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Models of Finseler Spaces With Given Geodesics

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Abstract: *In the present paper, we introduce the theory of four dimensional Finsler space and define geodesic equation with the basis of Finsler space. We also try to define geodesic equation to useful significance.*

I. Introduction

Finseler geometry is a kind of differential geometry, which was originated by P. Finselerin 1918. It is usually considered as a generalization of Riemannian geometry. The definition of finseler space-

1.1 Finseler Space:

Suppose that we are given a function L (x^i, y^i) of the line element (x^i, y^i) of a curve defined in R. We shall assume L as a function of class at least C^5 in all its 2n-arguments. If we define the infinitesimal distance ds between two points $P(x^i)$ and Q $(x^i + dx^i)$ of R by the relation

$$ds = L(x^i, dx^i) \qquad (1.1.1)$$

then the manifold M^n requipped with the fundamental function L defining the metrix (1.1.1) is called a Finsler space .if L (x^i, dx^i) satisfies the following conditions.

Condition A-

The function L (x^i, y^i) is positively homogeneous of degree one in y^i i.e.

$$L\,(x^i, ky^i) = k\, L\,(x^i, y^i),\ k > 0 \qquad (1.1.2)$$

Condition B-

The function L (x^i, y^i) is positively if not all y^i vanish simultaneously i.e.

$$L\,(x^i, y^i) > 0\ with\ \textstyle\sum_i (y^i)^2 \neq 0 \qquad (1.1.3)$$

Condition C-

The quadratic form

$$\dot{\partial}_j L^2(x,y)\varepsilon^i\varepsilon^j = \frac{\partial^2 L^2(x,y)}{\partial y^i \partial y^j}\varepsilon^i\varepsilon^j \qquad (1.1.4)$$

is assumed to be positive definite for any variable ε^i.

Form Euler's theorem on homogenous functions, we have

$$\dot{\partial}_i\, L\,(x,y)\dot{y}^i = L\,(x,y) \qquad (1.1.5)$$

$$\dot{\partial}_i\dot{\partial}_j L^2(x,y)y^i = 0 \qquad (1.1.6)$$

We put

$$g_{ij}\,(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x,y) \qquad (1.1.7)$$

Using the theory of quadratic form and the condition C, we deduce form (1.1.4) that-

$$g(x,y) = g_{ij}\,(x,y) > 0 \qquad (1.1.8)$$

for all line elements (x^i, y^i). If the function L is of particular form

$$L\,(x^i, dx^i) = [g_{ij}\,(x^k)dx^i dx^j]^{1/2} \qquad (1.1.9)$$

where the coefficients $g_{ij}\,(x^k)$ are independent of dx^i, the metric defined by this function is called Riemannian metric and manifold M^n is called a Riemannian space. Throughout the paper, $F^n\ or\ (M^n, L)$ will denote the n-dimensional finsler space, where as n-dimensional Riemannian space will be denoted by R^n.

1.2 Intrinsic Fields of Orthonormal Frames :

Berwald theory of two-dimensional Finsler space is developed based on the intrinsic field of orthonormal frame which consists of the normalized supporting element l^i and unit vector orthonormal to l^i. Following idea Moor introduced, in a three-dimensional Finsler space, the intrinsic field of orthonormal frame which consists of the normalized supporting elementl^i, the normalized torsion vector $C^i/C$and a unit vector orthogonal to them and developed a theory of three-dimensional Finsler spaces. Generalizing the Berwald's and Moor's ideas, Miron and Matsumoto[(1986), (1977), (1989)] developed a theory of intrinsic orthonormal frame fields on n-dimensional Finsler space as follows.

Let *L(x, y)* be the fundamental function of an n-dimensional Finsler space and introduce Finsler tensor fields of (0, 2α-1) type, α = 1,2,…..,n by

$$L_{i_1 i_2 \ldots\ldots i_{2\alpha-1}} = \frac{1}{2^\alpha}\dot{\partial}_{i_1}\dot{\partial}_{i_2}\ldots\ldots\dot{\partial}_{i_{2\alpha-1}}L^2$$

Then we get a sequence of covariant vectors

$L_{\alpha)i} = L_{ij_1j_2\ldots\ldots j_{2\alpha-3}j_{2\alpha-2}}g^{j_1}g^{j_1}\ldots\ldots g^{j_{2\alpha-3}j_{2\alpha-2}}$

Definition-1: If (n-1) covariant vectors $L_{\alpha)i}$, α = 1, 2,….., n-1 are linearly independent, the Finsler space is called strongly non-Riemannian.

Assuming above n-covectors$L_{\alpha)i}$ are linearly independent and put $e^i_{1)} = L^i_{1)}/L = l^i$. Here and in following we use raising and lowering of indices as$L^i_{1)} = g^{ij}L_{1)j}$.

Further putting $N_{1)ij} = g_{ij} - e_{1)i}e_{1)j}$ and matrix $N_{1)} = N_{1)ij}$ is of rank (n-1). Second vector $\boldsymbol{e_{2)}}$ is introduced by

$e^i_{2)} = L^i_{2)}/L_2$,

where, L_2 is the length of$L^i_{2)}$ relative to y^i. Next we put $N_{2)ij} = N_{1)ij} - e_{2)i}e_{2)j}$,$E^i_{3)}$=$N^i_{2)j}L^j_{3)}$ and so third vector$e_{3)}$ is defined by,

$e^i_{3)} = E^i_{3)}/E_3$,

where, $E_2$is the length of $E^i_{2)}$relative to $\boldsymbol{y^i}$.The repetition of above process gives a vector $e_{r+1)}$, r = 1, 2,……, n-1 defined by

$$e^i_{r+1)} = E^i_{r+1)}/E_{r+1}$$

where, $E^i_{r+1)}$=$N^i_{r)j}L^j_{r+1)}E_{r+1}$ is the length of $E^i_{r+1)}$ relative to $y^i$and $N_{r)ij} = N_{r-1)ij} - e_{r)i}e_{2)j}$.

Definition-2: The orthonormal frame $\{e_\alpha\}$, α = 1, 2,…..,n as above defined in every in every co-ordinate neighborhood of a strongly non-Riemannian Finsler space is called the 'Miron Frame'.

Consider the Miron frame $\{e_\alpha\}$, If a tensor T^i_j of (1, 1)-type, for instance, is given then

$T^i_j = T_{\alpha\beta}\, e^i_{\alpha)}e_{\beta)j}$

where, the scalars $T_{\alpha\beta}$ are defined as

$$T_{\alpha\beta} = T^i_j e_{\alpha)i}e^j_{\alpha)}$$

These scalars $T_{\alpha\beta}$ are called the scalar components of $T^i_j$with respect to Miron frame.

Let $H_{\alpha)\beta\gamma}$ be scalar components of the h-covariant derivatives $\boldsymbol{e^i_{\alpha)|j}}$ and $V_{\alpha)\beta\gamma}/L$ be scalar components of the v-covariant derivatives $e^i_{\alpha)}|_j$ with respect to CΓ of the vector $e^i_{\alpha)}$ belonging to the Miron frame. Then

$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma}\,e^i_{\beta)}e_{\gamma)j}$,

$e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma}\,e^i_{\beta)}e_{\gamma)j}$,

where, the scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ satisfying the following relations [Berwald (1947)].

$H_{1)\beta\gamma} = 0, H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}$,

$V_{\alpha)\beta\gamma} = \delta_{\beta\gamma} - \delta^1_\beta\delta^1_\gamma,\;\; V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}$

Definition-3: The scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called connection scalars.

If $C_{\alpha\beta\gamma}/L$be the scalar components of the (h)hv-torsion tensor C^i_{jk} i.e.,

$$LC^i_{jk} = C_{\alpha\beta\gamma}\,e^i_{\alpha)}e_{\beta)j}\,e_{\gamma)k}$$

then [Hambo,H(1934)], we have

Proposition-1:

i. $C_{1\beta\gamma} = 0$

ii. $C_{2\mu\mu} = LC,\;\; C_{3\mu\mu} = \cdots\ldots\ldots = C_{n\mu\mu} = 0$for $n \geq 3$, where $C$is the length of C^i.

Now, we consider scalar components of covariant derivatives of a tensor field, for instance, T^i_j. Let $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}/L$ be the scalar components of h-and v-covariant derivatives with respect to CΓ respectively of a tensor T^i_j *i*.e.,

$T^i_{j|k} = T_{\alpha\beta,\gamma}\,e^i_{\alpha)}e_{\beta)j}\,e_{\gamma)k}$ (1.2.1)

and

$LT^i_j|_k = T_{\alpha\beta;\gamma}\,e^i_{\alpha)}e_{\beta)j}\,e_{\gamma)k}$, (1.2.2)

then we have [Hambo,H(1934)],

$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta})e^k_{\gamma)} + T_{\mu\beta}\,H_{\mu)\alpha\gamma} + T_{\alpha\mu}\,H_{\mu)\beta\gamma}$ (1.2.3)

and $T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta})e^k_{\gamma)} + T_{\mu\beta}\,V_{\mu)\alpha\gamma} + T_{\alpha\mu}\,V_{\mu)\beta\gamma}$. (1.2.4)

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are called h-and v-scalar derivative of $T_{\alpha\beta}$ respectively.

(i) Two-dimensional Finsler space

The Miron frame $\{e_{1)}, e_{2)}\}$ is called the Berwald frame. The first vector $e^i_{1)}$is the normalized supporting element $l^i = y^i/L$ and the second vector $e^i_{2)} = m^i$ is the unit vector orthogonal to l^i. IfC^ihas non-zero length$C$,the$m^i = \pm C^i/C$. The connection scalars $H_{\alpha)\beta\gamma}$and $V_{\alpha)\beta\gamma}$ of a two-dimensional Finsler space are such that

$H_{\alpha)\beta\gamma} = 0,\ V_{\alpha)\beta 1} = 0,\ V_{\alpha)\beta 2} = \delta^{12}_{\alpha\beta}$,which implies

$l^i_{|j} = 0,\ m^i_{|j} = 0,\ Ll^i|_j = m^i m_j,\ Lm^i|_j = -l^i m_j$ (1.2.5)

There is only one surviving scalar components ofLC_{ijk} namelyC_{222}.If we put$I = C_{222}$.

Then $LC_{ijk} = I m_i m_j m_k$

The scalar $I$is called the main scalar of a two-dimensional Finsler space.

Proposition-2: In a two-dimensional Finsler space

i. The h-curvature tensor R_{hijk} of CΓ is written as,

$$R_{hijk} = R(l_h m_i - l_i m_h)(l_j m_k - l_k m_j)$$

ii. The hv-curvature tensor P_{hijk} of CΓis written as,

$P_{hijk} = I_{,1}(l_h m_i - l_i m_h) m_j m_k$

iii. The **(v)**hv-curvature tensor P_{ijk} is written as,

$P_{ijk} = I_{,1} m_i m_j m_k$

(ii) Three-dimensional Finsler space

The Miron frame of a three-dimensional Finsler space is called the Moor-frame. The first vector $e^i_{1)}$ of Moor-frame $\{e_{1)}, e_{2)}, e_{3)}\}$ is the normalized supporting element $\boldsymbol{l^i}$, the second vecto**r** $e^i_{2)}$ is the normalized torsion vector $m^i = C^i/C$and the third$e^i_{3)} = n^i$ is constructed by,

$n^i = \varepsilon^{ijk}\, e_{1)j}\, e_{2)k}$where, $\varepsilon^{ijk} = g^{-(1/2)}\delta^{ijk}_{123}$

Now, following two Finsler vector fields are defined [Mastsumoto(1986)]

$h_i = h_\gamma e_{\gamma)i}$and $v_i = v_\gamma e_{\gamma)i}$then we have, (1.2.6)

$$H_{\alpha)\beta\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h_\gamma \\ 0 & -h_\gamma & 0 \end{bmatrix}, \quad V_{\alpha)\beta\gamma} = \begin{bmatrix} 0 & \delta^2_\gamma & \delta^3_\gamma \\ -\delta^2_\gamma & 0 & v_\gamma \\ -\delta^3_\gamma & -v_\gamma & 0 \end{bmatrix}$$

$$\begin{cases} l^i_{|j} = 0 & Ll^i|_j = h^i_j \\ m^i_{|j} = n^i h_j & Lm^i|_j = -l^i m_j + n^i v_j \\ n^i_{|j} = -m^i h_j & Ln^i|_j = -l^i n_j - m^i v_j \end{cases} \quad (1.2.7)$$

Definition-4: The Finsler vector fields h_i and $v_i$defined in (2.2.6) are called the h-and v-connection vectors of a three-dimensional Finsler space.

The (h)hv-torsion tensor of a three-dimensional Finsler space is given by [Mastusmoto(1986)],

$LC_{ijk} = H m_i m_j m_k - J\pi_{(ijk)}(m_i m_j n_k) + I\pi_{(ijk)}(m_i n_j n_k) + J n_i n_j n_k$ (1.2.8)

The three scalar fields *H, I*and*J* of (1.2.8) are called the main scalars of a three-dimensional Finsler space and $\pi_{(ijk)}$ represent cyclic sum of the terms obtained by cyclic permutation of *i, j, k*.

The h-and v-connection vectors of a three-dimensional space has been firstly solved, in terms of main scalars explicitly, by Ikeda (1994).

(iii) Four-dimensional Finslar space

Prof. T. N. Pandey and D. K. Dwevidi developed the theory of four-dimensional Finsler spaces in the year 1997 in terms of scalars, taking l^i, m^i, n^i and a unit vector p^i perpendicular to l^i, m^i, n^i as $p^i = \epsilon^{ijkl} l_j m_k n_l$. The orthonormal frame (l^i, m^i, n^i, p^i) as above defined in every coordinate neighborhood of a strongly non-Riemannian Finsler space is called Miron frame.

M. Matsumoto defines the scalar component of a tensor in Miron frame as follows:-

If a tensor $T^i_{jk}$of (1, 2) type for instance is given, we define scalars

$$T_{\alpha\beta\gamma} = T^i_{jk}\, e_{\alpha)i}\, e^j_{\beta)}\, e^k_{\gamma)}$$

Then T^i_{jk} is written in the form,

$$T^i_{jk} = T_{\alpha\beta\gamma}\, e^i_{\alpha)}\, e_{\beta)j}\, e_{\gamma)k}$$

These $\mathbf{T}_{\alpha\beta\gamma}$ are called scalar components of $\mathbf{T}^{\mathbf{i}}_{\mathbf{jk}}$ with respect to Miron frame

$$e^i_{1)} = l^i, \quad e^i_{2)} = m^i, \quad e^i_{3)} = n^i, \quad e^i_{4)} = p^i.$$

From the equations

$$g_{ij}l^il^j = g_{ij}m^im^j = g_{ij}n^in^j = g_{ij}p^ip^j = 1$$

and

$$g_{ij}l^im^j = g_{ij}l^in^j = g_{ij}l^ip^j = g_{ij}m^in^j = g_{ij}m^ip^j = g_{ij}n^ip^j = 0$$

we have,

$$g_{ij} = l_il_j + m_im_j + n_in_j + p_ip_j$$

Next, the C-tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ satisfies $C_{ijk}l^i = C_{ijk}l^j = C_{ijk}l^k = 0$. So we have the expression of C_{ijk} in the form

$$LC_{ijk} = Hm_im_jm_k + Jn_in_jn_k + H'p_ip_jp_k + I\pi_{(ijk)}\{m_in_jn_k\} \tag{1.2.9}$$
$$+K\pi_{(ijk)}\{m_ip_jp_k\} + J'\pi_{(ijk)}\{n_ip_jp_k\} - (J + J')\pi_{(ijk)}\{n_im_jm_k\}$$
$$+I'\pi_{(ijk)}\{n_in_jp_k\} - (H' + I')\pi_{(ijk)}\{m_im_jp_k\} + K'\pi_{(ijk)}\{m_in_jp_k\}$$

where, *H, I, J, K*, H', I', J', K' are called main scalars satisfying $H + I + K = LC$.

Now we denote the h-and v-covariant differentiations of a tensor field with respect to

$\mathbf{C}^{\mathbf{i}}$ by the short line ($_{|i}$) and long line (**|i**) respectively, the following equations are derived

$$\begin{cases} L_{|i} = 0, \quad l_{i|j} = 0, \quad m_{i|j} = n_ih_j - p_ij_j, \\ n_{i|j} = p_ik_j - m_ih_j, \quad p_{i|j} = m_ij_j - n_ik_j \end{cases} \tag{1.2.10}$$

$$\begin{cases} Ll_{i|j} = h_{ij}, \qquad Lm_i|_j = -l_im_j + n_iu_j + p_iv_j \\ Ln_i|_j = -l_in_j - m_iu_j + p_iw_j, \quad Lp_{i|j} = -l_ip_j - m_iv_j - n_iw_j \end{cases} \tag{1.2.11}$$

where, h_i, j_i, k_i are components of vectors called h-connection vector and u_i, v_i, w_i are called components of v-connection vector respectively.

The equation (1.2.10) and (1.2.11) may be written as

$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma}e^i_{\beta)}e_{\gamma)j}$$
$$e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma}e^i_{\beta)}e_{\gamma)j}$$

The surviving scalar components of $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are given by

$V_{1)1\gamma} = V_{2)2\gamma} = V_{3)3\gamma} = V_{4)4\gamma} = 0$, $\quad V_{2)1\gamma} = V_{1)2\gamma} = -\delta_{2\gamma}$, $V_{2)3\gamma} = -V_{3)2\gamma} = u_\gamma$, $V_{2)4\gamma} = -V_{4)2\gamma} = v_\gamma$, $V_{3)4\gamma} = -V_{4)3\gamma} = w_\gamma$, $\quad V_{3)1\gamma} = -V_{1)3\gamma} = -\delta_{3\gamma}$, $V_{4)1\gamma} = -V_{1)4\gamma} = -\delta_{4\gamma}$, $\quad H_{2)3\beta} = h_\beta = -H_{3)2\beta}$, $\quad H_{4)2\beta} = j_\beta = -H_{2)4\beta}$, $\quad H_{3)4\beta} = k_\beta = -H_{4)3\beta}$.

where,h_α, j_α, k_α) and(u_α, v_α, w_α)(are scalar components of h_i and v_i respectively $h_\alpha = h_ie^i_{\alpha)}$, $\quad j_\alpha = j_ie^i_{\alpha)}$, $k_\alpha = k_ie^i_{\alpha)}$, $\quad v_\alpha = v_ie^i_{\alpha)}$, $\quad u_\alpha = u_ie^i_{\alpha)}$, $w_\alpha = w_ie^i_{\alpha)}$. The first scalar component $v_1 = v_il^i$ vanishes identically in a four-dimensional Finsler space.

The h-scalar derivative of the adopted components $T_{\alpha\beta}$ of the tensor T^i_j of (1, 1) type is defined as

$$T_{\alpha\beta,\gamma} = \frac{\delta T_{\alpha\beta}}{\delta x^k}e^k_{\gamma)} + T_{\mu\beta}H_{\mu)\alpha\gamma} + T_{\alpha\mu}H_{\mu)\beta\gamma}$$

where $\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - G^r_k\frac{\partial}{\partial y^r}$ and G^r_k are non-linear connection of $C\Gamma$.

Similarly the v-scalar derivative of the adopted components $T_{\alpha\beta}$ of *T* is defined as,

$$T_{\alpha\beta;\gamma} = L\frac{\partial T_{\alpha\beta}}{\partial y^\gamma}e^k_{\gamma)} + T_{\mu\beta}V_{\mu)\alpha\gamma} + T_{\alpha\mu}V_{\mu)\beta\gamma}$$

Thus $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are adopted components of $T^i_{j|k}$ and $LT^i_j|_k$ respectively. i. e.

$$T^i_{j|k} = T_{\alpha\beta,\gamma}e^i_{\alpha)}e_{\beta)j}e_{\gamma)k} \tag{1.2.12}$$
$$LT^i_j|_k = T_{\alpha\beta;\gamma}e^i_{\alpha)}e_{\beta)j}e_{\gamma)k} \tag{1.2.13}$$

II. Geodesics

The curve for shortest length, measured along the surface between any two points on the surface is called geodesic curve or geodesic.

Geodesic Equation from Geodesic Curve with Finsler Space

We know that

$$ds = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}}\,ds$$

$$\Rightarrow \int ds = \int (-g_{\alpha\beta} \frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds})^{1/2} ds$$

To extremize length take $\delta \int ds = 0$

According to the Euler's – Lagrange equation

$$\frac{d}{ds}\left(g_{\alpha\beta}\, u^{\alpha}\right) = \frac{1}{2} g_{\alpha\gamma,\beta} u^{\alpha} u^{\gamma} \qquad -(2.1)$$

Where $\left(-g_{\alpha\beta} \frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}\right) = 1$ and $u^{\alpha} = \frac{dx^{\alpha}}{ds}$

Hence $\frac{d}{ds}\left(g_{\alpha\beta}\, u^{\alpha}\right) = g_{\alpha\beta} \frac{du^{\alpha}}{ds} + g_{\alpha\gamma,\beta} u^{\alpha} u^{\gamma}$ $\qquad -(2.2)$

Hence equation (2.1) becomes

$$g_{\alpha\beta} \frac{d^2 x^{\alpha}}{ds^2} + u^{\alpha} u^{\gamma} (g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\alpha\gamma,\beta}) = 0 \qquad -(2.3)$$

Now we use

$$u^{\alpha} u^{\gamma} g_{\alpha\beta,\gamma} = u^{\alpha} u^{\gamma} \frac{1}{2} (g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha})$$

And multiply equation (2.3) by $g^{\beta\gamma}$ to obtain

$$\frac{d^2 x^{\tau}}{ds^2} + \frac{1}{2} g^{\beta\gamma} (g_{\alpha\beta,\gamma} + g_{\gamma\beta,\alpha} - g_{\alpha\gamma,\beta}) u^{\alpha} u^{\gamma} = \frac{d^2 x^{\tau}}{ds^2} + \Gamma^{\tau}_{\alpha\gamma} \frac{dx^{\alpha}}{ds}\frac{dx^{\gamma}}{ds} = 0 \qquad -(2.4)$$

Since a co-ordinate form

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}(g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda\alpha}) g^{\mu\alpha}$$

$$\Rightarrow \Gamma^{\tau}_{\alpha\gamma} = \frac{1}{2}(g_{\beta\alpha,\gamma} + g_{\beta\gamma,\alpha} - g_{\alpha\gamma,\beta}) g^{\tau\beta}$$

$$u^{\alpha} = \frac{dx^{\alpha}}{ds}, \qquad u^{\gamma} = \frac{dx^{\gamma}}{ds}$$

Again from (1.1.1) &(1.1.9) We have

$$ds = L(x^i, dx^i) = [g_{ij}(x^k) dx^i dx^j]^{\frac{1}{2}} \qquad -(2.5)$$

From geodesics integral

$$g_{\alpha\beta} = -g_{ij}(x^k) \frac{dx^i}{dx^{\alpha}} \frac{dx^j}{dx^{\beta}} \qquad -(2.6)$$

Again from (2.2) we have

$$g_{\alpha\beta,\gamma} = [\ \frac{d}{ds}\left(g_{\alpha\beta}\, u^{\alpha}\right) - g_{\alpha\beta} \frac{du^{\alpha}}{ds}] \frac{1}{u^{\gamma} u^{\alpha}} \qquad -(2.7)$$

$$g_{\gamma\beta,\alpha} = [\ \frac{d}{ds}\left(g_{\gamma\beta}\, u^{\gamma}\right) - g_{\gamma\beta} \frac{du^{\gamma}}{ds}] \frac{1}{u^{\alpha} u^{\gamma}} \qquad -(2.8)$$

$$g_{\alpha\gamma,\beta} = [\ \frac{d}{ds}\left(g_{\alpha\gamma}\, u^{\alpha}\right) - g_{\alpha\gamma} \frac{du^{\alpha}}{ds}] \frac{1}{u^{\alpha} u^{\beta}} \qquad -(2.9)$$

From (2.6),(2.7),(2.8)&(2.9) we have the geodesics equation of the form

$$\frac{d^2 x^{\tau}}{ds^2} + \frac{1}{2} g^{\beta\gamma} \left[g_{ik}(x^j) \frac{dx^i}{dx^{\alpha}}\frac{dx^k}{dx^{\gamma}} - g_{ij}(x^k) \frac{dx^i}{dx^{\alpha}}\frac{dx^j}{dx^{\beta}} - g_{kj}(x^i) \frac{dx^k}{dx^{\gamma}}\frac{dx^j}{dx^{\beta}} \right] = 0 \qquad -(2.10$$

Theorem-

The geodesic of the velocity space metric defined in $ds^2 = d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\emptyset^2)$.
where the magnitude of the velocity is v = tanhχ are paths of minimum fuel for a rocket ship changing its velocity.

Proof –the geodesic is the path between two velocities which minimizes the arc –length between them, but arc – length in the velocity space is just the magnitude of a small change of velocity. Science a rocket expends fuel monotonically for the boost it requires the geodesic of velocity space are paths of minimum fuel use.

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