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Pseudo Weakly N-Projective Modules

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Abstract: *A module M is said to be weakly projective iff it has a projective cover $\pi : P(M) \longrightarrow M$ and every mapping P(M) into a finitely generated module can be factored through M via an epimorphism. In particular, if M and N are two R-modules and assume M has a projective cover $\pi : P \longrightarrow M$, We say that M is pseudo weakly N-projective if for every map $\psi : P \longrightarrow N$ there exists an epimorphism $\sigma : P \longrightarrow M$ and a homomorphism $g : M \longrightarrow N$ such that $\psi = g.\pi$. In this paper we generalize the basic properties of pseudo weakly projective modules.*
***Keywords**: Pseudo projective module, pseudo weakly projective module, Projective cover.*

Definition:1
An R-module M is said to be pseudo projective if for any given R-module A, epimorphisms $g : M \to A$ and $f : M \to A$ there exists a homomorphism $h : M \to M$ such that $f = g.h$

Definition :2
We say that M is weakly M-Projective [Quasi Weakly Projective), If M has a projective cover π: $P(M) \to M$ and every homomorphism $\Psi : P(M) \to N$ can be factored through M via some epimorphism. Equivalently, a module M is weakly M projective if it has a projective cover $\pi : P(M) \to M$ and given any homomorphism $\Psi : P(M) \to N$ there exists $X \subseteq$ ker Ψ such that $\frac{P(M)}{X} \cong M$.

Remarks:
1.Every projective module is pseudo projective module.
2.If every module has a quasi projective cover then it has a pseudo projective cover.
3. If every module has a pseudo projective cover then it has a projective cover.
4. Every semi-simple projective module is pseudo projective module.

Theorem :1.1
Let M and N are two R-modules and assume M is N-projective cover P via an onto homomorphism π : P $\longrightarrow$M then M is N projective iff for every homomorphism ψ:P $\longrightarrow$ N, there exists a homomorphism ϕ: M $\longrightarrow$N such that $\phi.\pi=\psi$. Equivalently ψ (kerπ) =0
Proof :
Only if direction :- Let ψ : P $\longrightarrow$ N is a homomorphism. We shall first show that ψ(Kerπ)=0. Let T=ψ(Kerπ) and Let π_T : N $\longrightarrow \frac{N}{T}$ be the natural projection. Then ψ induces ϕ : $M \longrightarrow \frac{N}{T}$ defined by ϕ(m) = $\pi_T.\psi(\rho)$ where M=$\pi(\rho)$. Clearly $\phi\pi = \pi_T\psi$. Since M is N-projective, there exists a map β : M$\longrightarrow$ N such that $\phi = \pi_T.\beta$. Clearly $(\Psi - \beta\pi)P \subseteq T$. We claim that $\psi=\beta\pi$.
Let x={$\rho\in$P | $\psi(\rho) = \beta\pi(\rho)$}. We show that X = P. Let z $\in$ P.
Since (ψ-$\beta\pi$) (x) $\in$ T = ψ (Kerπ), there exists k$\in$Ker π such that
(ψ-$\beta\pi$) (x)= ψ(k). Therefore ψ(z-k) - $\beta\pi$ (z-k)=0, since $\beta\pi$(k)=0. Thus z-k$\in$X.
Therefore Kerπ + X = P which implies X = P
Since Ker π is small in P. Therefore (ψ - $\beta\pi$)P=0
In particular (ψ-$\beta\pi$) Kerπ=0, yielding ψ (Kerπ)=0. Equivalently, there exists ψ' : M $\longrightarrow$ N such that $\psi'_\pi=\psi$.

Conversely

Let $\psi : M \longrightarrow \frac{N}{k}$ is a homomorphism. Then by the projectivity of P there exists a homomorphism $\psi' : P \longrightarrow N$ such that $\psi\pi = \pi_k\psi'$.

It follows easily that $\pi_k\phi = \psi$ as desired. The above result is dual to a well-known characterization of relative injectivity.

Definition: <u>Pseudo weakly N-projective</u>

Let M and N are two R-modules and assume M has a projective cover $\pi : P \longrightarrow M$. We say that M is pseudo weakly N-projective if for every map $\psi : P \longrightarrow N$ there exists an epimorphism $\sigma : P \longrightarrow M$ and a homomorphism $g : M \longrightarrow N$ such that $\psi = g.\pi$.

Theorem -1.2

Let M and N are two R-modules and assume M has a projective cover $\pi : P \longrightarrow M$. Then the following statements are equivalent :

1. M is pseudo weakly N-projective.
2. For every sub module $K \subset N$, M is pseudo weakly projective.
3. For every sub module $K \subset N$, M is pseudo weakly $\frac{N}{K}$-projective.

Proof :

(i) (1) Implies (2) and (3) Assume M is pseudo weakly N-projective and let K is a sub-module of N and $\psi : P \longrightarrow K$ is a homomorphism. Then $\psi = i_\pi$. $\psi : P \longrightarrow N$ may be expressed as a composition $\psi = g\sigma$ for some homomorphism $g : M \longrightarrow N$ and epimorphism $\sigma : P \longrightarrow M$. Since σ is onto, the range of σ equals the range of g and so it is contained in K. Thus we may define $g : M \longrightarrow K$ via $\psi(m) = g(m)$ and then $\psi = g\sigma$, proving that M is pseudo weakly K-projective as claimed. Assume once again that M is pseudo weakly N-projective and let $f : P \longrightarrow N/K$ is a homomorphism. Since P is projective, there exists a map $\bar{f} : P \longrightarrow N$ such that $f = \pi_x.\bar{f}$. The weakly N-projective of M yields an epimorphism $\sigma : P \longrightarrow M$ and a homomorphism $h : M \longrightarrow N$ such that $\bar{f} = h.\sigma$. Let $\pi_{\underline{x}}.h = f_1$ then $f_1\sigma = \pi_k.h.\sigma = \pi\bar{f} = f$, proving that M is indeed pseudo weakly $\frac{N}{K}$ projective.

(ii) (2) or (3) implies (1) is trivially.

Remarks :

Let M and N are two R-modules and assume M has a projective cover $\pi : P \longrightarrow M$. Then M is pseudo weakly N-projective if and only if for every sub-module $K \subset N$ and for every epimorphism $\psi : P \longrightarrow K$ there exist epimorphism $\sigma : P \longrightarrow M$ and $g : M \longrightarrow N$ such that $\psi = g.\sigma$.

Theorem:1.3

Let M and N are two R-modules and assume M has a projective cover $\pi : P \longrightarrow M$. Then M is pseudo weakly N-projective iff for every map $\psi : P \longrightarrow N$ there exist a sub module $X \subset \mathrm{Ker}\psi$ such that $\frac{P}{X} \cong M$.

Proof :

Necessary condition :

Let $\psi : P \longrightarrow N$ is a homomorphism. Assume M is pseudo weakly N-projective and let the homomorphism $g : M \longrightarrow N$ and the epimorphism $\sigma : P \longrightarrow M$ be as in the definition of weakly relative projectivity. Since $\psi = g.\sigma$, $\mathrm{Ker}\,\sigma \subset \mathrm{Ker}\,\psi$. Also $\frac{P}{Ker\sigma} \cong M$ Thus the implication is proven by choosing $X = \mathrm{Ker}\,\sigma$.

Conversely :

If X ⊂ P satisfies the condition in the statement of the theorem, then the isomorphism $\frac{P}{X} \cong M$, composed with the natural projection π_k : P ⟶ P/X is an epimorphism σ : P ⟶ M satisfying that Ker σ = X ⊂ Ker ψ. It follows that the map g : M ⟶ N given by g(m)= ψ (ρ) whenever σ(ρ)=M is well defined and satisfies ψ=g.σ

Theorem :1.4

Let M and N are two R-modules and assumed M is supplemented and has a projective cover π : P ⟶M. Then M is pseudo weakly N-projective if and only if for every sub-module k⊂N and for every epimorphism ψ : P ⟶K there exists an epimorphism g : M ⟶K such that for every supplement L' of Kerg in M there exists a sub-module L⊂P such that $\frac{P}{L} \cong \frac{M}{L'}$ and L + Kerψ=P.

Proof :

Necessary Condition :

Assume M is pseudo weakly N-projective and Let ψ : P ⟶ K is an epimorphism onto a sub-module K⊂N. Then there exists epimorphism σ:P ⟶ M and g:M⟶ K such that ψ=gσ. Let L' is a supplement of Kerg in M and L=σ^{-1}(L'). For an arbitrary p∈P, σ(p) may be written as σ(P) = l'+k', with l'∈L' and k'∈Kerg. It follows then that

ψ (p) =gσ(p) = g(l') + g(k')=g(l').

Choose $p_1 \in \sigma^{-1}$ (l')⊂L. Then σ(p_1)=l'. On the other hand,

ψ (p_1)= gσ(p_1) = g'(l') = ψ(p). So P-P_1∈Ker ψ and so L+Ker ψ=P. The fact that $\frac{P}{L} \cong \frac{M}{L'}$ follows, since L is the kernel of the onto map π_L.σ: P⟶$\frac{M}{L'}$.

Sufficient Condition :

Let us assume that for every sub-module K ⊂N and for every epimorphism ψ : P ⟶K there exist an epimorphism g:M ⟶K such that for every supplement L' of Kerg in M there exist a sub-module L⊂P such that $\frac{P}{L} \cong \frac{M}{L'}$ and L+Kerg=P. Let ψ : P ⟶K is an epimorphism and g:M⟶K be the corresponding epimorphism. All we need is to produce another epimorphism σ:P⟶M such that ψ=gσ. Let L' is a supplement for Kerg and Let L be the corresponding sub-module of P. Let $\theta : \frac{P}{L} \longrightarrow \frac{M}{L'}$ is an isomorphism. By Chinese remainder theorem that the map M+Kerg ∩L' ⟶(M+Kerg, M+L') is an isomorphism between $\frac{M}{(Ker\, g \cap L')}$ and $\frac{M}{\left(Ker\, g \times \frac{M}{L'}\right)}$.

Also $\frac{M}{Ker\, g} \cong K$ via M+Ker g = g(m)

g = $\theta\pi_L$: P $\longrightarrow \frac{M}{L'}$. Since Ker ψ + L = P, the map α : P $\longrightarrow K \times \frac{M}{L'}$ given by α(p) = $(\psi(p), g(p))$ is onto.

The induced epimorphism $\alpha^1 = \beta^{-1}\alpha : P \longrightarrow \frac{M}{(Kerg \cap L')}$ may then the lifted to a map σ : P ⟶M. Since Kerg ∩L' ≤ M. σ is indeed an epimorphism. It only remains to show that g.σ= ψ. Let us refer for the rest of this proof to $\pi_{Kerg \cap L'}$ simply as π. We know that πσ = σ'= β'α hence βπσ=α. Let p∈P be arbitrary. Then β (α(P) +Ker g$\cap$L')=α(p) = (ψ(p), g(p)). On the other hand β(σ(P)+Kerg$\cap$L')=(g(σ(p)), σ(p)+L'). Comparing the first component in both expression yields the desired equality. Thus M is pseudo weakly N-projective.

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