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Con-S-K-Invariant Partial Orderings on Matrices

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Abstract: *In this paper it is shown that all standard partial orderings are preserved for con-s-k-EP matrices.*
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I. Introduction

Let c_{nxn} be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For $A\epsilon c_{nxn}$. Let $\bar{A}$, A^T, A^*, A^S , $\bar{A}^S$, $A^\dagger$, R(A), N(A) and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose , conjugate secondary transpose, Moore Penrose inverse range space, null space and rank of A respectively. A solution X of the equation AXA = A is called generalized inverse of A and is denoted by A^- . If A ϵ c_{nxn} then the unique solution of the equations A XA =A , XAX = X, $[AX]^*$ = AX , $(XA)^* = XA$ **[9]** is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$. A matrix A is called Con-s-$k-EP_r$ if $\rho(A)=r$ and N(A) = N(A^TVK) (or) R(A)=R(KVA^T). Throughout this paper let "k" be the fixed product of disjoint transposition in S_n= { 1,2,….n} and k be the associated permutation matrix .

Let us define the function k (x)=$\left(x_{k(1)},x_{k(2)},...,x_{k(n)}\right)$. A matrix A = (a_{ij}) ϵ c_{nxn} is s-k-symmetric if $a_{ij} = a_{n-k(j)+1,n-k(i)+1}$ for i, j = 1,2,…..n . A matrix A ϵ c_{nxn} is said to be Con-s-k-EP if it satisfies the condition A_x = 0 <=> A^s k (x) = 0 or equivalently N(A) =N(A^TVK). In addition to that A is con-s-k-EP <=> KVA is con-EP or AVK is con-EP and A is con-s-k-EP<=> A^T is con-s-k-EP$_r$ moreover A is said to be Con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer **[6]**.

Theorem 2 [2]

Let $A, B \in C_{nxn}$. Then we have the following:

(i) $R(AB) \subseteq R(A); N(B) \subseteq N(AB).$

(ii) $R(AB) = R(A) \Leftrightarrow \rho(AB) = \rho(A)$ and

$N(AB) = N(B) \Leftrightarrow \rho(AB) = \rho(B)$

(iii) $N(A) = N(A^*A)$ and $R(A) = R(A^*A)$

Theorem 2.1 [p.21, 8]

Let $A, B \in C_{nxn}$. Then

(i) $N(A) \subseteq N(B) \Leftrightarrow R(B^*) \subseteq R(A^*)$

$\Leftrightarrow B = BA^-A$ for all $A^- \in A\{1\}$

(ii) $N(A^*) \subseteq N(B^*) \Leftrightarrow R(B) \subseteq R(A) \Leftrightarrow B = AA^-B$ for every $A^- \in A\{1\}$.

Definition 2.1.1

For $A, B \in C_{n\times n}$,

(i) $A \geq_L B$ if $A - B \geq 0.$

(ii) $A \geq_T B$ if $B^T B = B^T A$ and $B^T B = AB^T$

(iii) $A \geq_{rs} B$ if $\rho(A-B) = \rho(A) - \rho(B).$

The relationship between the transpose and minus orderings is studied by Baksalary **[1]**, Mitra **[8]**, Mitra and Puri **[7]** and Hartwig and Styan **[4, 5]**.

In the sequel, the following known results will be used.

Result 2.1.2 [5]

For $A,B \in C_{n\times n}$, $A \geq_L B \Leftrightarrow \rho(A^{\dagger}B) \leq 1$ and $R(B) \subseteq R(A)$ where $\mathbf{r}(A) = \max\{|\lambda| : \lambda$ is an eigen value of $A\}$ is the spectral radius.

Result 2.1.3 [3]

For $A,B \in C_{n\times n}$, $A \geq_T B \Leftrightarrow A \geq_{rs} B$ and $(A-B)^{\dagger} = A^{\dagger} - B^{\dagger}$. For other conditions to be added to rank subtractivity to be equivalent to star order, one may refer **[1]**.

Result 2.1.4 [4]

For $A,B \in C_{n\times n}$, $A \geq_{rs} B \Leftrightarrow B = BA^{-}B = BA^{-}A = AA^{-}B.$

Definition 2.1.5

Let $A \in C_{n\times n}$, if $AA^S = A^S A = I$ then A is called s-orthogonal matrix.

Theorem 2.1.6

For $A,B \in C_{n\times n}$, K is the permutation matrix associated with 'k' the set of all permutations in $S = \{1,2,....,n\}$ and V is the secondary diagonal matrix with units in its secondary diagonal then,

(i) $A \geq_L B \Leftrightarrow KVA \geq_L KVB \Leftrightarrow AVK \geq_L BVK.$

(ii) $A \geq_T B \Leftrightarrow KVA \geq_T KVB \Leftrightarrow AVK \geq_T BVK.$

(iii) $A \geq_{rs} B \Leftrightarrow KVA \geq_{rs} KVB \Leftrightarrow AVK \geq_{rs} BVK.$

Proof

(i) $A \geq_L B \Leftrightarrow \mathbf{r}(A^{\dagger}B) \leq 1$ and $R(B) \subseteq R(A)$ **(by result (2.1.2))**

$\Leftrightarrow \mathbf{r}(A^{\dagger}VKKVB) \leq 1$ and $B = AA^{\dagger}B$ **(by Theorem (2.1))** $\Leftrightarrow \mathbf{r}(A^{\dagger}VKKVB) \leq 1$ and $(KVB) = (KVA)(A^{\dagger}VK)(KVB)$

$\Leftrightarrow \mathbf{r}((KVA)^{\dagger}(KVB)) \leq 1$ and $R(KVB) \subseteq R(KVA)$

(by (2.11) [6] and Theorem (2.1))

$\Leftrightarrow KVA \geq_L KVB$ **(by result (2.1.2))**

Also, $A \geq_L B \Leftrightarrow \mathbf{r}(A^{\dagger}B) \leq 1$ and $R(B) \subseteq R(A)$ **(by result (2.1.2))**

$\Leftrightarrow \mathbf{r}(KVA^{\dagger}BVK) \leq 1$ and $B = AA^{\dagger}B$ **(by Theorem(2.1))** $\Leftrightarrow \mathbf{r}((AVK)^{\dagger}(BVK)) \leq 1$ and

$(BVK) = (AVK)(AVK)^{\dagger}(BVK)$

$\Leftrightarrow \mathbf{r}((AVK)^{\dagger}(BVK)) \leq 1$ and $R(BVK) \subseteq R(AVK)$

(by Theorem (2.1))

$\Leftrightarrow AVK \geq_L BVK$ **(by Result (2.1.2))**

(ii) $A \geq_T B \Leftrightarrow B^T B = B^T A$ and $BB^T = AB^T$ **(by definition of transpose ordering)**

$\Leftrightarrow B^T VKKVB = B^T VKKVA$ and $KVBB^T VK = KVAB^T VK$

$\Leftrightarrow (KVB)^T (KVB) = (KVB)^T (KVA)$ and $(KVB)(KVB)^T = (KVA)(KVB)^T$

$\Leftrightarrow (KVA) \geq_T KVB$ **(by definition of transpose ordering)** Similarly it can be proved that, $A \geq_T B \Leftrightarrow AVK \geq_T BVK.$

(iii) $A \geq_{rs} B \Leftrightarrow \rho(A-B)=\rho(A)-\rho(B)$ **(by definition of minus ordering)**

$$\Leftrightarrow \rho(KV(A-B))=\rho(KVA)-\rho(KVB)$$
$$\Leftrightarrow \rho(KVA-KVB)=\rho(KVA)-\rho(KVB)$$
$$\Leftrightarrow KVA \geq_{rs} KVB.$$

Similarly it can be proved that, $A \geq_{rs} B \Leftrightarrow AVK \geq_{rs} BVK.$

Thus, all the three standard partial orderings are preserved for con-s-k-EP matrices.

The following results can be easily verified by using the **Theorem (2).**

Result 2.1.7

Lowener ordering is preserved under unitary similarity, that is, $A \geq_L B \Leftrightarrow P^T AP \geq_L P^T BP.$

Result 2.1.8

Star ordering is preserved under unitary similarity, that is, $A \geq_T B \Leftrightarrow P^T AP \geq_T P^T BP.$

Result 2.1.9

Rank subtractivity ordering is preserved under unitary similarity, that is, $A \geq_{rs} B \Leftrightarrow P^T AP \geq_{rs} P^T BP.$

Theorem 2.1.10

Lowener order, transpose order and rank subtractivity order are all preserved for s-k-orthogonal similarity.

Proof

(i) Lowener ordering is preserved for s-k-orthogonal similarity. We have to prove that, $A \geq_L B \Leftrightarrow KVP^{-1}KVAP \geq_L KVP^{1}KVBP$ for some orthogonal matrix P.

For $A \geq_L B \Leftrightarrow KVA \geq_L KVB$ **(by Theorem (2.1.6))**

$$\Leftrightarrow P^T KVAP \geq_L P^T KVBP.$$
$$\Leftrightarrow KVP^T KVAP \geq_L KVP^T KVBP.$$
$$\Leftrightarrow (KVP^{-1}KV)AP \geq_L (KVP^{-1}KV)BP$$
$$\Leftrightarrow C \geq_L D$$

Where $C = KVP^{-1}KVAP$ is orthogonaly s-k-similar to A

$D = KVP^{-1}KVBP$ is orthogonaly s-k-similar to B

Thus, Lowener ordering is preserved for s-k-orthogonal similarity.

(ii) Star ordering is preserved for s-k-orthogonal similarity, we have to prove that, $A \geq_T B \Leftrightarrow \left(KVP^{-1}KV\right)AP \geq_T \left(KVP^{-1}KV\right)BP$, for some orthogonal matrix P.

For $A \geq_T B \Leftrightarrow KVA \geq_T KVB$ **(by Theorem (2.1.6))**

$$\Leftrightarrow P^T KVAP \geq_T P^T KVBP \quad \textbf{(by result (2.1.8))}$$
$$\Leftrightarrow KVP^T KVAP \geq_T KVP^T KVBP \quad \textbf{(by Theorem (2.1.6))}$$
$$\Leftrightarrow \left(KVP^{-1}VK\right)AP \geq_T \left(KVP^{-1}VK\right)BP.$$

Thus transpose ordering is preserved for s-k-orthogonal similarity.

(iii) Rank subtractivity ordering is preserved for s-k-orthogonal similarity, we have to show that, $A \geq_{rs} B \Leftrightarrow (KVP^{-1}KV)AP \geq_{rs} (KVP^{-1}KV)BP$ for some orthogonal matrix P.

For, $A \geq_{rs} B \Leftrightarrow KVA \geq_{rs} KVB$ **(by Theorem (2.1.6))**

$$\Leftrightarrow P^T KVAP \geq_{rs} P^T KVBP \quad \textbf{(by result (2.1.9))}$$

$$\Leftrightarrow KVP^{T}KVAP \geq_{rs} KVP^{T}KVBP \qquad \textbf{(by Theorem (2.1.6))}$$

$$\Leftrightarrow (KVP^{-1}KV)AP \geq_{rs} (KVP^{-1}KV)BP$$

Thus rank subtractivity is preserved for s-k-orthogonal similarity. Thus all the three standard partial orderings are preserved for s-k-orthogonal similarity.

References

[1]. Baksalary, J.K., "Relationship between the star and minus orderings." Lin. Alg. Appl., 82 (1986), 163-168.

[2]. Ben- Israel, A. and Greville, T.N.E., "Generalized Inverses: Theory and Applications." 2nd Edition, Springer, New York (2003).

[3]. Hartwig, R.E. and Styan, G.P.H., "On some characterizations of the star partial ordering for matrices and rank subtractivity." Lin. Alg. Appl. 82 (1986), 145-161.

[4]. Hartwig. R.E. and Styan. G.P.H., "Partially ordered idempotent matrices." Proc. Second International Tampere Conference in Statistics, (1987), 361-383.

[5]. Hauke, J. and Markiewicz, A., "On partial orderings on the set of rectangular matrices." Lin. Alg. Appl., 219 (1995), 187-193.

[6]. Krishnamoorthy, S., Gunasekaran, K. and Muthugobal, B.K.N., "Con-s-k-EP matrices", Journal of Mathematical Sciences and Engineering Applications , Vol. 5, No.1, 2011, 353 – 364.

[7]. Mitra, S.K. and Puri, M.L., "The fundamental barded matrix of linear estimation and the Duffin Morley general linear electro mechanical systems." Applicable Analys., 14 (1983), 241-258.

[8]. Mitra. S.K., "Matrix partial orderings through generalized inverses: unified theory." Lin. Alg. Appl., 148 (1991), 237-263.

[9]. Rao, C.R. and Mitra, S.K., "Generalized Inverse of Matrices and Its Applications", Wiley and Sons, New York, 1971.