

On Application of Modified Lagrange Multipliers for Solving Optimization Problems in both Equality and Inequality Constraints

¹S. A. Olorunsola, ²T. E. Olaosebikan And ³K. J. Adebayo
^{1,2,3}Department of Mathematical Sciences, Ekiti State University, Ado Ekiti, Nigeria.

Abstract: This paper discusses the application of a Modified Lagrange Multipliers Method (MLM) in solving optimization problems with equality and inequality constraints. The method is aimed at circumventing the computational rigours undergone using the Lagrange multipliers method in solving this class of problems with equality and inequality constraints independently. Also, it aims at reducing the computer memory space occupied by the independent methods in solving these problems using the said methods.

In other to achieve this, a straightforward and simple to handle MLM algorithm has been developed to solve the same optimization problems with both equality and inequality constraints. Comparing the numerical results with that of the classical methods show some improvements.

Keywords: Lagrange Multipliers, Modified Lagrange Multipliers, Optimization Problems with Equality Constraints, Optimization Problems with Inequality Constraints.

I. Introduction

The general optimization problem to be considered is of the form [1] and [2] as:

$$\begin{array}{ll} \text{Optimize:} & f(x) & 1.1 \\ \text{Subject to:} & h_i(x) = 0 \quad i = 1, 2, \dots, m_1 & 1.2 \\ & g_j(x) \geq 0 \quad j = 1, 2, \dots, m_2 & 1.3 \\ & x \geq 0 & 1.4 \end{array}$$

where $x \in R^n$, $h_i(x)$, a vector equations of dimension m_1 , and $g_j(x)$ is a vector of inequality of dimension m_2 , such that the sum of the constraints $m = (m_1 + m_2)$. The functions $f(x)$, $h_i(x)$ and $g_j(x)$ are differentiable functions. Methods for solving this model have been developed, tested and successfully applied to many important problems of scientific and economic interest. However, in spite of the proliferation of the methods, there is no universal method for solving all optimization problems which calls for application of MLM for solving optimization problems that comprises of the equality and inequality constraints.

II. Computational Procedure of the MLM Algorithm

Before discussing the computational procedure of the modified Lagrange multipliers to solving this kind of optimization problems which is our major concern, it is expedient to briefly discuss how the equality and inequality type of the said problems have been solved independently with some supporting theorems which are advantageous in other to get the clear picture of what the analysis is all about and basis for the MLM algorithm.

Theorem 2.1:

Let f and $g_i \in C^1$, and given the existence of the Lagrange multiplier λ^* , then the following conditions have to be satisfied at the point x^* that corresponds to the solution of the primal problem:

$$\left. \begin{array}{l} \frac{\delta f}{\delta x_j}(x^*) + \sum_{i=1}^m \lambda^* \frac{\delta g_i}{\delta x_j} = 0, j = 1, 2, \dots, n \\ g_i(x^*) \leq 0, \quad i = 1, 2, \dots, m \\ \lambda^* g_i(x^*) = 0, \quad i = 1, 2, \dots, m \\ \lambda_i^* \geq 0, \quad i = 1, 2, \dots, m \end{array} \right\} \quad 2.1$$

Or in a more compact notation form:

$$\begin{array}{l} \nabla_x L(x^*, \lambda^*) = 0 \\ \nabla_\lambda L(x^*, \lambda^*) \leq 0 \\ \lambda^{*T} g(x^*) = 0 \\ \lambda^* \geq 0. \end{array}$$

These conditions are known as the Karush-Kuhn-Tucker (KKT) stationary conditions [3].

Proof:

First convert the inequality constraints into equality constraints by introducing the slack variables s_i :

$$g_i(x) + s_i = 0, i = 1, 2, \dots, m \tag{2.2}$$

$$s_i \geq 0.$$

Define the corresponding Lagrange function as:

$$L(x, s, \lambda) = f(x) + \sum_{i=1}^m \lambda_i (g_i(x) + s_i). \tag{2.3}$$

Assume that the solution (1.1), (1.2), (1.4) with the constraints (2.2) is given by x^*, s^* .

Now distinguish between the two possibilities:

(i) Let $s_i^* > 0$ for all i . In the case the problem is identical to the usual minimization problem with equality constraints which is solved using Lagrange multipliers. Here there are m additional variables

s_1, s_2, \dots, s_m . Hence the necessary conditions for the minimum are:

$$\frac{\delta L}{\delta x_j}(x^*, s^*, \lambda^*) = \frac{\delta f}{\delta x_j}(x^*) + \sum_{i=1}^m \lambda_i \frac{\delta g_i}{\delta x_j}(x^*) = 0, j = 1, 2, \dots, n \tag{2.4}$$

$$\frac{\delta L}{\delta s_i}(x^*, s^*, \lambda^*) = \lambda_i^* = 0, i = 1, 2, \dots, m. \tag{2.5}$$

As $\lambda_i^* > 0$, it also follows that $g_i(x^*) < 0$ and with the fact that $\lambda_i^* = 0$ this yields:

$$\lambda_i^* g_i(x^*) = 0, i = 1, 2, \dots, m. \tag{2.6}$$

Consequently all the conditions of the theorem hold for the case $s_i^* > 0$ for all i .

(i) Let $s_i^* = 0$ for $i = 1, 2, \dots, p$ and $s_i^* > 0, i = p + 1, \dots, m$.

In this case, the solution may be considered to the solution of an equivalent minimization problem with the following equality constraints:

$$g_i(x) = 0, i = 1, 2, \dots, p \tag{2.7}$$

$$g_i(x) + s_i = 0, i = p + 1, \dots, m. \tag{2.8}$$

Applying (2.4), (2.5) and (2.7) it follows that (2.6).

Obviously,

$$g_i(x^*) < 0, i = p + 1, \dots, m \tag{2.9}$$

And from (2.7) it follows that:

$$g_i(x^*) \leq 0, i = 1, 2, \dots, m. \tag{2.10}$$

However, no information concerning $\lambda_i^*, i = 1, \dots, p$ is available. This is obtained from the following additional argument. Consider feasible changes from x^*, s^* in all variables $x_1, \dots, x_n, s_1, \dots, s_m$. Again consider m of these as dependent variables and the remaining n as independent variables. If $p \leq n$ then, s_1, \dots, s_p can always be included in the set of independent variables. (Find λ^* by putting the partial derivatives of L at x^*, s^* with respect to the dependent variables, equal to zero and solving for λ^*). As $ds_i > 0 (s_i^* = 0)$ must apply for feasible changes in the independent variables s_1, \dots, s_p , it follows that in general for changes which are consistent with the equality constraints, that $df \geq 0$ for changes involving s_1, \dots, s_p . Thus if these independent variables are varied one at a time, since all the partial derivatives of L with respect to the dependent variables must be equal to zero, that:

$$df = \frac{\delta L}{\delta s_i}(x^*, s^*, \lambda^*) ds_i = \lambda_i^* ds_i \geq 0, i = 1, 2, \dots, p. \tag{2.11}$$

As $ds_i > 0$, it follows that, $\lambda_i \geq 0, i = 1, 2, \dots, m$. Thus, since it has already been proved that $\lambda_i^* = 0, i = p + 1, \dots, m$, it follows that indeed $\lambda_i \geq 0, i = 1, 2, \dots, m$. This completes the proof of the theorem.

Remark 1: Obviously, if an equality constraint, $h_k(x) = 0$ is also prescribed explicitly, then, s_k does not exist and nothing is known of the sign of λ_k^* as $\frac{\delta L}{\delta s_k}$ does not exist.

Case 1: Equality Constraints

In this case, (1.1) and (1.2) are to be considered only, the problem is to determine a point x^* which yields a strong relative minimum or maximum for (1.1) and also satisfies (1.2) and to achieve this using Lagrange method is by appending (1.2) to (1.1) with Lagrange multipliers:

$$\lambda_i, i = 1, 2, \dots, m. \tag{2.12}$$

The new (1.1) now becomes:

$$L(x, \lambda) \tag{2.13}$$

where (2.13) is called Lagrangian. Now, (1.1) and (2.13) becomes:

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) \tag{2.14}$$

(2.14) is called Lagrangian function which is now unconstrained. We now state the necessary conditions for stationarity as follows:

$$\frac{\delta L}{\delta x_j} = \frac{\delta f}{\delta x_j} + \sum_{i=0}^m \lambda_i \frac{\delta h_i}{\delta x_j} = 0, j = 1, 2, \dots, n, \tag{2.15}$$

$$\frac{\delta L}{\delta \lambda_i} = h_i(x) = 0, i = 1, 2, \dots, m, \tag{2.16}$$

$$(x^*, \lambda^*) \tag{2.17}$$

(2.15) and (2.16) yield a set of $m + n$ equations in $m + n$ unknowns (x, λ) to be solved for the (2.17).

$$L(x^*, \lambda^*) \tag{2.18}$$

Remark 2: The condition (2.16) guarantees that (1.2) is satisfied at (2.18).

$$f(x^*) \tag{2.19}$$

where (2.17), (2.18) and (2.19) are optimal value, optimal solution and optimal returns respectively. In this case, (1.2) and (2.17) of (2.14) is equal to (2.19) for (1.1).

Case 2: Inequality constraint

In this case, (1.1) and (1.3) are to be considered only. Here, a real-valued slack variable θ_j is been defined for each (1.3) such that:

$$\theta_j^2 = g_j(x) \geq 0, j = 1, 2, \dots, r \tag{2.20}$$

(1.3) is satisfied by satisfying (2.20) for all real-valued θ_j . Applying the Lagrange necessary conditions by [4] and [5], we obtain:

$$\left. \begin{aligned} L(x, \lambda, \theta) &= f(x) + \sum_{j=0}^m \lambda_j (g_j(x) - \theta_j^2) \\ \frac{\delta L}{\delta x_i} &= \frac{\delta f}{\delta x_i} + \sum_{j=0}^m \lambda_j \frac{\delta g_j}{\delta x_i} = 0, i = 1, 2, \dots, n \\ \frac{\delta L}{\delta \lambda_i} &= g_j(x) - \theta_j^2 = 0, j = 1, 2, \dots, m, \\ \frac{\delta L}{\delta \theta_j} &= -2\lambda_j \theta_j = 0, j = 1, 2, \dots, m. \end{aligned} \right\} \tag{2.21}$$

From (2.21), either $\lambda_j^* = 0$ or $\theta_j^* = 0$ or both.

Having gotten the clear knowledge of independent solution of this class of problems, we now want to see if giving a problem that comprises both cases discussed earlier in this paper, can be solved using the same method. Base on this, we state the MLM algorithm and procedure in the next section.

III. The MLM Algorithm for solving Equality and Inequality Optimization Problems

- Step 1: Change the inequality constraints into equality by introduction slack variables as in (2.20)
- Step 2: Equate the equality constraints to zero.
- Step 3: Introduce Lagrange multipliers as in (2.12)
- Step 4: Formulate Lagrangian function as in (2.14)
- Step 5: Apply the Lagrange multiplier method's necessary conditions (2.21) to the formulated Lagrange in step 4.
- Step 7: Hence, determine the optimal value, optimal solution and optimal function return as in (2.17), (2.18) and (2.19) respectively.

Remark 3: (1) The slack variable can either tend to or equal to zero. Also, either addition or subtraction can be used when converting the constrained to unconstrained problem by the Lagrange multiplier.

IV. Computational Results

The following problems were evaluated using the MLM algorithm thus:

Problem (P1):

$$\begin{aligned} \text{Maximize } f(x) &= 7x_1^2 + 6x_1 + 5x_2^2, \\ \text{Subject to: } & x_1 + 2x_2 \leq 10, \\ & x_1 - 3x_2 = 9, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Problem (P2):

$$\begin{aligned} \text{Maximize } f(x) &= 2x_1 - x_1^2 + x_2, \\ \text{Subject to: } & 2x_1 + 3x_2 = 6, \\ & 2x_1 + x_2 \leq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Problem (P3):

Minimize $f(x) = 2x_1 + 3x_2 - (x_1^2 + x_2^2 + x_3^2)$,

Subject to: $x_1 + x_2 \leq 1$,
 $2x_1 + 3x_2 = 6$,
 $x_1, x_2 \geq 0$.

Problem (P4):

Minimize $f(x) = 4x_1 + 9x_2 - x_1^2 - x_2^2$,

Subject to: $4x_1 + 3x_2 = 15$,
 $3x_1 + 5x_2 \geq 14$,
 $x_1, x_2 \geq 0$.

Problem (P5):

Minimize $f(x) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$,

Subject to: $x_1 + x_2 + x_3 \geq 15$,
 $2x_1 - x_2 + 2x_3 = 20$,
 $x_1, x_2, x_3 \geq 0$.

Problem (P6):

Minimize $f(x) = x_1^2 + x_2^2 + x_3^2$,

Subject to: $x_1 + x_2 + 3x_3 \leq 2$,
 $5x_1 + 2x_2 + x_3 = 5$,
 $x_1, x_2, x_3 \geq 0$.

V. Conclusion

In this paper, the algorithm developed has been tested on some problems with corresponding results shown in Table 1 below. From the table, problem 1 with numerical result 702.920 coincide with the analytical result which is 702.920, problem 2 with numerical result 1.750 is close to the analytical result which is 2.444, problem 5 with the numerical result 32.000 is less than the analytical result which is 91.111, also, problem 6 with numerical result of 0.783 is close to the analytical result which is 0.847. However, problem 3 and 4 with numerical results 2.309 and 22.56 are greater than the analytical results 2.125 and 11.000 respectively. This shows that the method has worked for problem 1, 2, 5, 6 and is been violated slightly by problem 3 and 4. With these, it is invariably justified clearly that, the MLM can be used to solve optimization problems of both equality and inequality type.

References

- [1]. RAO, S. S., (1978), Optimization Theory and Applications, Willy and Sons.
- [2]. THOMAS, F.E., and DAVID, M.H., (2001), Optimization of Chemical Processes, McGraw Hill
- [3]. IGOR GRIVA STEPHEN G and NASH ARIELA SOFER (2009), Linear and Nonlinear Optimization, George Mason University Fairfax, Virginia.
- [4]. JAN A. SNYMAN (2005), Practical Mathematical Optimization, University of Pretoria, Pretoria, South Africa.
- [5]. RICHARD BRONSON and GOVINDASAMA NAADIMUTHU.(2007), Operations Research, Tata McGraw-Hill Publishing Company Limited, New Delhi.

Table of Results:

Table 1:

Problems	Numerical Results				Analytical Results			
	x_1	x_2	x_3	$f(x)$	x_1	x_2	x_3	$f(x)$
1	9.600	0.200	-	702.920	9.600	0.200	-	702.920
2	1.500	1.000	-	1.750	0.667	1.556	-	2.444
3	0.462	0.693	0.000	2.309	0.250	0.750	0.000	2.125
4	0.960	3.720	-	22.56	3.000	1.000	-	11.000
5	2.000	0.000	8.000	32.000	3.667	3.333	8.000	91.111
6	0.808	0.323	0.162	0.783	0.804	0.348	0.283	0.847