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On Pairwise Completely Regular Ordered Spaces

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Abstract: *In this paper, we introduce the concept of pairwise 0-completely regular filters on a pairwise completely regular ordered bitopological space; we define the category of bitopological ordered K-spaces, which is isomorphic to that found among both bitopological and ordered spaces.*
Key words & Phrases: *A bitopological ordered space; a bitopological partially ordered space; a pairwise completely regular ordered space; a pairwise 0-completely regular filter; pairwise continuous isotone; a pairwise compact ordered space; pairwise G_k-set; pairwise k-compact; pairwise k-Lindelöf; bitopological P-spaces*

I. Introduction

In (1963) Kelly, J. C. [**2**] initiated the study of bitopological spaces. A set equipped with two topologies is called a bitopological space. Since sevsral others authors continued investigating such spaces; among them recently [**4**]. In (1965) Nachbin, L. [6] initiated the study of topological ordered spaces. A topological ordered space is a triple $(X, \tau, \leq)$, where τ is a topology on X and $\leq$ is a partial order on X. In (1971) Singal, M. K. and Singal, A. R. [**9**] introduced the concept of a bitopological ordered space, and they studied some separation axioms for such spaces. Raghavan, T. G. [7, 8] and various other authors have contributed to development and construction some properties of such spaces. In (1976) Choe, T. H. and Hong, Y. H. [1] introduced the concept of 0-completely regular filters on a completely regular ordered space and gave some results with this concept. Kopperman, R. and Lawson, J. D. [5] defined bitopological and topological ordered K-spaces in order to handle the requirements of domain theory in theoretical computer science. The aim of this paper is to introduce the concept of pairwise 0-completely regular filters on a pairwise completely regular ordered bitopological space; we define the category of bitopological ordered K-spaces, which is isomorphic to that found among both bitopological and ordered spaces.

II. Preliminaries and notations

Let $(X, \leq)$ be a partially ordered set (i.e. a set X together with a reflexive, antisymmetric and transitive relation $\leq$). For a subset $A \subseteq X$, we write:

$L(A) = \{y \in X: y \leq x \text{ for some } x \in A\}$, and

$M(A) = \{y \in X: x \leq y \text{ for some } x \in A\}$.

In particular, if A is a singleton set, say $\{x\}$, then we write L(x) and M(x) respectively. A subset A of X is said to be decreasing (resp. increasing) if A = L(A) (resp. A = M(A)). The complement of a decreasing (resp. an increasing) set is an increasing (resp. a decreasing) set. A mapping $f: (X, \leq) \to (X^*, \leq^*)$ from a partially ordered set $(X, \leq)$ to a partially ordered set $(X^*, \leq^*)$ is increasing (resp. a decreasing) if $x \leq y$ in X implies $f(x) \leq^* f(y)$ (resp. $f(y) \leq^* f(x)$). f is called **isotone** if its monotone icreasing and therefore order-preserving. f is called an order isomorphism if it is an increasing bijection such that f^{-1} is also increasing.

A bitopological ordered space [9] is a quadruple consisting of a bitopological space (X, τ_1, τ_2), and a partial order $\leq$ on X; it is denoted as $(X, \tau_1, \tau_2, \leq)$. The partial order $\leq$ said to be **c**ontinuous(resp. weakly continuous) [7] if its graph $G(\leq) = \{(x, y): x \leq y\}$ is closed in the product topology $\tau_i \times \tau_j$ (resp. $\tau_1 \times \tau_2$) where i, j =1, 2; i $\neq$j , or equivalently, if L(x) and M(x) are τ_i-closed, where i, j =1, 2 (resp. L(x) is τ_1- closed and M(x) is τ_2-closed), for each $x \in X$. If the bitopological space equipped with (weakly) continuous partial order, then the space is (weakly) pairwise Hausdorff. If a bitopological space (X, τ_1, τ_2) is equipped with a continuous partial order $\leq$; then $(X, \tau_1, \tau_2, \leq)$ will be called a bitopological partially ordered space
For a subset A of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$,

$H_i^l(A) = \cap\{F \mid F \text{ is } \tau_i\text{-decreasing closed subset of X containing A}\}$,

$H_i^m(A) = \cap\{F \mid F \text{ is } \tau_i\text{-increasing closed subset of X containing A}\}$,

Clearly, $H_i^m(A)$ (resp. $H_i^l(A)$) is the smallest τ_i-increasing closed set containing A. Further A is τ_i -increasing (resp. τ_i-decreasing) closed if and only if $A = H_i^m(A) =$ (resp. $H_i^l(A)$).

A map f: (X, τ_1, τ_2, ≤) → (X^*, τ^*_1, τ^*_2, $\leq^*$) is pairwise continuous isotone if f: (X, τ_1, τ_2) → (X^*, τ^*_1, τ^*_2) is pairwise continuous and f: (X, ≤) → (X^*, $\leq^*$) is isotone.
The category of bitopological ordered spaces and pairwise continuous isotone functions will be denoted by BTOS. Let (X, τ_1, τ_2, ≤) a bitopological ordered space. Let $\Im$ (resp. $\Re$) be a filter in X consisting of τ_i-decreasing (resp. τ_j-increasing) closed subsets of X where i, j =1, 2; i ≠j. A pair ($\Im$, $\Re$) of $\Im$, $\Re$ is called a pairwise filter on X **[7]** if F∩G ≠ ø for any F ∈ $\Im$ and G∈ $\Re$.
For given two $(\Im_1, \Re_1) \subseteq (\Im_2, \Re_2)$ $(\Im_1, \Re_1)$ and $(\Im_2, \Re_2)$ we defined a relation $(\Im_1, \Re_1) \subseteq (\Im_2, \Re_2)$ if and only if $\Im_1 \subseteq \Im_2$ and $\Re_1 \subseteq \Re_2$, we can easily remark by Zorn's Lemma that every pairwise filter is contained in a maximal pairwise filter.

III. Pairwise completely regular ordered spaces

Let I denote [0, 1], considered as a set ([0, 1], σ, ω); as a bitopological space, where σ = { (a, 1] | a ∈ [0, 1]} ∪ {[0, 1]}, and ω = { [0, a) | a ∈ [0, 1]} ∪ {[0, 1]}.
Definition 3.1: A bitopological partially ordered space (X, τ_1, τ_2, ≤) will be called pairwise completely separated provided that whenever x ≤y in X, there exists a pairwise continuous isotone f: X → I such that f(x) >f(y) where I is the unit interval with the usual order and bitopology.
The category of bitopological partially ordered spaces and pairwise continuous isotone functions will be denoted by BTPOS.
Definition 3.2: A pairwise completely separated bitopological partially ordered space (X, τ_1, τ_2, ≤) is said to be a pairwise completely regular ordered space if for any point x ∈X and for any τ_i-open neighborhood V of x there exists pairwise continuous isotone f: X → I and a pairwise continuous ant-isotone g: X → I such that f(x)=1 g(x) and $v_i^c \subseteq f^{-1}(0) \cup g^{-1}(0)$ where v_i^c denotes the complement of V in X with respect to τ_i, i = 1, 2.
The category of pairwise completely regular ordered spaces and pairwise continuous isotone functions will be denoted by PCROS.
Definition 3.3: Let (X, τ_1, τ_2, ≤) be a pairwise completely regular ordered space. a pairwise filter ($\Im$, $\Re$) on X is said to be pairwise 0-completely regular if $\Im$ (resp. $\Re$) has a τ_i-(resp. τ_j-) open base B, satisfying that for each u ∈ B, there exists many pairwise continuous isotones $f_1, f_2, \ldots, f_n$: X → [-1, 1] such that $f_r(v) = 0$ for each r = 1, 2 ,….., n and u_i^c (resp. u_j^c) is contained in $\cup f_r^{-1}$({-1, 1}), i, j =1, 2; i ≠ j.

By a maximal pairwise 0-completely regular filter on X is meant a pairwise 0-completely regular filter not contained in any other pairwise 0-completely regular filter.
Remark 3.4: For every pairwise 0-completely regular filter, there exists by Zorn's Lemma, a maximal pairwise 0-completely regular filter containing it. In particular, a pairwise 0-completely regular filter ($\Im$, $\Re$) on a pairwise completely regular ordered space X is a maximal pairwise 0-completely regular filter iff for any pair of τ_i-(resp. τ_j-) open sets u and v with $v \subseteq u$ and finitely many pairwise continuous isotones $f_1, f_2, \ldots, f_n$: X → [-1, 1] such that $f_r(v) = 0$ for each r = 1, 2 ,….., n and u_i^c (resp. u_j^c) is contained in $\cup f_r^{-1}$({-1, 1}), either $u \in \Im$ (resp. $\Re$) or $u \notin \Im$ (resp. $\Re$) and there exists some F ∈ $\Im$ (resp. $\Re$) with F ∩ v = ø , i, j =1, 2; i ≠ j.
Theorem 3.5: A pairwise filter ($\Im$, $\Re$) on a pairwise completely regular ordered space (X, τ_1, τ_2, ≤) contains a maximal pairwise 0-completely regular filter iff
f ($\Im$, $\Re$) is convergent for each pairwise continuous isotone f: X → [-1, 1].
Proof. Since pairwise filter containing a convergent pairwise filter is a gain convergent, its enough to show that every maximal pairwise 0-completely regular filter satisfies the necessary conditions. Let ($\Im$, $\Re$) be a maximal pairwise 0-completely regular filter on X and f a member of pairwise hom (X, [-1, 1]).
Since [-1, 1] is pairwise compact, ∩{ H_i^l(f (F)) | F∈ $\Im$ }(resp. ∩{ H_j^m (f(F))| F∈ $\Re$ }) 0≠ ø. Using the above remark, it is easy to show that ∩{ H_i^l(f (F))| F∈ $\Im$ }(resp. ∩{ H_j^m (f(F)) | F∈ $\Re$ }is a singleton set and f($\Im$, $\Re$) converges to the point.
Conversely, let (μ, υ) be pairwise filter on X such that for any f∈ of pairwise hom (X, [-1, 1] f (μ, υ) is convergent. Let (x_f, y_f)= lim f (μ, υ). Then $f^{-1}(N_i$ (x_f)) ⊆ μ (resp. $f^{-1}(N_j$ (y_f))⊆ υ, where N_i (x_f) (resp. N_j (y_f)) is the τ_i-(resp. τ_j-) neighborhood filter of x_f -(resp. y_f)).Hence ∪{ $f^{-1}(N_i$ (x_f))}(resp.

$\cup\{f^{-1}(N_j$ (y_f))} | f∈ pairwise hom (X, [-1, 1])}generates a pairwise filter; let $\mathfrak{I} = \vee f^{-1}(N_i$ (x_f)) (resp. $\mathfrak{R} = \vee f^{-1}(N_j$ (y_f))) | f∈ pairwise hom (X, [-1, 1]). It is easy to show that a join of pairwise 0-completely regular filters is a gain pairwise 0-completely regular filter and that ($f^{-1}(N_i$ (x_f)), $f^{-1}(N_j$ (y_f))) is a pairwise 0-completely regular filter base. Hence ($\mathfrak{I}, \mathfrak{R}$) is a pairwise 0-completely regular filter. Using the above remark, ($\mathfrak{I}, \mathfrak{R}$) is a maximal pairwise 0-completely regular filter contained in (μ, υ), i, j =1, 2; i ≠j. □

Remark 3.6: For a maximal pairwise 0-completely regular filter ($\mathfrak{I}, \mathfrak{R}$) on a pairwise completely regular ordered space (X, τ_1, τ_2, ≤) and f ∈ pairwise hom (X, [-1, 1]), let x_f = lim $f(\mathfrak{I})$ (resp. y_f = lim $f(\mathfrak{R})$). Then $\mathfrak{I} = \vee f^{-1}(N_i$ (x_f)) (resp. $\mathfrak{R} = \vee f^{-1}(N_j$ (y_f))) | f ∈ pairwise hom (X, [-1, 1]).

By the definition of pairwise completely regular ordered space and the above theorem, we have,

Corollary 3.7: Every neighborhood pairwise filter of pairwise completely regular ordered space is a maximal pairwise 0-completely regular filter.

IV. Pairwise compact and k- compact ordered spaces

The following definition of pairwise compactness is due to Kim [3].

Definition 4.1: Let (X, τ_1, τ_2, ≤) be a bitopological ordered space. Let τ(i, V) ={ø, X, { U ∪V | U∈τ_i }} where V∈τ_j, i, j =1, 2; i ≠j. If τ(i, V) is compact for every V∈τ_j. hen the space is called pairwise compact.

Definition 4.2: A pairwise compact bitopological space (X, τ_1, τ_2) equipped with a continuous partial order is called a pairwise compact ordered space.

Note that, if a continuous partial order is replaced by a weakly continuous partial order in the above definition, then we obtain on the definition of a pairwise G- compact space due to Raghavan, T. G. [8].

The proofs of the two following lemmas are similar to which in [**7**]

Lemma 4.3: If (X, τ_1, τ_2) is bitopological space equipped with a continuous partial order, then if K is τ_i-(resp. τ_j-) compact then L(K) (resp. M(K)) is τ_j-(resp. τ_i-) closed , i, j =1, 2; i ≠ j.

Lemma 4.4: Let (X, τ_1, τ_2, ≤) be a pairwise compact ordered space. If P is an increasing τ_i-(resp. a decreasing τ_j-) set and V is a τ_i-(resp.τ_j-) neighborhood of P, then there exists an increasing τ_i-(resp. a decreasing τ_j-) open set U such that P ⊂ U ⊂ V, i, j =1, 2; i ≠ j.

Definition 4.5: Let k be an infinite cardinal. A pairwise completely regular ordered space (X, τ_1, τ_2, ≤) is called pairwise k-compact if every maximal pairwise 0-completely regular filter on X with the k-intersection property is convergent.

Definition 4.6 [1]: Let k be an infinite cardinal, and let (X, τ) be a topological space. A subset of X is called a G_k-set if it is an intersection of fewer than k-open subsets of X. A subset of X is called k-closed if it is closed with respect to the topology generated by all G_k-sets of X.

Let (X, τ_1, τ_2) be a bitopological space, A ⊂ X, we say that A is a pairwise G_k-set if A is G_k-set with respect to both τ_1 and τ_2. A subset of X is called pairwise closed if it is closed with respect to both τ_1 and τ_2.

For a pairwise completely regular ordered space (X, τ_1, τ_2, ≤). Let $\beta_o X$ be the set of all maximal pairwise 0-completely regular filters on X, endowed with the topologies τ_i (resp.τ_j) generated by { U^* | U^* = {(μ, υ)∈ $\beta_o X$ | U ∈ μ (resp.∈ υ), U is an increasing τ_i-(resp. a decreasing τ_j-) open set, i, j =1, 2; i ≠ j. And a relation $\leq^*$ defined as follows: (μ_1, υ_1) $\leq^*$ (μ_2, υ_2) in $\beta_o X$ if lim $f(\mu_1)$ ≤ lim $f(\mu_2)$ and lim $f(\upsilon_1)$ ≤ lim $f(\upsilon_2)$ for all f ∈hom (X, [-1, 1] It is abvious that ($\beta_o X$, ≤) is a patially ordered set and that { U^* | U^*∈ $\beta_o X$ | U is a τ_i-open set, i =1, 2.

Let β_o: X → $\beta_o X$ be a map defined by $\beta_o(x)$ = N_i (x) for x ∈X, i =1, 2. By the construction of $\beta_o X$, $\beta_o X$ is precisely the strict extension of X with all maximal pairwise 0-completely regular filters of X as the pairwise filter trace. Furthermore for any x ∈X and any f ∈hom (X, [-1, 1], lim f(N_i (x)) = f (x) and X is pairwise completely regular; it follows that β_o is order isomorphism. Consequently the map β_o: X → $\beta_o X$ is a dense embedding.

Lemma 4.7: The space ($\beta_o X$, ≤, τ^*_1, τ^*_2) is a bitopological partially ordered space.

Proof. We wish to prove that the partial order ≤ is continuous in the product $\tau^*_i \times \tau^*_j$, i, j =1, 2; i ≠ j. Let (μ_1, υ_1), (μ_2, υ_2) are two elements of $\beta_o X$ with (μ_1, υ_1)$\not\leq$(μ_2, υ_2) in $\beta_o X$, and we wish to find τ_i-increasing (resp. τ_j-decreasing) neighborhood of (μ_1, υ_1), (resp. (μ_2, υ_2)). Since (μ_1, υ_1)$\not\leq$(μ_2, υ_2) implies to μ_1, $\not\leq$ μ_2 and υ_1 $\not\leq$ υ_2 , there are f ∈ hom (X, [-1, 1] with $f(\mu_2)\not\leq$ lim $f(\mu_1)$ and $f(\upsilon_2)\not\leq$ lim $f(\upsilon_1)$.

Let r_1 and r_2 be elements of [-1, 1] with lim $f(\mu_2, \upsilon_2) < r_1 < r_2 <$ lim $f(\mu_1, \upsilon_1)$ and let U = f^{-1} ([-1, r_1[and V = f^{-1}( ] r_2 , 1]). Then it is obvious that U^* (resp. V^*) is a τ_i – (resp. τ_j-) neighborhood of (μ_1 , υ_1), (resp. (μ_2, υ_2)) and that for any ($\Im_1, \Re_1) \in V^*$ and any $(\Im_2, \Re_2) \in U^*$, $(\Im_1, \Re_1) \not\leq (\Im_2, \Re_2)$; i .e. U^* (resp. V^*) is an increasing (resp. decreasing) as a desired. □

Lemma 4.8: A pairwise completely regular ordered space (X, τ_1, τ_2, ≤) is called a pairwise k-compact iff it is pairwise k-closed in $\beta_o X$.

Proof. Noting that $\beta_o X$ is the strict extension of X with all pairwise 0-completely regular filters as the pairwise filter trace, the proof is immediate from Lemma 4.3. □

For an infinite cardinal k, the category of pairwise k-compact ordered spaces and pairwise conttinuouss isotones will be denoted by PKCOS.

Definition 4.9 [1]: Let k be an infinite cardinal. A Hausdorff space is said to be k- Lindelöf if every filter with the k-intersection property has a cluster point.

Definition 4.10: Let (X, τ_1, τ_2) be a bitopological space. Let $\tau(i, V) = \{ø, X, \{ U \cup V \mid U \in \tau_i \}\}$ where $V \in \tau_j$, i, j =1, 2; i ≠ j. If $\tau(i, V)$ is k-Lindelöf for every $V \in \tau_j$. then the space is called pairwise k-Lindelöf.

The notion of P-space is well-known. A topological space is called a P-space if and only if every G_δ-set is open . A space (X, τ_1, τ_2) is called a bitopological P-space [8] if and only if (X, τ_1) and (X, τ_2) are both P-spaces.

Definition 4.11: A bitopological P- orderd space (X, τ_1, τ_2, ≤) is called pairwise k-ordered Lindelöf if and only if the (X, τ_1, τ_2) is pairwise k-Lindelöf and the partial order ≤ is continuous.

Proposition 4.12: A pairwise k-Lindelöf completely regular ordered space X a is pairwise k-compact ordered space.

Proof. For any maximal pairwise 0-completely regular filter ($\Im$, $\Re$) on X with the k-intersection property, let x (resp.y) be a cluster point of $\Im$ (resp. $\Re$). Then N(x) ∨ $\Im$ (resp. N(y) ∨ $\Re$) exists; N(x) = $\Im$ (resp. N(y) = $\Re$). Hence ($\Im$, $\Re$) is convergent as a desired.□

Proposition 4.13: If a pairwise filter ($\Im$, $\Re$) on a pairwise completely regular ordered space (X, τ_1, τ_2, ≤) contains a maximal pairwise 0-completely regular filter with the countable intersection property then $f(\Im)$ (resp. $f(\Re)$) is convergent for any pairwise continuous isotone f: X → R.

Proof. It is enough to show that for any maximal pairwise 0-completely regular filter ($\Im$, $\Re$) with the countable intersection property and a pairwise continuous isotone f: X → R, $f(\Im)$ (resp. $f(\Re)$) is convergent. Since $f(\Im)$ (resp. $f(\Re)$) is a filter τ_i - (resp. τ_j-) base with the countable intersection property and R is pairwise Lindelöf, $f(\Im)$ (resp. $f(\Re)$) has a cluster point. Moreover, by the same argument as that in the proof of theorem 3.4, one can easily show that $f(\Im)$ (resp. $f(\Re)$) has only one cluster point and that $f(\Im)$ (resp. $f(\Re)$) converges to the point.□

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