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On A Two Dimensional Finsler Space Whose Geodesics Are Semi-Elipses and Pair of Straight Lines

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Abstract: *It is an interesting problem to find the fundamental function of a two dimensional Finsler space whose geodesics constitute a given family of curves. M. Matsumoto [**5**],[**6**],[**8**][1)] obtained the fundamental function of a two dimensional Finsler space whose geodesics are special conic sections. The aim of the present paper is to obtain the fundamental metric function of two dimensional Finsler space whose geodesics are semi-elipses and pair of straight lines. We show that such space is locally Minkowskian.*

I. Preliminaries.

Let $F^n = (M^n, L(x, y))$ be an n-dimensional Finsler space on an underlying smooth manifold M^n with the fundamental function L(x,y). The fundamental tensor $g_{ij}(x,y)$, the angular metric tensor $h_{ij}(x,y)$ and the normalized supporting element $l_i(x,y)$ are defined respectively by

$g_{ij} = h_{ij} + l_i l_j$, $\quad h_{ij} = LL_{(i)(j)}$, $\quad l_i = L_{(i)}$

where $L_{(i)} = \dfrac{\partial L}{\partial y^i}$ and $L_{(i)(j)} = \dfrac{\partial^2 L}{\partial y^i \partial y^j}$.

The geodesic, the extremal of the length integral $s = \int_{t_0}^{t} L(x, y)dt$ where $t \geq t_0$, $\quad y^i = \dot{x}^i = \dfrac{dx^i}{dt}$ along an oriented curve $C: x^i = x^i(t)$ from point $P = x^i(t_o)$ to point $\quad Q= x^i(t)$, $t > 0$, is given by the Euler equation

(1.1) $$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - L_i = 0,$$

where $L_i = \dfrac{\partial L}{\partial x^i}$.

In terms of $F(x, y) = L^2(x, y)/2$, (1.1) is written in the form

(1.2) $$\frac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = h(t)\frac{dx^i}{dt}$$

where we put

(1.3) $$2g_{ij}\, G^i(x,y) = y^r \left(\frac{\partial^2 F}{\partial x^r \partial yJ}\right) - \frac{\partial F}{\partial x^J},$$

and $h(t) = (d^2s/dt^2) / (ds/dt)$.

Now we consider two dimensional Finsler space and use the notation (x,y) and (p,q) respectively, instead of (x^1,x^2) and (y^1,y^2). The fundamental function L(x,y;p,q) are positively homogeneous of degree one in p and q. Therefore, we have

(1.3 a) $\quad L_x = pL_{xp} + qL_{xq}$, $\quad L_y = pL_{yp} + qL_{yq}$.

Also from homogeneity of L_p and L_q we have

(1.3 b) $$\frac{L_{pp}}{q^2} = \frac{-L_{pq}}{pq} = \frac{L_{qq}}{p^2} = W(say),$$

where W is called Weierstrass invariant. Consequently the two equations represented by (1.1) reduce to the single equation

(1.4) $\quad L_{xq} - L_{yp} + (p\dot{q} - q\dot{p})W = 0,$

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1) Numbers in square brackets refer to the references at the end of the paper.

Which is called Weierstrass form of geodesic equation. Now consider the associated fundamental function A(x,y,z), $z=y' = \frac{dy}{dx}$ defined as follows:

(1.5) $A(x,y,z) = L(x,y;1,z), \quad L(x,y;p,q) = A(x,y,\frac{q}{p})p.$

Therefore $L_p=A-zA_z$, $L_q=A_z$, $L_{yp} = A_y - zA_{yz}$, $L_{xq} = A_{xz}$, $L_{pp} = (z^2/p) A_{zz}$, $L_{pq} = (-z/p) A_{zz}$, $L_{qq}= (1/p) A_{zz}$.
On using these values in (1.4), we have

(1.6) $A_{zz} y'' + A_{yz} y' + A_{xz} - A_y = 0, \; z=y',$

which is called the Rashevsky form of geodesic equation.

We observe that $y'=q/p$ gives $y''=(p\dot{q}-q\dot{p})/p^3$. Hence from (1.2) we have another form of geodesic equation

(1.7) $y''= \frac{2}{p^3}\left(qG^1 - pG^2\right).$

Now in n-dimensional Finsler space F^n we have the Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j\right)$; defined by $G^i_j = \partial G^i / \partial y^j$, $G^i_{jk} = \partial G^i_j / \partial y^k$ and two kinds of covariant differentiation of Finslerian vector field V^i(x,y) given by

$V^i_{(j)} = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial y^r} G^r_J + V^r G^i_{rJ}, \; V^i_{,j} = \frac{\partial v^i}{\partial y^j},$

Called the h- and v–covariant derivative of V^i respectively. Since BΓ is L-metrical, i.e. $L_{(i)}=0$, we have

(1.8) $L_i = l_r G^r_i .$

Further from (1.8) we have

(1.9) $L_{i,j} = (1/L) h_{rj} G^r_i + l_r G^r_{ij} .$

Next from (1.3) we have $2g_{ij}G^i y^J = \left(\frac{\partial F}{\partial x^r}\right) y^r$, that is

(1.10) $2G^i l_i = L^r y_r .$

Now we shall return to the two-dimensional case since the matrix (h_{ij}) is of rank one, we get $\varepsilon = \pm 1$ and the vector field $m_i(x,y)$ satisfying

(1.11) $h_{ij} = \varepsilon m_i m_j,$

so we have

(1.12) $g_{ij} = l_i l_j + \varepsilon m_i m_j ,$

therefore, we get easily

(1.13) $l_i l^i = 1, \; l_i m^i = m_i l^i = 0, \; m_i m^i = \varepsilon .$

Thus we obtain the orthonormal frame field (l^i, m^i), called the Berwald frame. Therefore, we have scalar fields h(x, y; p, q) and k(x, y; p, q) such that

(1.14) $(m_1, m_2) = h(-l^2, l^1), \quad (m^1, m^2) = k(-l_2, l_1), \quad hk = \varepsilon.$

The equations (1.12) and (1.14) give

(1.15) $g(=\det g_{ij}) = g_{11}g_{22} - g^2_{12} = \varepsilon (l_1 m_2 - l_2 m_1)^2 = \varepsilon h^2,$

and

(1.16) $\frac{1}{h}(m^1, m^2) = \frac{1}{g}(-l_2, l_1).$

We already have $h_{11} = L L_{pp} = L W q^2$ and $h_{11} = \varepsilon (m_1)^2 = \varepsilon h^2 (l^2)^2 = \varepsilon h^2 \left(\frac{q}{l}\right)^2$. Consequently, we have

(1.17) $L^3 W = \varepsilon h^2 = g.$

Now we try to find the expression for G^i in the Berwald frame [**4**] the equations (1.9) and (1.14) give

$L_{xq} - L_{yp} = \frac{1}{L}\left(h_{rj} G^r_i - h_{ri} G^r_j\right)$

$= \frac{\varepsilon m_r}{L}\left(m_j G_i^r - m_i G_j^r\right)$

$= \frac{\varepsilon m_r}{L}\left(m_2 G_1^r - m_1 G_2^r\right)$

$= \frac{\varepsilon m_r}{L} h\left(l^1 G_1^r + l^2 G_2^r\right)$

$= \frac{\varepsilon m_r}{L^2} h\left(G_i^r y^i\right).$

Due to homogeneity of G^r in y^i, we have $G_i^r y^i = 2G^r$, therefore

$2G^i m^i = (L_{xq} - L_{yp})L^2/\varepsilon\, h.$

Using $2G^i(l_i l^j + \varepsilon\, m_i\, m^j) = (L_r y^r) l^j + m^j (L_{xq} - L_{yp}) L^2 / h$ and (1.10) leads to

(1.18) $2G^i = (L_r y^r) l^i + (L^2/h)\,(L_{xq} - L_{yp}) m^i.$

If we put

(1.19) $L_r y^r = L_o$ and $L_{xq} - L_{yp} = M$ in (1.18), and using (1.16) and (1.17) we get

(1.20) $2G^1 = \frac{1}{L}\left(L_0 p - \frac{M}{W} L_q\right)$, $2G^2 = \frac{1}{L}\left(L_0 q - \frac{M}{W} L_p\right)$.

II. From geodesics to the Finsler metric.

Let us consider a family of curves $\{C(a,b)\}$ on the (x,y)–plane R^2, given by the equation

(2.1) $y = f(x,a,b)$,

with two parameters (a, b). Differentiating (2.1) with respect to x, we get

(2.2) $z\ (=y') = f_x(x,a,b)$.

Solving (2.1) and (2.2) for a, b, we find

(2.3) $a = \alpha(x,y,z),\ b = \beta(x,y,z)$.

In view of (2.3) the differentiation of (2.2) leads to

(2.4) $z' = f_{xx}(x,a,b) = u(x,y,z)$,

which is precisely the second order differential equation of y characterizing $\{C(a, b)\}$.

Now we are concerned with the Rashevsky form (1.6) of geodesic equation $L_{qq} = A_{zz} / p$ and $A_{zz} = Wp^3$, hence from (1.5) and (1.17) we have

(2.5) $L^3 W = A^3 A_{zz} = g$.

Thus it is suitable to call A_{zz} the associated Weierstrass invariant. If we put $B = A_{zz}$, then the differentiation of (1.6) with respect to z gives

(2.6) $B_x + B_y z + B_z u + B u_z = 0$,

which is first order quasilinear partial differential equation. Its auxiliary equations are given by

(2.7) $$\frac{dx}{1} = \frac{dy}{z} = \frac{dz}{u} = \frac{dB}{-Bu_z}.$$

Now defining U(x,a,b) and V(x,y,z) by

(2.8) $U(x;\ a,\ b) = \exp \int u_z(x, f, f_x)dx$, $V(x,\ y,\ z) = U\ (x;\ \alpha,\ \beta)$,

we obtain

(2.9) $$B(x,y,z) = \frac{H(\alpha,\beta)}{V(x,y,z)},$$

where H is an arbitrary non-zero function of two arguments.

From $A_{zz} = B$ we get A in the form

(2.10) $$\begin{cases} A(x,y,z) = A^*(x,y,z) + C(x,y) + D(x,y)z, \\ A^* = \int\left\{\int B(x,y,z)dz\right\}dz = \int_{z_0}^{z}(z-t).B(x,y,t)dt. \end{cases}$$

where C and D are arbitrary but must be chosen so that A may satisfy (1.6), that is

(2.11) $C_y – D_x = A^*_{zz}u + A^*_{yz}z + A^*_{xz} - A^*_y$.

If a pair (C_0, D_0) has been chosen so as to satisfy (2.11). Then $(C–C_0)_y=(D–D_0)_x$, so that we have locally a function E(x,y) satisfying $E_x=C–C_0$ and $E_y=D–D_0$. Thus (2.10) is written as

$A=A^*+C_0+D_0 z+E_x+E_y z$.

Therefore (1.5) leads to fundamental function

(2.12) $$\begin{cases} L(x,y;p,q) = L_0(x,y;p,q) + e(x,y,p,q), \\ L_0 = A^*(x,y,\frac{q}{p})p + C_0(x,y)p + D_0(x,y)q \end{cases}$$

where e is the derived form given by

(2.13) $e(x,y;p,q)=E_x dx+E_y dy$.

Thus, we see that the Finsler metric is uniquely determined when the functions H and E of two arguments are chosen. Further, for different choice of the function H we obtain Finsler spaces which are projective to each other because each one has the same geodesics {C(a,b)}.

III. Family of semi-elipses.

Let us consider the family of semi-elipses {C(a,b)} given by the equation

(3.1) $x^2 + ay^2 = b$, y, b>0,

on the semiplane R^2_+ .

From (3.1) we have

(3.2) $2x + 2ayz = 0$, $z = y'$.

Consequently the functions α(x,y,z), β(x,y,z) and u(x,y,z) of the preceding sections are given by

(3.3) $$a = -\frac{x}{yz} = \alpha(x,y,z),\ b = x^2 - \frac{xy}{z} = \beta(x,y,z),$$

(3.4) $$z' = \frac{z}{x} - \frac{z^2}{y} = u(x,y,z).$$

From (3.4) we get

(3.5) $$y'' = \frac{y'}{x} - \frac{(y')^2}{y},$$

which characterizes the family {C(a,b)}.

Now we find the function U(x;a,b) and V(x;y,z) defined by (2.8). Differentiating (3.4) we get

$$u_z = \frac{1}{x} - \frac{2z}{y} = \frac{1}{x} - \frac{2x}{x^2 - b} \quad \text{and}$$

$$U(x;a,b) = \exp\int\left(\frac{1}{x} - \frac{2x}{x^2-b}\right)dx = \frac{x}{x^2-b},\quad V(x,y,z) = \frac{z}{y}.$$

Thus, (2.9) implies that

$$B(x,y,z) = H(\alpha,\beta)\frac{y}{z} = -\frac{y^2}{x}\alpha\, H(\alpha,\beta).$$

On account of the arbitrariness of H, we may write B as $B=H(\alpha,\beta)\frac{y^2}{x}$ and A^* of (2.10) is written in the form

(3.6) $$A^*(x,y,z) = \frac{y^2}{x}\iint H(\alpha,\beta)(dz)^2,$$

or

(3.6)′ $$A^*(x,y,z) = \frac{y^2}{x}\int_1^z (z-t)F(t)dt,\qquad F(t) = H\left(-\frac{x}{yt},\ x^2 - \frac{xy}{t}\right).$$

We have taken the limit of integration from 1 to z instead of 0 to z because t is in the denominator of F(t).

Now if we put $F_1(t)=H_\alpha\left(-\frac{x}{yt},\ x^2-\frac{xy}{t}\right)$, $F_2(t)=H_\beta\left(-\frac{x}{yt},\ x^2-\frac{xy}{t}\right)$,

then from (2.10) and (2.11) we have

(3.7) $C_y–D_x$ $=\left(\frac{y^2z}{x^2}-\frac{yz^2}{x}\right)H(\alpha,\beta)-\frac{y^2}{x^2}\int_1^z F(t)dt+\int_1^z F_1(t)dt+y^2\int_1^z F_2(t)dt$

$-\frac{y}{x}\int_1^z \frac{F_1(t)}{t}dt-\frac{y^3}{x}\int_1^z \frac{F_2(t)}{t}dt+\frac{2y}{x}\int_1^z tF(t)dt$.

Therefore we have:

Theorem 1. Every associated fundamental function A(x,y,z) of a Finsler space $\left(R_+^2, L(x,y;p,q)\right)$ having the semi-elipses (3.1) as the geodesics is given by
$A(x,y,z)= A^*(x,y,z)+C(x,y)+D(x,y)z$,
where A^* is defined by (3.6)′, H is an arbitrary function of (α,β) given by (3.3) and the function (C,D) must be chosen so as to satisfy (3.7).

Example. In particular, we first put $H(\alpha,\beta)=(\alpha)^n$ for a real number n. Then $F(t)=\left(-\frac{x}{yt}\right)^n$, hence from (3.6)′ and (3.7) we have

(3.8) $$A^*=\frac{y^2}{x}\int_1^z (z-t)\left(-\frac{x}{yt}\right)^n dt,$$

or

(3.8)′ $$A^*=\frac{(-1)^n x^{n-1}}{(1-n)(2-n)\,y^{n-2}}\left[z^{2-n}-(2-n)z+(1-n)\right]$$

and

(3.9) $$C_y-D_x=(-1)^n\left[\frac{x^{n-2}}{y^{n-2}}-\frac{x^{n-1}}{y^{n-1}}\right].$$

If we choose $C=-\frac{(-1)^n x^{n-1}}{y^{n-2}(2-n)}$ and $D=\frac{(-1)^n x^{n-1}}{y^{n-2}(1-n)}$, we have

(3.10) $$A(x,y,z)=\frac{(-1)^n x^{n-1} z^{2-n}}{(1-n)(2-n)y^{n-2}}.$$

Therefore it follows from (1.5) that, the fundamental function

(3.11) $L(x,y;p,q)= x^{n-1} y^{2-n} p^{n-1} q^{2-n}$, $n\neq 1, 2$

where $\frac{(-1)^n}{(1-n)(2-n)}$ was omitted.

Case I. If n = 1, then (3.8) and (3.9) gives

$A^*=-y\,(z\log|z|-z+1)$, $C_y–D_x=-\frac{y}{x}-1$.

Choosing $C=\frac{-y^2}{2x}$, D = x we have $A(x,y,z)=z\left(y-y\log|z|+x\right)-y-\frac{y^2}{2x}$,

Consequently, we obtain fundamental function

(3.12) L(x,y. p,q) = $q\left(y - y\log\left|\frac{q}{p}\right| + x\right) + yp - \frac{y^2 p}{2x}$.

Therefore, L_{xq} = 1, L_{yp}=$\frac{q}{p} - \frac{y}{x} + 1$, $L_{pp} = \frac{-qy}{p^2}$. From (1.3b) we have $W = \frac{-y}{qp^2}$, so using these values in (1.4), where we use $(p\dot{q} - q\dot{p}) = y'' p^3$ and $\frac{q}{p} = z = y'$. Then it leads to (3.5) immediately.

Case II. If n = 2, we have similarly A* = $x(z - \log|z| - 1)$, C_y–D_x= $-\frac{x}{y} + 1$.

Choosing C=y and D=$\frac{x^2}{2y}$, we have A(x,y,z)= $x\left(z - 1 - \log|z| + \frac{xz}{2y}\right) + y$. Therefore, we obtain the metric

(3.13) L(x,y; p,q) = qx – px–xp log $\left|\frac{q}{p}\right| + \frac{x^2 q}{2y} + yp$.

Now we shall return to the general case with the Finsler metric (3.11). If we refer to the new co-ordinate system $(\bar{x}, \bar{y}) = \left(\frac{x^2}{2}, \frac{y^2}{2}\right)$, then we have (p,q)= $\left(\frac{\bar{p}}{\sqrt{2\bar{x}}}, \frac{\bar{q}}{\sqrt{2\bar{y}}}\right)$ and the metric (3.11) can be written in the form

(3.11)′ $\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q} = (-1)^n (\bar{q})^{2-n} (\bar{p})^{n-1}$.

Since $\bar{L}$ does not depend on $\bar{x}$ and $\bar{y}$, this is a simple metric, called a locally Minkowskian metric and ($\bar{x}, \bar{y}$) is an adapted co-ordinate system to the structure. Further its main scalar I is constant. Since (3.11)′ is of the form (i) or (iv) of Theorem 3.5.3.2 of [1], we have directly as follows:

(i) $\varepsilon = 1$, $I^2 > 4$, (2 – n) (n – 1) < 0, $\frac{I}{\sqrt{I^2 - 4}}$ +1=2(2–n),

(ii) $\varepsilon = -1$, (2 – n) (n – 1)> 0, $\frac{I}{\sqrt{I^2 + 4}}$ +1=2(2–n).

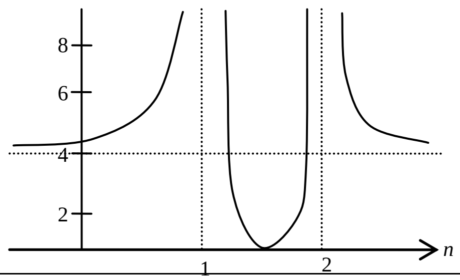
Therefore we have:

Proposition 1. The Finsler space ($R^2_+, L(x, y, p, q)$) with a metric (3.11) is locally Minkowskian and has the signature ε and the constant main scalar I as follows:

(1) $n < 1, \ > 2$: ε = 1, I^2= $\frac{(2n-3)^2}{(n-2)(n-1)}$,

(2) $1 < n < 2$: ε = –1, I^2 = $-\frac{(2n-3)^2}{(n-2)(n-1)}$.

Remark. $\frac{(2n-3)^2}{(n-2)(n-1)}$ = 4+$\frac{1}{n-2} - \frac{1}{n-1}$. The graph of I^2 is shown in figure.



Since a Finsler space of dimension two is Riemannian if and only if I = 0. Therefore, The Finsler space $\left(R_+^2, L(x,y,p,q)\right)$ under consideration is Riemannian if and only if n=3/2.

IV. Family of pair of straight lines.

Let us consider the family of pair of straight lines {C(a,b)} given by the equation

(4.1) $(x-a)^2 - by^2 = 0, \quad y, \quad b>0,$

on the semiplane R_+^2.

From (4.1) we have

(4.2) $2(x-a)= 2byz = 0, \quad z = y'.$

Consequently the functions $\alpha(x,y,z)$, $\beta(x,y,z)$ and $u(x,y,z)$ of the preceding sections are given by

(4.3) $$a = x - \frac{y}{z} = \alpha(x,y,z)\,,\ b = \frac{1}{z^2} = \beta(x,y,z)\,,$$

(4.4) $$z' = 0 = u(x,y,z)\,.$$

From (4.4) we get

(4.5) $$y'' = 0\,,$$

which characterizes the family {C(a,b)}.

Now we find the function U(x;a,b) and V(x;y,z) defined by (2.8). Differentiating (4.4) we get

$u_z = 0$ and $U(x;a,b) = \exp\int 0dx = 1, \qquad V(x,y,z) = 1.$

Thus, (2.9) implies that $B(x,y,z)= H(\alpha,\beta)$.

Therefore, A^* of (2.10) is written in the form

(4.6) $$A^*(x,y,z) = \iint H(\alpha,\beta)(dz)^2\,,$$

or

(4.6)′ $$A^*(x,y,z) = \int_1^z (z-t)F(t)dt, \qquad F(t) = H\left(x - \frac{y}{t},\ \frac{1}{t^2}\right).$$

We have taken the limit of integration from 1 to z instead of 0 to z because t is in the denominator of F(t).

Now if we put $F_1(t)=H_\alpha\left(x-\frac{y}{t},\ \frac{1}{t^2}\right)$, $F_2(t)=H_\beta\left(x-\frac{y}{t},\ \frac{1}{t^2}\right)$, then from (2.10) and (2.11) we have

(4.7) $$C_y - D_x = \int_1^z F_1(t)dt - \int_1^z F_2(t)dt\,.$$

Therefore we have:

Theorem 1. Every associated fundamental function A(x,y,z) of a Finsler space $\left(R_+^2, L(x,y;p,q)\right)$ having the pair of straight lines (4.1) as the geodesics is given by

$A(x,y,z)= A^*(x,y,z)+C(x,y)+D(x,y)z,$

where A^* is defined by (4.6)′, H is an arbitrary function of (α,β) given by (4.3) and the function (C,D) must be chosen so as to satisfy (4.7).

Example. In particular, we first put $H(\alpha,\beta)=(\beta)^n$ for a real number n. Then $F(t)=\left(\frac{1}{t^2}\right)^n$, hence from (4.6)′ and (4.7) we have

(4.8) $$A^* = \int_1^z (z-t)\left(\frac{1}{t^2}\right)^n dt\,,$$

or

(4.8)′ $$A^* = \frac{1}{(1-2n)(2-2n)}\left[z^{2-2n} - (2-2n)z + (1-2n)\right]$$

and

(4.9) $$C_y - D_x = 0.$$

If we choose C = $-\dfrac{1}{(2-2n)}$ and D = $\dfrac{1}{(1-2n)}$, we have

(4.10) $\quad A(x,y,z)=\dfrac{z^{2-2n}}{(1-2n)(2-2n)}$.

Therefore it follows from (1.5) that the fundamental function

(4.11) $\quad L(x,y;\, p,q)= p^{2n-1} q^{2-2n},\ n\neq 1/2,\ 1,$

where $\dfrac{1}{(1-2n)(2-2n)}$ was omitted.

Case I. If n = 1/2, then (4.8) and (4.9) gives
$A^{*}= (z \log|z| - z +1)$, $C_y - D_x = 0$.
Choosing D = – C =1, we have $A(x,y,z)= z\log|z|$,
Consequently, we obtain fundamental function

(4.12) $\quad L(x,y.\ p,q) = q\log\left|\dfrac{q}{p}\right|$.

Therefore, $L_{xq}=0$, $L_{yp}=0$, $L_{pp}=\dfrac{q}{p^2}$. From (1.3 b) we have $W=\dfrac{1}{qp^2}$, so using these values in (1.4), where we use $\left(p\dot{q}-q\dot{p}\right)=y''p^3$ and $\dfrac{q}{p}=z=y'$. Then it leads to (4.5) immediately.

Case II. If n = 1, we have similarly $A^{*} = (z-\log|z|-1)$, $C_y - D_x = 0$.
Choosing C= –D=1, we have $A(x,y,z)=-\log|z|$. Therefore, we obtain the metric

(4.13) $\quad L(x,y;\, p,q) = p\log\left|\dfrac{q}{p}\right|$.

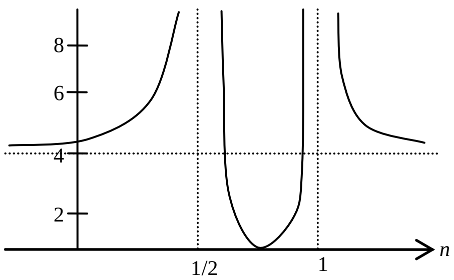
Now we shall return to the general case with the Finsler metric (4.11). Since it is of quite similar form to (3.11)′ we get also the result similar to proposition 1 as follows:

Proposition 2. The Finsler space $(R_{+}^{2}, L(x,y,p,q))$ with a metric (3.11) is locally Minkowskian and has the signature ε and the constant main scalar I as follows:

(1) $\quad n<\dfrac{1}{2},\ >1$: $\varepsilon = 1$, $I^2=\dfrac{(4n-3)^2}{2(2n-1)(n-1)}$,

(2) $\quad \dfrac{1}{2}<n<1$: $\varepsilon = -1$, $I^2=-\dfrac{(4n-3)^2}{2(2n-1)(n-1)}$.

Remark. $\dfrac{(4n-3)^2}{2(2n-1)(n-1)} = 4+\dfrac{1}{2(n-1)}-\dfrac{1}{2n-1}$. The graph of I^2 is shown in figure.



Since a Finsler space of dimension two is Riemannian if and only if I = 0. Therefore, The Finsler space $\left(R_{+}^{2}, L(x,y,p,q)\right)$ under consideration is Riemannian if and only if n=3/4.

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