IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 4 Ver. IV (Jul-Aug. 2014), PP 38-39
www.iosrjournals.org

A New Bound for the Gamma FunctionIn the Direction of W.K.Hayman

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Abstract: *In this paper I have extended a result of W.K.Hayman to Euler's gamma function which is known to be a logarithmically Convex function.*
Key Words: *Euler's gamma function,convex function.*

I. Introduction And Main Results

By Bohr – Mollerup- Artin theorem, we can define the Euler gamma function as follows.

<u>Theorem A:</u> Let $f : R^+ \to R^+$ satisfy each of the following properties.

i) Log f(x) is a convex function

ii) $f(x+1) = x f(x) \quad \forall\ x \in R^+$

iii) $f(1) = 1$

Then, $f(x) = \Gamma(x)$, $\forall\ x \in R^+$

But,Usually the Euler Gamma function is introduced as a function of a real variable and is defined via an integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt\ ,\ x>0$$

Here, we can observe that $\Gamma(x+1) = x!$

One can easily verify the following properties.

Property 1 :

$\Gamma(x)$ is a logarithmatically convex function

(or)

f(x) = log $\Gamma(x)$ is a convex function.

The proof of this property follows directly from the following definition.

Definition :

Let Φ (x) be a real valued function on [a, b] and let $\Phi^{||}$ (x) $\geq$ 0 for all x $\in$ [a, b]. Or, Equivalently,Let $\Phi^{|}$ be non decreasing on [a, b].Then, Φ is a convex function on [a, b].

Or Φ is said to be convex on [a, b] iff

$$\begin{vmatrix} \Phi(x_1) & x_1 & 1 \\ \Phi(x_2) & x_2 & 1 \\ \Phi(x_3) & x_3 & 1 \end{vmatrix} \leq 0$$

i.e. $\Phi(x_1)\ (x_3 - x_2) + \Phi(x_3)\ (x_2 - x_1) \geq \Phi(x_2)\ (x_3 - x_1)$ ---→ (1)

Clearly, one can observe that $\Phi(x) = \log\Gamma(x)$ statisfies (1) and hence log $\Gamma(x)$ is a convex function on [a, b].

We wish to establish the following theorem.

Theorem :

Suppose that g(x) = $\log\Gamma(x)$ is a strictly increasing and convex function of x for x $\geq$ x_0. Then given k > 1 there exists a sequence $x_n \to \infty$ such that If f(x) is any other positive increasing and convex function of x such that f(x) < g(x) for x $\geq$ x_0,then we have,

$$\Gamma(x_n) f^{|}(x_n) < e^k \Gamma^{|}(x_n) \quad \textbf{(n=1,2, …..)}$$

Here, $f^{|}(x)$ denotes the right derivative of f(x) and $\frac{\Gamma^{|}(x)}{\Gamma(x_n)}$ is the left derivative of g(x).

To prove the above result, we require the following lemma [Hayman].

Lemma[1] : Suppose that $\Phi(x)$ is positive for $x \geq x_0$ and bounded in every interval $[x_0, x_1]$ when $x_0 < x_1 < \infty$. Then given $k > 1$ there exists a sequence $x_n \to \infty$ such that

$$\Phi(x) < k\,\Phi(x_n) \text{ for } x_n < x < x_n + \frac{1}{\log^{+}[\Phi(x_n)]^{k}} + \frac{1}{\Phi(x_n)}$$

Proof of the theorem :

Since $g(x) = \log\Gamma(x)$ is convex, $g^{|}(x)$ is non decreasing. Since g(x) is strictly increasing, $g^{|}(x) > 0$ for $x > x_0$. Also $g^{|}(x)$ is bounded above in any finite interval (x_0, x_1) for $x_1 > x_0$. Thus, we may apply the above lemma to the function $\Phi(x) = \frac{g^{|}(x)}{g(x)}$ and hence we can find a sequence $x_n \to \infty$ such that.

$$\Phi(x) < k\,\Phi(x_n) \quad \text{for } x_n \leq x \leq x_n + \frac{1}{\Phi(x_n)}$$

Also, if $x_n^{|} = x_n + \frac{1}{\Phi(x_n)}$,We have,

$$\text{Log } g(x_n^{|}) - \log g(x_n) = \int_{x_n}^{x_n^{|}} \frac{g^{|}(x)}{g(x)}dx < (x_n^{|} - x_n)\,k\,\Phi(x_n) = k.$$

Hence, $g(x_n^{|}) < e^{k} g(x_n)$

Then, Since $f^{|}(x)$ is increasing, we have

$$f^{|}(x_n) \leq \frac{1}{x_n^{|} - x_n}\int_{x_n}^{x_n^{|}} f^{|}(x)dx = \frac{f(x_n^{|}) - f(x_n)}{x_n^{|} - x_n} \leq \Phi(x_n)\, f(x_n^{|})$$

$$\leq \Phi(x_n)\, g(x_n^{|})$$

$$\leq e^{k}\,\Phi(x_n)\, g(x_n)$$

$$= e^{k} g^{|}(x_n)$$

Thus, $f^{|}(x_n) \leq e^{k} g^{|}(x_n)$

Or $\quad f^{|}(x_n) \leq e^{k} \frac{\Gamma^{|}(x_n)}{\Gamma(x_n)}$

Or $\quad \Gamma(x_n) f^{|}(x_n) < e^{k}\Gamma^{|}(x_n)$ (n = 1, 2, 3……)

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