteq X$.

Claim 2.$W = \{U(x)/x \in D\}$covers X.

Proof. For any $y \in X$ find $m \in N$ such that $\in I(m)$. If $y \in J(m)$then $\in U(y) \subseteq \bigcup W$. If $y \notin J(m)$ then

$$y \in (\bigcup\{U(x)/i < m, x \in J(i)\}) \bigcup (\bigcup\{U(x)/x \in J(m), x < y\}) \subseteq \bigcup W$$

Claim 3.$D$is a closed discrete set in $\bigcup W$ and hence in X.

Proof: This follows if we show that for any$t \in D, D\backslash\{t\}$ is closed. To this end we may assume$Z = \bigcup W$ and let$x \in Z \backslash (D\backslash\{t\})$. It suffices to find a weak neighbourhood$V$of$x$ such that$V \bigcap (D\backslash\{t\}) = \emptyset$. Let$m$ be the first element of$\mathbb{N}$ such that there is a first element$y$ of $J(m)$where$x \in U(y)$. Now for all$z \in D$ with$y < z$ we have,$z \notin U(y)$. Also for all$z \in D$ with$z < y$ we have, $B(z, 1/m) \subseteq B(x, 1/k_2) \subseteq U(z)$ and$x \notin U(z)$, so $z \notin B(x, 1/m)$. That is, $U(y) \bigcap B(x, 1/m)$ is a weak neighbourhood of $x$with $U(y) \bigcap B(x, 1/m) \bigcap (D\backslash\{y\}) = \emptyset$. If $t = y$ claim 3 stands proved.If $\neq y$, then since $y \in D$ we know $x \neq y$ and there is $j \in \mathbb{N}$ such that $y \notin B(x, 1/j)$. This gives $V = B(x, 1/j) \bigcap U(y) \bigcap B(x, 1/m)$ as the weak neighbourhood of x with $V \bigcap (D\backslash\{t\}) = \emptyset$. Hence D is closed and discrete in . Therefore X is a D-space.

Corollory 1.16. The quotient compact image of a metrizable space is a D-space.
Proof: Since the quotient compact image of a metrizable space X is symmetrizable, X is a D-space.

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Definition 1.17.[3] A collection of subsets$B = \bigcup\{B_x / x \in X\}$ of a topological space X is said to be a weak base provided

- To each$x \in X$, every member ofB_x contains$x$
- For any 2 membersB_1 andB_2 ofB_x there exists$B_3 \in B_x$ such that$B_3 \subset B_1 \bigcap B_2$
- B determines the topology of$X$ in the following way: A set$U \subset X$ is open in$X$ if and only if for all$z \in U$, there exists$B \in B_z$ with$B \subseteq U$.

Definition 1.18.[6] A space X is said to be sequential if for every non closed subset $A \subseteq X$ there exists a sequence $\{x_n\}_{n<\omega}$ in A which converges to some $z \in X\backslash A$.
Definition 1.19.[3] If X is a sequential space and $\in W \subseteq X$, we say that W is a weak neighbourhood of x if whenever $\{x_n\}_{n<\omega}$ converges to$x$ then$\{x_n\}_{n<\omega}$ is eventually in W.
The next proposition says that in a sequential space the collection of weak neighbourhoods is a weak base for X.
Proposition 1.20. If X is a sequential space then a subset $U \subseteq X$ is open if only if for all $x \in U$ there exists a weak neighbourhood W of x such that $W \subseteq U$.
Definition 1.21.[3] A collection W of subsets of a sequential space X is said to be a ω-system for the topology on X if whenever $x \in U \subseteq X$, with U open, there exists a subcollection$V \subseteq W$ such that $x \in \bigcap V$, $\bigcup V$ is a weak neighbouhood of x and $\bigcup V \subseteq U$.
Proposition 1.22. If $f: Z \to X$ is a quotient map from a space$Z$ onto aT_2 sequential space$X$ and$\mathbb{B}$ is any base for the topology on$Z$ then$W = \{f(B)/B \in \mathbb{B}\}$ is aω-system for X.
Proof. Let $\subseteq X$, with U open. We need to find a subcollection $V \subseteq W$ such that $x \in \bigcap V, \bigcup V$ is a weak neighbourhood of x, and $\bigcup V \subseteq U$. In Z, let $C = \{B \in \mathbb{B}/B \bigcap f^{-1}(x) \neq \emptyset$ and$B \subseteq f^{-1}(U)\}$ and let$V = \{f(B)/B \in C\}$. Clearly$x \in \bigcap V$and$\bigcup V \subseteq U$. To show that$\bigcup V$ is a weak neighbourhood of$x$, suppose$\{y_n\}_{n<\omega}$ converges to$x$; then we need to show that$\{y_n\}_{n<\omega}$ is eventually in$\bigcup V$. If this is not the case there would be a subsequence missing$\bigcup V$ completely so without loss of generality we may assume $\{y_n/n < \omega\} \bigcap (\bigcup V) = \emptyset$. Since$X$ isT_2 a we see that$\{y_n/n < \omega\} \bigcup \{x\}$ is closed in$X$ and$f^{-1}(\{y_n/n < \omega\} \bigcup \{x\})$ is closed in Z. Now$f^{-1}(x) \subseteq \bigcup C$, and$\bigcup\{f^{-1}(y_n)/n \in \omega\} \bigcap (\bigcup C) = \emptyset$ implies that$\bigcup\{f^{-1}(y_n)/n \in \omega\}$ is a closed saturated set in Z. This says$\{y_n/n < \omega\}$ is a closed set in X, a contradiction. Hence$W = \{f(B)/B \in \mathbb{B}\}$ is aω-systemfor X.
Theorem 1.23. A sequential space X with a point countable ω-system is a D-space.
Proof. Let $\mathbb{W}$ be a point countable ω-system for X and for each $x \in X$ let $\mathbb{W}_x$denote $\{W \in \mathbb{W}/x \in W\}$. Suppose $\mathbb{U} = \{U(x)/x \in X\}$ is the range space of neighbourhood assignment $U$for X. For every $x \in X$ pick a subcollection $\mathbb{V}_x \subseteq \mathbb{W}_x$, such that $x \in \bigcap \mathbb{V}_x$, and $V(x) = \bigcup \mathbb{V}_x$ is a weak neighbourhood of x and $V(x) \subseteq U(x)$. For $t \in X$, let $\mathbb{H}_t$ denote the countable set $\{W \in \mathbb{W}/t \in W \in \bigcup_{x \in X} \mathbb{V}_x\}$. Consider $\mathbb{H}_t$ to be well ordered with an order type as a subset of ω. Identify the centers of elements of $H \in \mathbb{H}_t$ by letting

- $c(H) = \{x \in H/H \in \mathbb{V}_x\}$
- $C(t) = \bigcup\{c(H)/H \in \mathbb{H}_t\}$.

By a recursion process we will identify an ordinal μ, countable sets $A_\alpha \subseteq X, \alpha < \mu$ and open sets $O_\alpha = \bigcup\{U(x)/x \in A_\alpha\}$ so that $\bigcup_{\alpha<\mu} O_\alpha = X$ and $D = \bigcup_{\alpha<\mu} A_\alpha$ is closed and discrete in X. So $\{U(x)/x \in D\}$ would be the subcover of U satisfying D-space property.
For an ordinal β, assuming that A_α and O_α, for all $\alpha < \beta$, have been defined, continue the process as follows:
If $\bigcup_{\alpha<\beta} O_\alpha = X$, we stop and let $\mu = \beta$.
If $\bigcup_{\alpha<\beta} O_\alpha \neq X$, pick some $z_\beta \in X\backslash \bigcup_{\alpha<\beta} O_\alpha$.
Next we find by induction on ω an increasing sequence $\left\{F_n^\beta\right\}_{n\in\omega}$ of finite subsets of X, with the initial $F_0^\beta = \{z_\beta\}$, as follows:
Given that F_n^β is defined and $t \in F_n^\beta$, let

$$R(t) = (C(t) \setminus \bigcup_{\alpha \in F_n^\beta} U(s) \setminus \bigcup_{\alpha < \beta} O_\alpha \quad , \qquad E_n^\beta = \left\{ t \in F_n^\beta / R(t) \neq \emptyset \right\}$$

For $t \in E_n^\beta$, let $k(t,n) = min\{n, |\{W \in \mathbb{H}_t / R(t) \cap C(W) \neq \emptyset\}|\}$. Now let $W_{t,i,} i = 1, 2, \ldots, k(t,n)$, be the first$k(t,n)$ elements of$\mathbb{H}_t$ such that$R(t) \cap C(W) \neq \emptyset$ and pick$x(t,i) \in R(t) \cap c(W_{t,i,})$, for each$i$. We let

$$F_{n+1}^\beta = F_n^\beta \cup \left\{ x(t,i) / t \in E_n^\beta, 1 \leq i \leq k(t,n) \right\}.$$

If some $E_n^\beta = \emptyset$ then $F_n^\beta = F_{n+1}^\beta = F_{n+2}^\beta = \cdots$ In any case the resulting $F_n^\beta, m \in \omega$, form an increasing sequence of finite sets. Now we let $A_\beta = \bigcup_{n \in \omega} F_n^\beta$.

That concludes the recursion process which defines the countable sets$A_\alpha \subseteq X$, for$\alpha < \mu$, and open sets$O_\alpha = \bigcup\{U(x) / x \in A_\alpha\}$. It is clear from the construction that$\bigcup_{\alpha < \mu} O_\alpha = X$. The following two observations follow from the construction above:

References

[1]. E.K. Van Douwen, W.F. Pfeffer, Some properties of the Sorgenfrey line and related spaces, Pacific J.Math. No.2vol.81 (1979), 371-377.

[2]. R. Buzyakova, On D-property of strong ∑–spaces, comment. Math. Univ. Car. Vol.43 (2002), 493-495.

[3]. D.K. Bruke, Weak bases and D-spaces, Comment.MatUniv.Car.vol.48(2007), 281-289.

[4]. G. Gruenhage, Generalized metric spaces, in the Handbook of set-theoretic topology, K. Kunen and J.E. Vaughan, eds., North-Holland, Amsterdam, 1984, 423-501.

[5]. G. Gruenhage, A survey of D-spaces, in: L. Balinkostova, A. Caicedo, S.Geschke, M. Scheepers (Eds.), Set Theory and its Applications, Contemp. Math. (2011). 13-28.

[6]. [J.R.Boone, On irreducible spaces, Bull. Austral. Math. Soc. Vol.12(1975), 143-148.