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# Stability of Evolution Operators on Hilbert Space

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***Abstract:*** *We present a general stability concept of linear evolution operators which are defined on a Hilbert space. The stability concept include situations among which are non-uniform stability and non-uniform exponential stability.*
***Keywords:*** *Banach space, evolution, operator, stability*

## I. Introduction

In recent years the classical ideas on exponential stability and other asymptotic properties concerning the solutions of differential equations have witnessed significant development. The general case of evolutionary processes have been studied in [2] by Datko for exponential stability and by Buse [3] and Megan and Buse [5] for exponential dichotomy.

The aim of this paper is to characterize the uniform exponential stability of evolutionary processes and also to establish some sufficient conditions for the non—uniform exponential stability as they apply to Hilbert spaces in the spirit of the ideas due to Datko [2] and Preda, Latcu and Preda [1].

Let X be a real or a complex Banach space and L(X) the Banach algebra of all linear and bounded operators acting on X. The norms on X and on L(X) will be denoted by $\|\cdot\|$. Let T be the set defined by:

$$T=\{(t,s): 0 \le s \le t < \infty\}. \quad (1)$$

**Definition 1.1**
An application $\Phi(\cdot,\cdot)$: T→L(X) is said to be an evolutionary process if the following statements hold:
i) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ for all $0 \le t_0 \le s \le t$;
ii) $\Phi(t,t)x=x$ for all x∈X and t≥0;
iii) $\Phi(\cdot,s)x$ is continuous on [s,∞) for all s≥0, x∈
$\Phi(t,\cdot)x$ is continuous on [0,t] for all t≥0, x∈X;
iv) there exists a non-decreasing function $p(\cdot): \Re_+ \to (0,\infty)$ such that $\|\Phi(t,s)\| \le p(t-s)$, for all t≥s≥0 $(\Re_+ = [0,\infty))$.
Condition (iv) can be replaced by:
v) there are M, ω>0 such that
$\|\Phi(t,s)\| \le Me^{\omega(t-s)}$ for all t≥s≥0.

**Remark 1.2**
i) If the process Φ satisfies (i), (ii), (iii) and (v) then Φ is called an evolutionary process of the class $C(0,e)$, also called a reversible evolutionary process.
ii) If the process satisfies (i), (ii), (iii) and $\Phi(t,s) = \Phi(t-s,0)$ for all t≥s≥0, then it is called a semigroup of class $C_0$.

**Example 1.3** Let $A:\Re_+ \to L(X)$ be strongly measurable function such that:

$$\sup\left\{\int_t^{t+1}\|A(s)\|ds: t \ge 0\right\} < \infty \quad (2)$$

The unique solution $Y(\cdot)$ of the Cauchy problem $\dot{Y} = A(t)Y$, Y(0)= I where I denotes the identity operator on X, has the property that $\Phi(t,s) = Y(t)Y^{-1}(s)$, t≥s≥0, is a reversible evolutionary process on X.

**Definition 1.4** The evolutionary process $\Phi(t,s)$ will be called non-uniformly exponentially stable, if there are $\nu > 0$ and a function $N(\cdot): \Re_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_o)\| \le N(t_o)e^{-\nu(t-t_o)}, \text{ for all } t \ge t_o \ge 0. \quad (3)$$

**Remark 1.5** We observe along with Preda et al [1] that uniform exponential stability is a special case of the non-uniform exponential stability; specifically

$$\|\Phi(t,t_o)\| \le Ne^{-\nu(t-t_o)} \quad (4)$$

where N, ν>0 for all $t \ge t_o \ge 0$. The following result is due to Datko [2].

**Theorem 1.6** Let p∈ $[1, \infty)$ be arbitrary.The evolutionary process $\Phi$ is uniformly exponentially stable if and only if there is k>0 such that

$$\int_{t_o}^{\infty} \|\Phi(t, t_o)x\|^p dt \le k\|x\|^p \quad (5)$$

for all $t_o \ge 0$ and x ∈ X.

We observe that the necessary and sufficient conditions for the uniform exponential stability as well as some sufficient conditions for the non-uniform exponential stability were treated in Preda et al [1], Buse [3], Ichikawa [4] etc. We note the following:

**Lemma 1.7** Let $\Phi$ be an evolutionary process. If there are $r > 0$ and a continuous function g: $[r, \infty) \to (0, \infty)$ with $\inf_{t>r} g(t) < 1$ such that

$$\|\Phi(t, t_o)\| \le g(t - t_o) \text{for all } t_o \ge 0, t \ge t_o + r \quad (6)$$

then $\Phi$ is uniformly exponentially stable.

Proof:

Let $\delta > r$ be such that $g(\delta) < 1$. For $t \ge t_o \ge 0$ there is n ∈ N such that $n\delta \le t - t_o \le (n + 1)\delta$. Then

$$\|\Phi(t, t_o)\| \le Me^{\omega(t-n\delta-t_o)}\|\Phi(t_0 + n\delta, t_o)\| \quad (7)$$

$$\le Me^{\omega(t-n\delta-t_o)}g(\delta)^n,$$

Denoting $\nu = \frac{-\ln g(\delta)}{\delta} > 0$, it follows that

$$\|\Phi(t - t_o)\| \le Me^{\omega t} \cdot e^{\nu\delta} \cdot e^{-\nu(t-t_o)} \quad (8)$$

Denoting N = $Me^{(\omega+\nu)\delta}$ we obtain

$$\|\Phi(t, t_o)\| \le Ne^{-\nu(t-t_o)} \text{for all } t \ge t_o \ge 0. \quad (9)$$

**Theorem 1.8** [1, Thm 2.3]

Let $\Phi$ be an evolutionary process. If there are $\alpha > 0$ and a function H : $\Re_+ \to (0, \infty)$ such that

$$\int_t^{\infty} \left(\int_u^{u+1} \|\Phi(s, t)x\| ds\right) du \le H(t)\|x\| \quad (10)$$

for all t ≥ 0 and x ∈ X ; then there is a function N : $\Re_+ \to (0, \infty)$ such that

$$\|\Phi(t, t_o)\| \le N(t_o)e^{-\alpha(t-t_o)} \text{for all } t \ge t_o \ge 0. \quad (11)$$

Hence, $\Phi$ is non-uniformly exponentially stable.

## II. Uniform And Non-Uniform Exponential Stability Of Evolutionary Processes on Hilbert Space

Let $\Phi : T \to L(X)$ be an evolutionary process on a Banach space X, and p ≥ 1 a fixed real number.

**Definition 2.1** A function V: X × [0,∞)→ [0,∞) is called a p-Lyapunov function for the evolutionary process $\Phi$ if

there is $\alpha > 0$ such that

i) $V(0, t) = 0$ for all t ≥ 0, (12)

ii) $\frac{d}{dt} V(\Phi(t, s)x, t) \le -\exp(p\alpha t)\|\Phi(t, s)x\|^p, \ \forall t \ge s \ge 0, x \in X.$ (13)

We reproduce the following result of Buse [3, Thm 3.1], the proof has been adequately treated in the paper quoted.

**Theorem 2.2** The evolutionary process $\Phi$ is exponentially stable if and only if there exists a p-Lyapunov function for $\Phi$ satisfying Definition 2.1 above.

Let X be a Hilbert space and $\Phi$an evolutionary process of C(0,e) class on X i.e. $\Phi$ satisfies items i), ii), iii) and v) of Definition 1.1. The inner product on X will be denoted by$\langle \cdot, \cdot \rangle$. Throughout this section we shall by X denote a Hilbert space.

**Definition 2.3** The function B : [0,∞) → L(X) is Lyapunov function for $\Phi$ if there is $\alpha > 0$ such that

i) B(t) > 0 (i.e. $\langle B(t)x, x\rangle > 0$ for all x ≠ 0 and t ≥ 0) (14)

ii) $\frac{d}{dt}\langle B(t)\Phi(t, s)x, \Phi(t, s)x\rangle \le -\exp(2 \alpha t)\|\Phi(t, s)x\|^2$ for all t ≥ s ≥ 0 and x ∈ X(15)

**Proposition 2.4** The evolutionary process Φ of class C(0,e) on X is exponentially stable if and only if there exist a Lyapunov function for Φ.

Proof
Let $\nu > 0, \alpha \in (0,\nu)$ and put

$$B(t) = \int_t^{\infty} \exp(2\alpha u)\Phi(u,t)^*\Phi(u,t)du \tag{16}$$

Since,

$$\langle B(t)x, x\rangle = \int_t^{\infty} \exp(2\alpha u)\|\Phi(u,t)x\|^2 du$$

$$\leq [N_\nu(t)]^2[2(\nu-\alpha)]^{-1}\|x\|^2 \tag{17}$$

then B (t) ϵ L(X) for all t ≥ 0 and $\langle B(t)x, x\rangle > 0$ for x≠0. Moreover,

$$\frac{d}{dt}\langle B(t)\Phi(t,s)x, \Phi(t,s)x\rangle = \frac{d}{dt}\left(\int_t^{\infty} \exp(2\alpha u)\|\Phi(u,s)x\|^2 du\right)$$

$$= -\exp(2\alpha t)\|\Phi(t,s)x\|^2 \tag{18}$$

Type equation here.

Φ is of class C(0,e) implies that

$$\|\Phi(t,s)x\| \leq M_\alpha \exp[\alpha(t\text{-}s)]\|x\|. \tag{19}$$

Hence we obtain

$$\frac{d}{dt}\langle B(t)\Phi(t,s)x, \Phi(t,s)x\rangle = -\exp(2\alpha t)\|\Phi(t,s)x\|^2$$

$$\leq -M_\alpha^2 \cdot \exp[2\alpha(t-s])]\|x\|^2 \cdot \exp(2\alpha t)$$

$$= -M_\alpha^2 \cdot \exp(2\alpha t)\cdot\exp(-2\alpha s)\|x\|^2 \tag{20}$$

Let $K = M_\alpha^2 \exp(2\alpha t)$ then,

$$\frac{d}{dt}\langle B(t)\Phi(t,s)x, \Phi(t,s)x\rangle \leq -K \exp(-2\alpha s)\|x\|^2 \text{Type equation here.} \tag{21}$$

Therefore

$$\Rightarrow B(t) \in L(X). \tag{22}$$

Type equation here.

**Proposition 2.5** The evolutionary process Φ defined on the Hilbert space X, is uniformly exponentially stable if and only if there is $K \in (0,\infty)$ such that

$$\int_t^{\infty}\left(\int_u^{u+1}\|\Phi(s,t)x\|^2 ds\right)du \leq K\|x\|^2 \tag{23}$$

for all $t \geq 0$ and $x \in X$.

Proof
Let Φ be an evolution operator which satisfies for $K \in (0,\infty)$ the conditions of the theorem. We have, without loss of generality

$$\|\Phi(t,t_o)x\|^2 \leq M\, e^{-\omega(t-s)}\|\Phi(s,t_o)x\|^2$$

$$\Rightarrow \quad e^{\omega s}\|\Phi(t,t_o)x\|^2 \leq Me^{\omega t}\|\Phi(s,t_o)x\|^2 \text{ for all } t \geq s \geq 0 \text{ and } x \in X \tag{24}$$

Let $t \geq t_o + 1$. Integrating successively the relation (24) we obtain

$$\frac{1}{\omega}(e^{\omega}-1)e^{\omega u}\|\Phi(t,t_o)x\|^2 \le Me^{\omega t}\int_u^{u+1}\|\Phi(s,t_o)x\|^2 ds. \quad (25)$$

For u ϵ [$t_o$, t-1], it follows that

$$\frac{e^{\omega}-1}{\omega^2}\left(e^{\omega(t-1)}-e^{\omega t_o}\right)\|\Phi(t,t_o)x\|^2 \le Me^{\omega t}\int_{t_o}^{t-1}\left(\int_u^{u+1}\|\Phi(s,t_o)x\|^2 ds\right)du$$
$$\le MKe^{\omega t}\|x\|^2 \quad (26)$$

Then,

$$e^{-\omega t}\|\Phi(t,t_o)x\|^2 \le e^{-\omega(t-t_o)}\|\Phi(t,t_o)x\|^2 + \frac{MK\omega^2}{e^{\omega}}\|x\|^2$$

$$\le M\left(1+\frac{K\omega^2}{e^{\omega}-1}\right)\|x\|^2 \quad (27)$$

For $0 \le t_o \le t \le t_o+1$ we have

$$\|\Phi(t,t_o)x\|^2 \le Me^{\omega(t-t_o)}\|x\|^2 \le Me^{\omega}\|x\|^2 \le Me^{\omega}\|x\|^2 \text{for } x \in X. \quad (28)$$

Since $0 \le t - t_o \le 1$. Denoting $L = Me^{\omega}\left(1+\frac{K\omega^2}{e^{\omega}-1}\right)$ we obtain that

$$\|\Phi(t,t_o)x\|^2 \le L\|x\|^2 \text{ for all } t \ge t_o \ge 0 \text{ and } x \in X. \quad (29)$$

It follows that

$$\|\Phi(t,t_o)x\|^2 = \|\langle\Phi(t,t_o)x,\Phi(t,t_o)x\rangle\| = \|\langle\Phi(t,s)\Phi(s,t_o)x,\Phi(t,s)\Phi(s,t_o)x\rangle\|$$

$$\le L\|\langle\Phi(s,t_o)x,\Phi(s,t_o)x\rangle\|$$

$$=L\|\Phi(s,t_o)x\|^2 \quad (30)$$

For $t \ge s \ge t_o \ge 0$ and x ∈ X.

When $t \ge t_o+1$ we obtain

$$\|\Phi(t,t_o)x\|^2 \le L\int_u^{u+1}\|\Phi(s,t_o)x\|^2 ds \text{ for all } u \in [t_o,\ t-1]. \quad (31)$$

And so,

$$(t-1-t_o)\|\Phi(t,t_o)x\|^2 \le L\int_{t_o}^{t-1}\left(\int_u^{u+1}\|\Phi(s,t_o)x\|^2 ds\right)du$$

$$\le L\cdot K\|x\|^2 \quad (32)$$

**Proposition 2.6**

The evolutionary process Φ defined on the Hilbert space X, is uniformly exponentially stable if and only if there is K > 0 such that

$$\int_{t_o}^{t}\left(\int_{u-1}^{u}\|\Phi(t,s)\|^2 ds\right)du \le K \quad (33)$$

for all $t \ge t_o \ge 1$.

Proof

Let $t \ge t_1 \ge 1$ and $t_o = t_1 - 1$ we have, without loss of generality

$$\|\Phi(t,t_o)\|^2 \le Me^{\omega(s-t_o)}\|\Phi(t,s)\|^2 \quad (34)$$

Which implies that

$$\|\Phi(t,t_o)\|^2 e^{-\omega s} \le Me^{-\omega t_o}\|\Phi(t,s)\|^2 \text{ for } s \in [t_o,t]. \quad (35)$$

Integrating first with respect to s on [u-1,u] where u ϵ [$t_o+1,t$] and after that with respect to u on [[$t_o+1,t$], we obtain that

$$\frac{e^{\omega}-1}{\omega^2}\left(e^{\omega(t_o+1)}-e^{\omega t}\right)\|\Phi(t,t_o)\|^2 \le Me^{-\omega t_o}\int_{t_o+1}^{t}\left(\int_{u-1}^{u}\|\Phi(t,s)\|^2 ds\right)du \le MKe^{-\omega t_o} \quad (36)$$

It follows that

$$\|\Phi(t,t_o)\|^2 \le e^{\omega}\left(\frac{MK\omega^2}{e^{\omega}-1} + e^{\omega(t-t_o)}\|\Phi(t,t_o)\|^2\right) \le Me^{\omega}\left(\frac{K\omega^2}{e^{\omega}-1}+1\right). \quad (37)$$

For $0 \le t_o \le t \le t_o + 1$ we have $\|\Phi(t,t_o)\|^2 \le Me^{\omega}$, so it follows that

$$\|\Phi(t,t_o)\|^2 \le K_o \text{ for all } t \ge t_o \ge 0, \quad (38)$$

where $K_o = Me^{\omega}\left(\frac{K\omega^2}{e^{\omega}-1}+1\right)$.

Let again $t_o \ge 0$ and $t \ge t_o + 1$. Then for $s \in [t_o, t]$ we have

$$\|\Phi(t,t_o)\|^2 \le K_o\|\Phi(s,t_o)\|^2 \quad (39)$$

It follows that

$$\|\Phi(t,t_o)\|^2 \le K_o \int_{u-1}^{u}\|\Phi(t,s)\|^2 ds \text{ for } u \in [t_o+1, t] \quad (40)$$

and hence

$$(t - t_o - 1)\|\Phi(t,t_o)\|^2 \le K_o \cdot \int_{t_o+1}^{t}\left(\int_{u-1}^{u}\|\Phi(t,s)\|^2 ds\right)du \le K_o \cdot K \quad (41)$$

For all$t \ge t_o + 1$. So that $\Phi$ is uniformly exponentially stable.

**Proposition 2.7**

Let $\Phi$ be an evolutionary process on a Hilbert space X. If there is $\alpha > 0$ and a function $H: \Re_+ \to (0,\infty)$ such that

$$\int_t^{\infty}\left(\int_u^{u+1} e^{\alpha(s-t)}\|\Phi(s,t)x\|^2 ds\right)du \le H(t)\|x\|^2 \quad (42)$$

for all $t \ge 0$ and $x \in X$, then there is a function $N:\Re_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_o)\|^2 \le N(t_o)e^{\alpha(t-t_o)} \text{ for all } t \ge t_o \ge 0. \quad (43)$$

Hence, $\Phi$ is non-uniformly exponentially stable.

Proof

Let $t_o \ge 0, t \ge t_o + 1$ and $x \in X$. We have

$$\|\Phi(t,t_o)x\|^2 \le Me^{\omega(t-s)}\|\Phi(s,t_o)x\|^2 \quad (44)$$

And by integration we infer

$$e^{-\alpha t_o} \cdot \frac{e^{\omega+\alpha}-1}{(\omega+\alpha)^2}\left(e^{(\omega+\alpha)(t-1)} - e^{(\omega+\alpha)t_o}\right)\|\Phi(t,t_o)x\|^2$$

$$\le Me^{\omega t}\int_u^{u+1} e^{\alpha(s-t_o)}\|\Phi(s,t_o)x\|^2 ds \quad (45)$$

for $u \in [t_o, t-1]$, and so

$$e^{-\alpha t_o} \cdot \frac{e^{\omega+\alpha}-1}{(\omega+\alpha)^2}\left(e^{(\omega+\alpha)(t-1)} - e^{(\omega+\alpha)t_o}\right)\|\Phi(t,t_o)x\|^2$$

$$\le Me^{\omega t}\int_{t_o}^{t-1}\left(\int_u^{u+1} e^{\alpha(s-t_o)}\|\Phi(s,t_o)x\|^2 ds\right)du \quad (46)$$

from which

$$\left(e^{\alpha(t-t_o)-(\omega+\alpha)} - e^{-\omega(t-t_o)}\right)\|\Phi(t,t_o)x\|^2$$

$$\leq \ M\frac{(\omega+\alpha)^2}{e^{(\omega+\alpha)}-1}H(t_o)\|x\|^2. \tag{47}$$

It follows that

$$e^{\alpha(t-t_o)}\|\Phi(t,t_o)x\|^2 \ \leq \ e^{(\omega+\alpha)}\left(\frac{(\omega+\alpha)^2}{e^{(\omega+\alpha)}-1}H(t_o)+M\right)\|x\|^2. \tag{48}$$

DenotingType equation here.

$$N(t_o) = \ Me^{(\omega+\alpha)}\left(\frac{(\omega+\alpha)^2}{e^{(\omega+\alpha)}-1}H(t_o)+1\right) \tag{49}$$

we obtain

$$\|\Phi(t,t_o)x\|^2 \ \leq \ N(t_o)e^{-\alpha(t-t_o)} \text{ for all } t \geq t_o \ \geq 0. \tag{50}$$

**Proposition 2.8**

Let $\Phi$ be an evolutionary process on a Hilbert space X. If there are $\alpha \ > 0$ and a function $H:\Re_+ \to (0,\infty)$ such that

$$\int_{t_o}^{t}\left(\int_{u-1}^{u} e^{\alpha(t-s)}\|\Phi(t,s)\|^2 ds\right)du \ \leq \ H(t_o) \tag{51}$$

for all $t \geq t_o \geq 1$, then there is a function $N:\Re_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_o)\|^2 \ \leq \ N(t_o)e^{-\alpha(t-t_o)} \text{ for all } t \geq \ t_o \geq 0. \tag{52}$$

Proof

$\alpha$

$$\|\Phi(t,t_o)\|^2 \ \leq \ Me^{\omega(s-t_o)}\|\Phi(t,s)\|^2 \tag{53}$$

So that

$$e^{-(\omega+\alpha)s}\cdot e^{\omega t_o}\cdot e^{\alpha t}\|\Phi(t,t_o)\|^2 \ \leq \ Me^{\omega(t-s)}\|\Phi(t,s)\|^2 \text{for } s \in [t,t_o]. \tag{54}$$

Integrating first with respect to s on [u-1,u], where u $\epsilon$ $[t_o+1,t]$ and then with respect to u on $[t_o+1,t]$, we obtain

$$\frac{e^{\omega+\alpha}-1}{(\omega+\alpha)^2}e^{\alpha t+\omega t_o}\left(e^{-(\omega+\alpha)(t_o+1)}-e^{-(\omega+\alpha)t}\right)\|\Phi(t,t_o)\|^2 \ \leq \ MH(t_o+1). \tag{55}$$

Consequently,

$$\|\Phi(t,t_o)\|^2 e^{\alpha(t-t_o)}\cdot e^{-(\omega+\alpha)} \leq \ \frac{M(\omega+\alpha)^2 H(t_o+1)}{e^{\omega+\alpha}-1}+ \ e^{-\omega(t-t_o)}\|\Phi(t,t_o)\|^2 \tag{56}$$

So it follows that

$$\|\Phi(t,t_o)\|^2 e^{\alpha(t-t_o)} \ \leq \ Me^{\omega+\alpha}\left(\frac{H(t_o+1)(\omega+\alpha)^2}{e^{\omega+\alpha}-1}+1\right). \tag{57}$$

Let

$$N(t_o) \ \leq \ Me^{\omega+\alpha}\left(\frac{H(t_o+1)(\omega+\alpha)^2}{e^{\omega+\alpha}-1}+1\right) \tag{58}$$

then it follows that

$$\|\Phi(t,t_o)\|^2 \ \leq \ N(t_o)e^{-\alpha(t-t_o)} \ \text{ for } t \ \geq \ t_o+1. \tag{59}$$

If $0 \ \leq \ t_o \ \leq t \ \leq \ t_o+1$, we have

$$\|\Phi(t, t_o)\|^2 \leq Me^{\omega+\alpha} \cdot e^{\alpha(t-t_o)} \leq N(t_o)e^{-\alpha(t-t_o)} \tag{60}$$

So that

$$\|\Phi(t, t_o)\|^2 \leq N(t_o)e^{-\alpha(t-t_o)} \text{ for all } t \geq t_o \geq 0. \tag{61}$$

## III. Conclusion

We considered stability concepts of linear evolution operators defined on a Hilbert space. The stability concepts we specialized to a general Hilbert space include non-uniform stability and non-uniform exponential stability. The results obtained specialized similar concepts obtained by Buse [4], Datko [2], Ichikawa [3], Pandolfi [5], Preda, Latcu and Preda [6] for non-uniform exponential stability.

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