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KS - Graph on Commutative KS-Semigroup

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Abstract : *In this paper, we introduce the concept of KS-graph of commutative KS-semigroup. We also introduce the notion of L-prime, zero divisors of commutative KS – semigroup and investigated its related properties. We also discuss the concept of KS-graph of commutative KS-semigroup and provide some examples and theorems.*
Keywords: *commutative KS-semigroup, connected graph, KS- graph, L- prime of commutative KS- semigroup P- ideal, zero divisors.*

I. Introduction

In abstract algebra, mathematical system with one binary operation called group and two binary operations called rings were investigated. In 1966, Y.Imai and K.Iseki [2] defined a class of algebra called BCK-algebra [2]. A BCK – algebra is named after the combinators B,C and K by Carew Arthur Merideth, an Irish logician. At the same time, Iseki [3] introduced another class of algebra called BCI- algebra, which is a generalization of the class of BCK- algebra and investigated its properties. For the general development of BCI/BCK –algebras, the ideal theory and graph plays an important role. In 2006, Kyung Ho Kim [7] introduced a new class of algebraic structure called KS-semigroup "On Structure of KS-semigroup". Also define a new class algebras related to BCK-algebras, commutative properties and semigroup, called a commutative KS-semigroup. Then we introduced the concept of G(X) is KS-graph on commutative KS-semigroup. It is connected G(X) is complete graph. Finally, we discussed the relation between some operations on graph and commutative KS-semigroup.

II. Preliminaries

We need some definitions and properties that will be useful in our results **BCK-algebra.**

Definition: 2.1 [7]

A BCI-algebra is a triple (X,*,0) where X is a non empty set, "*" is a binary operation on X. $0 \in X$ is an element such that the following axioms are satisfied for every $x,y,z \in X$.

I. $[(x*y)*(x*z)]*(z*y)=0$; $\forall\, x,y \in X$.
II. $[x*(x*y)]*y=0$; $\forall\, x,y \in X$.
III. $x*x=0$; $\forall\, x,y \in X$.
IV. $x*y=0$ and $y*x=0 \Rightarrow x=y$; $\forall\, x,y \in X$.
if a BCI- algebra X satisfies the following identity:
V. $0*x=0$ $\forall\, x \in X$, then X is called a BCK-algebra.

If X is a BCK-algebra, then the relation $x \le y$ iff $x*y=0$ is a partial order on X, which will called the natural ordering on X. Any BCK- algebra X satisfies the following conditions

I. $x*0 = x$ for all $x \in X$.
II. $(x*y)*z = (x*z)*y$ for all $x,y,z \in X$.
III. $x \le y \Rightarrow x*z \le y*z$ and $z*y \le z*x$;for all $x,y,z \in X$.
IV. $(x*z)*(y*z) \le x*y$; for all $x,y,z \in X$.

Example: 2. 2 [6]

Let X = {0,a,b,c} be a set with *-operation given by **Table,**

*	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then (X,*,0) is a BCK-algebra.

Definition: 2.3 [7]

A non- empty subset I of a BCK-algebra is called an ideal if it satisfies

1. $0 \in X$.
2. $x * y \in X$ and $y \in X$ imply $x \in X$ for all $x,y \in X$.

Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

Example: 2.4 [6]

Let X = {0,a,b,c}be a set with the *-operation given by **Table**,

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Then (X,*,0) is a BCK-algebra. The set I={0,b} is an ideal of X.

Definition: 2.5 [6]

Let X denote BCK-algebra, for any subset A of X , we will use the notation U(A) and L(A) to denote the sets,

$U(A) = \{ x \in X / a * x = 0 ,\text{for all } a \in A\}$,

$L(A) = \{ x \in X / x*a= 0 , \text{for all } a \in A\}$,

i.e. $U(A) = \{ x \in X / a \leq x \ \forall \ a \in A\}$ and $L(A) = \{ x \in X / x \leq a \ \forall \ a \in A\}$.

Example: 2.6 [6]

Let X = {0,a,b,c} be set with the *-operation given by **Table**.

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Then , X is a BCK- algebra.

Then, $L(A) = L(\{0,a\}) = L(\{0,b\}) = L(\{0,c\}) = L(\{a,b\}) = L(\{b,c\}) = L(\{a,c\}) = \{0\}$

Definition: 2.7 [6]

Let $x \in X$. we will use the notation Z_x to denote the set of all elements $y \in X$, such that $L(\{x,y\})=\{0\}$. That is, $Z_x = \{ y \in X / L(\{x,y\})=\{0\}\}$.which is called the set of zero divisors of x.

Example : 2.8 [6]

Let X = {0,a,b,c } be set with the *-operation given by **Table** .

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

$Z_0 = \{ y \in X / L(\{0,y\})=\{0\}\}$.

$Z_0 = \{0,a,b,c, \} Z_a = \{0,a\}, \ Z_b = \{0,b\}$.

Then , (X,*,0) is a BCK- algebra. Therefore Z_x is zero divisor of x.

Definition: 2.9 [6]

Let X is a BCK- algebra and Γ(X) be a simple graph vertices are just the elements of X and for distinct, x, y $\in X$,there is an edge connecting x and y denoted by xy iff $L(\{x,y\})=\{0\}$ then, Γ(X) is called a BCK- graph of X.

Example : 2.10 [6]

Let X = {0,a,b,c,d} be set with the *-operation given by **Table.**

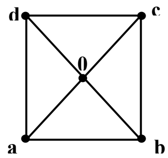
*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

L(A) = L({0,a}) = L({0,b}) = L({0,c}) =L({0,d}) = L({a,b}) = L({b,c}) = L({c,d}) =L({d,a}) = {0}

And so E (G(X)). = {0a, 0b, 0c, od, ab, bc, cd, da}.

Therefore, G (X) is a BCK – graph of X is given by the **Figure 1**.

Figure: 1



III. Commutative KS- Semigroup

Definition: 3.1 [7]

A semigroup is an ordered pair (S,*), where S is a nonempty set and "*" is an associative binary operation on S.

Definition: 3.2 [7]

An commutative KS-semigroup is a non –empty set X with two binary operations "*" and "•" and constant 0 satisfying the axioms;

i) (X,*,0) is BCK-algebra.
ii) (X,•) is semigroup.
iii) x • (y*z) = (x • y)*(x • z) and (x*y) • z = (x • z)*(y • z) ∀ x,y,z∈X.
iv) x*(x*y) = y*(y*x) ∀ x,y ∈X.

Example: 3.3 [7]

Let X= {0,a,b,c} be a set with the '*' and '•' operations given by **Table 1**.

Table: 1 "*" and "•" operations

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then (X,*, • , 0) is a commutative KS-semigroup.

Definition: 3.4 [7]

A non empty subset A of a semigroup (X, •) is said to be left and right stable if xa∈A and ax∈A whenever x ∈X and a∈A. Both left and right stable is a two sided stable or simply stable.

Example: 3.5 [7]

Let X ={0,a,b,c}be a commutative KS- semigroup be a set with the '*' and '•' operations from the **Table 1**. If A={0,a,b}.then, A is an stable of commutative KS-semigroup of X.

Definition: 3.6 [7]

A non empty subset A of a commutative KS-semigroup X is called a left and right ideal of X if
(i) A is left and right stable subset of (X , •). (ii) ∀ x, y∈ X, x *y ∈ A and y∈A ⇒ x∈A.
A subset which is both left and right ideal is called a two sided ideal or simply on ideal.

Example: 3.7 [7]

Let X ={0,a,b,c}be a commutative KS- semigroup be a set with the '*' and '.' operations given by **Table 2.**

Table: 2 "*" and "." Operations

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

•	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	0	0	c

If A={0,a}. Then, A is an ideal of commutative KS-semigroup of X.

Definition: 3.8 [7]

A non-empty subset A of a commutative KS-semigroup X is called a left (respectively right) P-ideal of X if i) A is a left (respectively right \ stable subset of (x, .)).

ii) $\forall$ x,y,z $\in$X, (x*y) * z $\in$ A and (y*z) $\in$ A $\Rightarrow$ x*z $\in$ A.

A subset of X which is both left and right P-ideal is called P-ideal of commutative KS-semigroup X. A P-ideal is always an ideal.

Example: 3.9[7]

Let X = {0,a,b,c}. X is a commutative KS –semigroup be a set with the '*' and '•' operations given by **Table 3.**

Table: 3 "*" and "." operations

*	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

•	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	c	0	0

If A={0,a}. Then, A is a P-ideal of commutative KS-semigroup of X.

Definition: 3.10

Let X denote the commutative KS- semigroup , for any subset A of X , we will use the notation U(A) and L(A) to denote the sets

U(A) = { x $\in$ X / a * x = 0 and a. x = 0 $\forall$ a $\in$ A}.

L(A) = { x $\in$ X / x*a= 0 and x.a=0 $\forall$ a $\in$ A} .

i.e. U(A) = { x $\in$ X / a $\le$ x $\forall$ a $\in$ A} and L(A) { x $\in$ X / x $\le$ a $\forall$ a $\in$ A}.

Example: 3.11

From the example 3.3, we have

L(A) = L({0,a}) = L({0,b}) = L({0,c}) = L({a,b}) = L({b,c}) = L({a,c}) = {0}.

Definition: 3.12

A P-ideal A of commutative KS-semigroup X is said to be L-prime if it satisfies

(i) A is a proper (i.e) A $\neq$ X.

(ii) ($\forall$x,y$\in$X), L({x,y}) $\subseteq$ A $\Rightarrow$ x $\in$ A or y $\in$ A

Example : 3.13

From the example 3.3 , A = {0} is a L –prime.

Definition: 3.14

Let x$\in$X. X is a commutative KS - semigroup . We will use the notation Z_x to denote the set of all elements y$\in$X such that L({x,y})={0}. That is Z_x = { y$\in$X / L({x,y})={0}}, which is called the set of zero divisors of x.

Example : 3.15

From the example 3.3 ,we define

$Z_0 = \{ y \in X / L(\{0,y\})=\{0\}\}$, $Z_0 = \{0,a,b,c\}$

$Z_a = \{ y \in X / L(\{a,y\})=\{0\}\}$, $Z_a = \{0\}$

$Z_b = \{ y \in X / L(\{b,y\})=\{0\}\}$, $Z_b = \{0\}$

Theorem: 3.16

Z_x is a P- ideal of commutative KS- semigroup X, for any $x \in X$

Proof:

Let $x \in X$ .Suppose that Z_x is a P- ideal of commutative KS- semigroup X.

Let A is a Z_x . i) $\forall$ $x \in X$ and $a \in A$ such that $x.a \in A$ and $a.x \in A$

ii) $\forall$ x,y, z $\in$X, $(x*y) * z \in A$ and $(y*z) \in A \Rightarrow x*z \in A$ therefore, Z_x is a P- ideal of commutative KS- semigroup X.

Example: 3.17

Let X = {0,a,b,c,d} be a set with the '*' and '•' operations given by **Table 4**.

Table: 4 "*" and "•" operations

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	d	0

•	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

$Z_a = \{ y \in X / L(\{a,y\})=\{0\}\}$, $Z_a = \{0,a\}$

$Z_b = \{ y \in X / L(\{b,y\})=\{0\}\}$, $Z_b = \{0,b\}$

$Z_c = \{ y \in X / L(\{c,y\})=\{0\}\}$, $Z_c = \{0,b\}$

Therefore, Z_x is a P- ideal of commutative KS- semigroup of X.

Theorem : 3.18

Let X is a commutative KS-semigroup, then $L(\{x,0\})=\{0\}$ for all $x \in X$.

Proof:

Suppose let $a \in L(\{x,0\})$

$a*x=0$ & $a*0=0$

$a \bullet x=0$ & $a \bullet 0=0$

which is contradiction to $a*0=a$. Therefore, $L(\{x,0\})=\{0\}$

Theorem: 3.19

For any elements a & b of a commutative KS-semigroup X, if $a*b = 0$, $a \bullet b = 0$, then $L(\{a\}) \subseteq L(\{b\})$ and $Z_b \subseteq Z_a$.

Proof:

Assume that $a*b=0$, $a \bullet b=0$.

Let $x \in L(\{a\})$,

then $x*a=0$ & $x \bullet a=0$.

And so, $(x*b)*(x*a) = 0$ by (a2)[4]

$(x*b)*(x*a) = 0$ and $(x \bullet b) \bullet (x \bullet a) = 0$

$(x*b)*0 = 0$ and $(x \bullet b) \bullet 0 = 0$

$(x*b) = 0$ and $(x \bullet b) = 0$

Thus, $x \in L(\{b\})$, which shows that $L(\{a\}) \subseteq L(\{b\})$

Obviously, $Z_b \subseteq Z_a$

Theorem: 3.20

For any element X of a commutative KS – Semigroup, the set of zero divisors of x is a P-ideal of X containing the zero element 0. Moreover, if Z_x is maximal in $\{Z_a / a \in X, Z_a \neq X \}$, then Z_x is L – prime.

Proof:

We have $0 \in Z_x$ Let $a \in X$ and $b \in Z_x$ be such that $a * b = 0$, $a . b = 0$

We have $L(\{x,a\}) = L(\{x\}) \cap L(\{a\}) \subseteq L(\{x\}) \cap L(\{b\}) = L(\{x,b\}) = \{0\}$ Therefore $L(\{x,a\}) = \{0\}$

Hence a $\in Z_x$. Therefore Z_x is P-ideal of X.

Let a,b $\in$ X be such that L ({a,b}) $\subseteq Z_x$ and a $\notin Z_x$

Then L ({a,b,x}) = {0}. Let $0 \neq y \in$ L ({a,x}) be an arbitrarily element.

L({b,y}) $\subseteq$ L ({a,b,x}) = {0} , and so L ({b,y}) = {0}

ie., b$\in Z_y$. Since y$\in$ L ({a,x}), we have y*x=0, y . x = 0,it follows from Theorem 3.19,

$Z_x \subseteq Z_y \neq X$, so from the maximality of $Z_x = Z_y$.

Hence, b $\in Z_x$ shows that Z_x is L– prime.

Definition 3.21

By the KS-graph of a commutative KS-semigroup X, denoted G(X), we mean the graph whose vertices are just the elements of X and for distinct x,y$\in$G(X), there is an edge connecting x and y, iff L({x,y})={0}.

Example : 3.22

Let X = {0,a,b,c} be a set with "*" and "•" operations from the example 3.3. Then, X is a commutative KS-semigroup.

L(A) = L({0,a}) = L({0,b}) = L({0,c}) = L({a,b}) = L({b,c}) = L({a,c}) = {0}

And so E(G(X)). = { 0a,0b,0c,ab,ac,bc}.

Therefore , G(X) is a KS- graph in **Figure 1**.

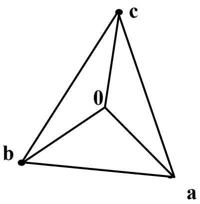


Figure 1

Example: 3.23

Let X = {0,1,2}. Then, X is a commutative KS – semigroup. Define the operations "*" and "•" by the **Table 5**. L({0,1) = L({0,2}) = L({1,2})= {0}.

Table: 5 "*" and "•" operations

*	0	1	2
0	0	0	0
1	1	0	1
2	2	1	0

•	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

The G(X) is a complete KS- graph in **Figure 2.**

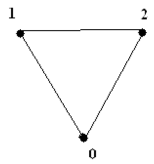


Figure 2

Example : 3.24

Let X = {0,a,b,c,d}. X is a commutative KS – semigroup. Define the operations "*" and "•" from the example 3.17.The G(X) is KS –graph in **Figure 3.**

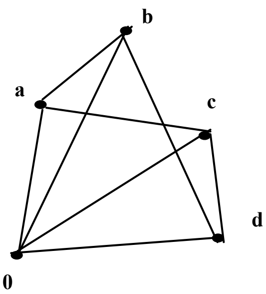


Figure 3

Theorem: 3.25

G(X) is a connected graph, for any x$\in$X.

Proof:

Let 0 $\in$ X and x, y $\in$ X.

x0, y0 $\in$ E(G(X)) and so there is a path from x to y in G(X).

Theorem: 3.26

The KS-graph of a commutative KS-semigroup is connected in which every non-zero vertex is adjacent to 0.

It is follows by theorem 3.18.

Theorem: 3.27

Let G(X) be the KS-graph of commutative KS – semigroup X. For any x,y $\in$ G(X) ,if Z_x and Z_y are distinct L – prime P-ideals of X ,then there is an edge connecting x and y.

Proof:

It is sufficient to show that L({x,y}) = {0}. If L({x,y}) $\neq$ {0},then x $\notin Z_y$ and y $\notin Z_x$ .For any a $\in Z_x$, we have L({x,a}) = {0} $\subseteq Z_y$ .since Z_y is L – prime, it follows that a$\in Z_y$ so that $Z_x \subseteq Z_y$ .similarly, $Z_y \subseteq Z_x$. Hence $Z_x = Z_y$. which is a contradiction . Therefore, x is adjacent to y.

Theorem : 3.28

Let X be a finite length of commutative KS-semigroup and 0 $\in$ X, then

G(X) is a cycle iff X = {0}

Proof :

X is a commutative KS - semigroup.

G(X) is a connected graph . If X = {0}, then clearly , G(X) is a tree.

Let X $\neq$ {0}, x ,y $\in$ X - {0} and so L({x,y}) = {0}.

Hence E(G(X)) = { x 0/ x$\in$ X - {0}}does not have tree. Therefore, G(X) is a cycle .

Example: 3.29

Let X = {0,a,b,c} be a commutative KS- semigroup. Define the operation "*" and "•" by the example 3.3. X = {0,a,b,c}.

E(G(X)) = { x 0/ x$\in$ X - {0}}

L({a,b}) = {0}, L({b,c}) = {0}, L({c,a}) = {0}

E(G(X)) ={ a0,b0,c0,ab,bc,ca},

Therefore , G(X) is a cycle.

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