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A New Notion of Generalized Closed Sets in Topological Spaces

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Abstract: *In this paper, We introduce a new Class of Closed set namely $\widehat{P}g$ Closed Sets . We give characterizations for $\widehat{P}g$ Closed Sets and Open sets Further We investigate some of their Fundamental properties.*
Keywords: $\widehat{P}g$ -Closed sets, $\widehat{P}g$ -Open sets , Closed sets,Open set , $\widehat{P}g$ Closure, $\widehat{P}g$ Interior.

I. Introduction:

The concept of generalized Closed sets introduced by Levine[7] plays a significant role in General Topology. This notion has been studied extensively in recent years by many topologists. The investigation of generalized Closed sets has led to several new and interesting concepts, e.g. new covering Properties and new separation axioms weaker than T_1. Some of these Separation axioms have been found to be useful in computer science and digital topology. As an example, the well-known digital line is a $T_{3/4}$ space but fails to be a T_1 space. After the introduction of generalized Closed sets there are many research papers which deal with different types of generalized Closed sets. Dunham [5] [6] further investigated the Properties of $T_{½}$ spaces and defined a new Closure operator Cl* by using generalized Closed sets. Levine [8] introduced the concept of Semi Open sets and Semi continuity in a topological space. Bhattacharya and Lahiri introduced a new Classes of Semi generalized Open sets by means of Semi Open sets introduced by Levine. A subset $S \subset X$ is called PreOpen [11] (resp. α-Open [12] and β-Open [4] (or Semi-PreOpen [1])),if $S \subset Int(Cl(S))$ (resp. $S \subset Int(Cl(Int(S))$ and $S \subset Cl(Int(Cl(S)))$. The complement of a PreOpen set is called a PreClosed [11]set. The family of all PreOpen (resp. PreClosed) sets of a space X is denoted byPO(X, τ) (resp. PC(X, τ)). The intersection of all PreClosed sets containing S is called the PreClosure of S [11] and is denoted by pCl(S). The Preinterior [11] of S is defined by the union of all PreOpen sets contained in S and is denoted by pInt(S). In 1996, H.Maki, J. Umehara and T. Noiri [16] introduced the Class of Pregeneralized Closed sets and used them to obtain Properties of Pre-$T_{1/2}$ spaces. The modified forms of generalized Closed sets and generalized continuity were studied by K. Balachandran, P. Sundaram and H. Maki [3]. M.K.R.S veera kumar et.al [17]introduced a new Classes of Open sets namely g* - Closed sets .This characterization paved a new pathway. Throughout this paper, spaces (X , τ) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X . The Closure of A and the interior of A are denoted by Cl(A) and Int(A) respectively.

The purpose of this paper is to define a new Class of generalized Closed sets called $\widehat{P}g$ Closed sets in topological spaces and study their basic properties. We obtain many interesting results ,To substantiate these results, suitable examples are given at the respective places.

Definition2.1: - A subset A of a topological space (X, τ) is called.
(1) a **Semi-Open** set[8] if $A \subseteq Cl(int(A))$ and a **Semi-Closed** set if $int(Cl(A)) \subseteq A$.
(2) a **Pre-Open** set[11] if $A \subseteq int(Cl(A))$ and a **Pre-Closed** set if $Cl(int(A)) \subseteq A$.
(3) an **α -Open** set[12] if $A \subseteq int(Cl(int(A)))$ and an **α -Closed** set if $Cl(int(Cl(A))) \subseteq A$.
(4) a **semi-PreOpen** set[1](= β **-Open**[1] if $A \subseteq Cl(int(Cl(A))))$ and a **semi-PreClosed** set (= **β-Closed**) if $int(Cl(int(A))) \subseteq A$.
(5) a **Semi*Open** set [14]if $A \subseteq Cl^*(int(A))$ and a S**emi*-Closed** set if $int^*(Cl(A)) \subseteq A$.
(7) a **Pre*-Open** set[15] if $A \subseteq int^*(Cl(A))$ and a **Pre*-Closed** set if $Cl^*(int(A)) \subseteq A$

Definition 2.2: - A subset A of a topological space (X,τ) is called
(1) a **generalized Closed** set(briefly **g-Closed**)[7] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is Open in (X,τ). The complement of a g-Closed set is called a **g-Open** set.
(2) a **Semi-generalized Closed** set(briefly **Sg-Closed**)[4] if $SCl(A) \subseteq U$ whenever $A \subseteq$ and U is Semi-Open in (X, τ).The complement of a Sg-Closed set is called a S**g-Open** set.
(3) a **generalized Semi-Closed** set(briefly **gs-Closed**)[2] if $SCl(A) \subseteq U$ whenever $A \subseteq U$ and U is Open in (X, τ).The complement of a gs-Closed set is called a **gs-Open** set.

(4) a **generalized α -Closed** set(briefly **gα-Closed**)[10] if **α** Cl(A) ⊆ U whenever A ⊆ U and U is Open in (X, τ).The complement of a **gα** -Closed set is called a **gα-Open** set

(5) an **α-generalized Closed** set(briefly **α g-Closed**)[9] if **α** Cl(A) ⊆ U whenever A ⊆ U and U is **α** Open in (X, τ).The complement of a **αg** -Closed set is called a **αg-Open** set.

(6) a **regular ω Closed** set(briefly **RW-Closed**)[3] if Cl(A) ⊆ U whenever A ⊆ U and U is regular SemiOpen in (X, τ). The complement of a **RW** -Closed set is called a **RW-Open** set.

(7) a ⍰- **Closed** set[13] if SCl(A) ⊆ U whenever A ⊆ U and U is μ-Open in (X, τ). The complement of a ⍰-Closed set is called a ⍰- **Open** set.

Definition 3.1: $\hat{P}$g **Closed set:**

A subset A of a topological space (X, τ) is called **$\hat{P}$g Closed set** if Pre*Cl(A) ⊆ U whenever A ⊆ U where U is Pre* Open and Pre* Cl(A) = {∩ F/ F is Pre* Closed and A ⊆ F }.The set of all $\hat{P}$g Closed sets is denoted by $\hat{P}$gC(X,τ)

Theorem3.2: Every Closed set is $\hat{P}$g Closed
The converse of the above theorem is not true in general as it can be seen from the following example.

Example 3.3:
X={a,b,c,d} τ ={Φ,{a},{b},{a,b},{b,c},{a,b,c},X}
Here {c} is $\hat{P}$g Closed but not Closed .

Theorem3.4: Every g Closed set is $\hat{P}$g Closed
The converse of the above theorem is not true in general as it can be seen from the following example.

Example 3.5:
X={a,b,c,d} τ ={Φ,{a,b},X}
Here {a} is $\hat{P}$g Closed but not g-Closed .

Theorem3.6: Every Pre Closed set is $\hat{P}$g Closed
The converse of the above theorem is not true in general as it can be seen from the following example.

Example 3.7:
X={a,b,c} τ ={Φ,{a},{a,b},X}
Here {b} is $\hat{P}$g Closed but not Pre Closed .

Theorem 3.8:
1. $\hat{P}$g Closedness is independent from Semi Closedness
2. $\hat{P}$g Closedness is independent from Semi* Closedness
3. $\hat{P}$g Closedness is independent from α-Closedness
4. $\hat{P}$g Closedness is independent from Semi generalized Closedness
5. $\hat{P}$g Closedness is independent from RW Closedness
6. $\hat{P}$g Closedness is independent from β Closedness
7. $\hat{P}$g Closedness is independent from Semi Pre Closedness
8. $\hat{P}$g Closedness is independent from S_{g*} Closedness
9. $\hat{P}$g Closedness is independent from ⍰- Closedness

Proof:
It follows from the following examples..

Example 3.9:
Let X={a,b,c,d} and τ ={Φ,{a},{b},{a,b},{a,b,c},X}
Here {a} is Semi Closed set but not $\hat{P}$g Closed set.
{a,b,d} is $\hat{P}$g Closed set but not Semi Closed set

Example 3.10:
Let X={a,b,c,d} and τ ={Φ,{a},{b},{a,b},{b,c},{a,b,c},X}
Here {a} is Semi *Closed set but not $\hat{P}$g Closed set.
{a,b,d} is $\hat{P}$g Closed set but not Semi *Closed set

Example 3.11:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},X}
Here {a,d} is $\hat{P}g$ Closed set but not α- Closed set and
X={a,b,c,d} τ={Φ,{a},{a,b},{a,b,c},X}
{b} is α- Closed set but not $\hat{P}g$ Closed set

Example 3.12:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},{a,b,d},X}
Here {a} is Sg-Closed set but not $\hat{P}g$ Closed set.
{a,b,d} is $\hat{P}g$ Closed set but not Sg- Closed set

Example 3.13:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},{a,b,d},X}
Here {c} is $\hat{P}g$ Closed set but not RW Closed set.
{a,b},{a,b,c} are $\hat{P}g$ Closed set but not RW Closed set

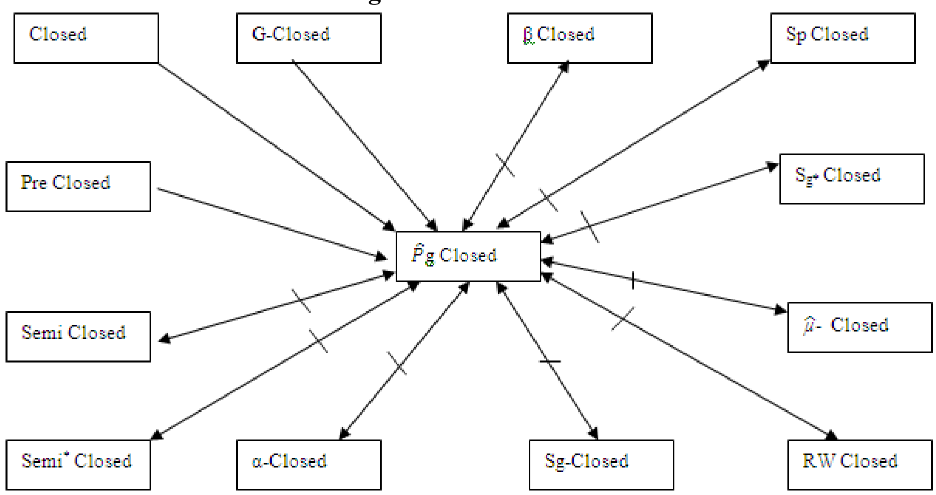
Example 3.14:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},{a,b,d},X}
Here {a},{b},{b,c},{a,c} are β- Closed set but not $\hat{P}g$ Closed set.
{a,b,d} is $\hat{P}g$ Closed set but not β- Closed set.

Example 3.15:
Let X={a,b,c} and τ={Φ,{a},X}
Here {a,b} and{a,c} are $\hat{P}g$ Closed. but not Semi Pre Closed and
Let X={a,b,c} τ={Φ,{a},{b},{a,b},X}.
{a},{b} are Semi Pre Closed but not $\hat{P}g$ Closed

Example 3.16:
Let X= {a, b, c, d} and τ={ Φ,{a, b},X}
Here {a,b} is $\hat{P}g$ Closed but not $S\,g^*$- Closed.
Let X= {a, b, c, d} and τ={ Φ,{a},{a, b},{a,b,c},X}
{c} is $S\,g^*$- Closed but not $\hat{P}g$ Closed

Example 3.17:
Let X={a,b,c} and τ={Φ,{a},{b},{a,b},X}
Here {a} ⍰- Closed set but not $\hat{P}g$ Closed set and
Let X={a,b,c} and τ={Φ,{a},{a,b},X}
{a,c} is $\hat{P}g$ Closed set but not ⍰- Closed set.
From the above results and discussions we have the following diagram,

Diagram 3.18



A $\longrightarrow$ B Which represents A implies B but not conversely.
A $\nleftrightarrow$ B Which represents A and B are independent.

Remark 3.19: The Union of Two $\widehat{P}$g Closed sets need not be $\widehat{P}$g Closed .
Proof:
The Proof is shown by the following example.
Let X= {a,b,c,d} and τ={ ϕ ,{a,b},{a,b,c},X}. In this space,
P*O(X, τ)= ρ(X)/{{d},{c,d}}
P*C(X, τ)= ρ(X)/{{a,b},{a,b,c}}
$\widehat{P}$gC(X, τ)= ρ(X)/{{a,b},{a,b,c}}
Here {a},{b} are $\widehat{P}$g Closed sets but their Union {a,b} is not $\widehat{P}$g Closed.

Definition 3.20: $\widehat{P}$g Closure

The intersection of all $\widehat{P}$g Closed sets containing A is called **$\widehat{P}$g Closure** of A and denoted by $\widehat{P}$g Cl(A). That is $\widehat{P}$g Cl(A) = $\cap$ {F: A $\subseteq$ F and F$\in\widehat{P}$g C(X)}

Theorem 3.21: Let A be a subset of X. Then A is $\widehat{P}$g Closed if and only if $\widehat{P}$g Cl(A)=A.
Proof: The Proof is obvious.

Theorem 3.22: Let A and B be subsets of (X, τ). Then the following results hold.
(i) $\widehat{P}$g Cl(φ)=φ and $\widehat{P}$g Cl(X)=X.
(ii) A$\subseteq\widehat{P}$g Cl(A).
(iii) If A$\subseteq$B, then $\widehat{P}$g Cl(A)$\subseteq$ $\widehat{P}$g Cl(B).
(iv) A$\subseteq\widehat{P}$g Cl(A)$\subseteq$PCl(A)$\subseteq$Cl(A).
Proof: the Proof is obvious.

.
Lemma 3.23:

For a space (X, τ), each x$\in$X {x} is Pre* Closed or its complement $\{x\}^C$ is $\widehat{P}$g Closed .

Proof:

Suppose that {x} is not Pre* Closed (X, τ).
Since $\{x\}^C$ is not Pre* Open , the space X itself is the only Pre* Open set containing $\{x\}^C$.
Therefore Pre*Cl($\{x\}^C$) $\subseteq$ X holds and so $\{x\}^C$ is $\widehat{P}$g Closed.

Theorem 3.24:

A set A is $\widehat{P}$g Closed iff Pre*Cl(A) - A contains no non empty Pre* Closed.

Proof :

Necessary Condition:

Let A be a $\widehat{P}$g Closed set in a topological space (X, τ).
Let F be Pre* Closed set such that F $\subseteq$ Pre*Cl(A) - A .
Since F is Pre* Closed, F^C is Pre* Open.
By the definition of $\widehat{P}$g Closed set we have ,
Pre*Cl(A) $\subseteq$ F^C , this implies F $\subseteq$ Pre*Cl(A)^C.
Then F $\subseteq$ Pre*Cl(A) $\cap$ Pre*Cl(A)^C.
Therefore F $\subseteq$ ϕ.

Sufficiency condition:

Let A be a set in (X, τ) with Pre*Cl(A) - A contains no non empty Pre* Closed set.
Let A $\subseteq$ O ,O is Pre* Open set .
If Pre*Cl(A) $\cap$ O^C $\neq$ ϕ
Then by the fact Pre*Cl(A) $\cap$ O^C $\subseteq$ Pre*Cl(A) - A , We obtain a contradiction.
This proves the sufficiency and hence the theorem.

Defnition4.1: $\widehat{P}$g Open set:

A Subset is said to be **$\widehat{P}$g Open** in (X,τ) if its complement (X-A) is $\widehat{P}$g Closed in (X,τ). The set of all $\widehat{P}$g Open sets is denoted by $\widehat{P}$gO(X,τ).

Lemma 4.2:
For x∈X, x ∈ $Pre^*Cl(A)$ iff V∩A = ϕ for every subset V ∈ $P^*O(X,\tau)$ such that x ∈ V.

Proof:
Necessity condition:
Suppose that there exists a subset V ∈ $P^*O(X,\tau)$ such that x ∈ V and V∩A = ϕ .
Then V ∈ (X-A) This implies $Pre^*Cl(A) \subseteq X/V$
Therefore x ∉ $Pre^*Cl(A)$.
Sufficiency condition:
Suppose that x ∉ $Pre^*Cl(A)$, then there exists a subset F of X such that A ⊆ F,
X \F ∈ $P^*O(X,\tau)$ and x ∉ F.
Since X \F contains x and X \F ∈ $P^*O(X,\tau)$.
This implies X \F ∩ A = ϕ.
Hence Proved.

Theorem4.3: Every Open set is $\hat{P}g$ Open
The converse of the above theorem is not true in general as it can be seen from the following example.

Example 4.4:
Let X={a,b,c,d} and τ ={Φ,{a},{b},{a,b},{b,c},{a,b,c},X}
Here {a,b,c} is $\hat{P}g$ Open but not Open .

Theorem4.5: Every g Open set is $\hat{P}g$ Open
The converse of the above theorem is not true in general as it can be seen from the following example.

Example 4.6:
Let X={a,b,c,d} and τ ={Φ,{a,b},X}
Here {b,c,d} is $\hat{P}g$ Open but not g-Open .

Theorem4.7: Every Pre Open set is $\hat{P}g$ Open
The converse of the above theorem is not true in general as it can be seen from the following example.

Example 4.8:
Let X={a,b,c} τ ={Φ,{a},{a,b},X}
Here {a,c} is $\hat{P}g$ Open but not Pre Open .

Theorem 4.9:
1. $\hat{P}g$ Openness is independent from Semi Openness
2. $\hat{P}g$ Openness is independent from $Semi^*$ Openness
3. $\hat{P}g$ Openness is independent from α-Openness
4. $\hat{P}g$ Openness is independent from Semi generalized Openness
5. $\hat{P}g$ Openness is independent from RW Openness
6. $\hat{P}g$ Openness is independent from β Openness
7. $\hat{P}g$ Openness is independent from Semi Pre Openness
8. $\hat{P}g$ Openness is independent from S_{g*} Openness
9. $\hat{P}g$ Openness is independent from [illegible]- Openness

Proof:
It follows from the following examples.
Example 4.10:
Let X={a,b,c,d} and τ ={Φ,{a},{b},{a,b},{a,b,c},X}
Here {b,c,d} is SemiOpen set but not $\hat{P}g$ Open set.
{c} is $\hat{P}g$ Open set but not Semi Open set

Example 4.11:
Let X={a,b,c,d} and τ ={Φ,{a},{b},{a,b},{b,c},{a,b,c},X}
Here {b,c,d} is $Semi^*$Open set but not $\hat{P}g$ Open set.
{c} is $\hat{P}g$ Open set but not $Semi^*$Open set

Example 4.12:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},X}
Here {b,c} is $\widehat{P}g$ Open set but not α- Open set.and
Let X={a,b,c,d} τ={Φ,{a},{a,b},{a,b,c},X}
Here {a,c,d} is α- Open set but not $\widehat{P}g$ Open set

Example 4.13:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},{a,b,d},X}
Here {b,c,d} is Sg-Open set but not $\widehat{P}g$ Open set.
{c} is $\widehat{P}g$ Open set but not Sg- Open set

Example 4.14:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},{a,b,d},X}
Here {a,b,d} is $\widehat{P}g$ Open set but not RW Open set.
{d},{c,d} are $\widehat{P}g$ Open set but not RW Open set

Example 4.15:
Let X={a,b,c,d} and τ={Φ,{a},{a,b,c},{a,b,d},X}
Here {a,d},{b,d},{b,c,d},{a,c,d} are β- Open set but not $\widehat{P}g$ Open set.
{c} is $\widehat{P}g$ Open set but not β- Open set.

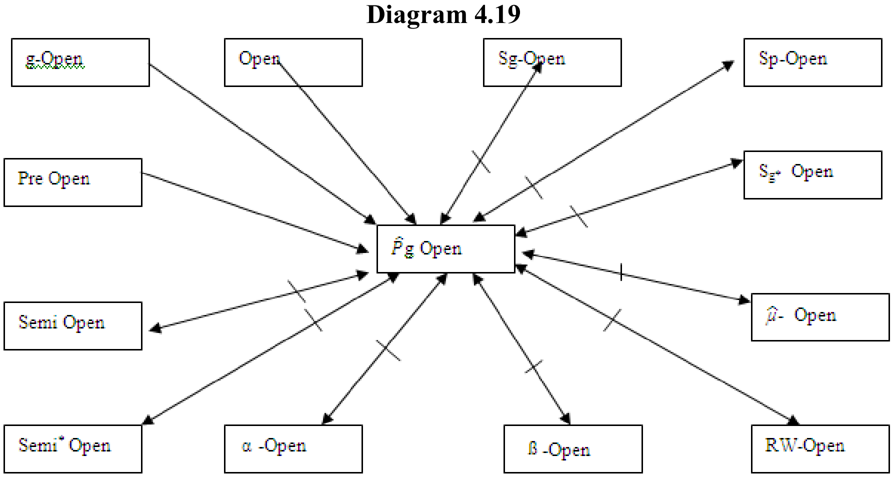
Example 4.16:
Let X={a,b,c} and τ={Φ,{a},X}
Here {c,d} and{b,d} are $\widehat{P}g$ Open. but not $\widehat{P}g$ Open Semi Pre Open and
Let X={a,b,c} and τ={Φ,{a},{b},{a,b},X}.
Here {a,c,d},{b,c,d} are Semi Pre Open but not $\widehat{P}g$ Open

Example 4.17:
Let X= {a, b, c, d} and τ={ Φ,{a, b},X}
Here {c,d} is $\widehat{P}g$ Closed but not S g^*- Closed and
Let X= {a, b, c, d} and τ={ Φ,{a},{a, b},{a,b,c},X}
Here {a,b,d} is S g^*- Closed but not $\widehat{P}g$ Closed

Example 4.18:
Let X={a,b,c} and τ={Φ,{a},{b},{a,b},X}
Here {b,c} is ⍰- Open set but not $\widehat{P}g$ Open set and
Let X={a,b,c} and τ={Φ,{a},{a,b},X}
Here {b} is $\widehat{P}g$ Open set but not ⍰- Open set

From the above results and discussions we have the following diagram,

Diagram 4.19



A ⟶ B Which represents A implies B but not conversely.
A ↮ B Which represents A and B are independent

Definition 4.20 :
Let A be a subset of X. Then $\widehat{P}g$ interior of A is defined as the union of all $\widehat{P}g$ Open sets contained in it.
$\widehat{P}g\ int(A) = \cup\{V: V \subseteq A \text{ and } V \in \widehat{P}g\ O(X)\}$.

Theorem 4.21. Let A and B be subsets of (X, τ). Then the following results hold.
(i) $\widehat{P}g\ int(\varphi)=\varphi$ and $\widehat{P}g\ int(X)=X$.
(ii) $\widehat{P}g\ int(A) \subseteq A$.
(iii) If $A \subseteq B$, then $\widehat{P}g\ int(A) \subseteq \widehat{P}g\ int(B)$.
(iv) $A \subseteq int(A) \subseteq Pint(A) \subseteq \widehat{P}g\ int(A)$
Proof: (i), (ii), (iii) follows from definition 4.20. (iv) follows from Theorem 4.7

Theorem 4.22. Let A and B be subset of a space X. Then
(i) $\widehat{P}g\ int(A) \cup \widehat{P}g\ int(B) \subseteq \widehat{P}g\ int(A \cup B)$.
(ii) $\widehat{P}g\ int(A \cap B) \subseteq \widehat{P}g\ int(A) \cap \widehat{P}g\ int(B)$.

Proof: (i) and (ii) follows from Theorem 4.21(iii)

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