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Common Fixed Point Theorem for Occasionally Converse Commuting Maps in Complex-Valued Metric Space

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Abstract: *In this paper, we proved a common fixed point theorem for occasionally converse commuting (OCC) self-maps without continuity of maps for non-complete complex-valued metric space.*
Keywords: *Converse commuting maps, occasionally converse commuting maps, fixed point and complex-valued metric space*

I. Introduction

Lii [2] introduced the concept of converse commuting maps and proved some common fixed point results. Liu and Hu [3] used this concept for multi-valued mappings. Later on H.K. Pathak and R.K. Verma [6] introduced occasionally converse commuting maps and proved common fixed point theorems for occasionally converse commuting maps in symmetric spaces.

A. Azam, B. Fisher and M. Khan [1] introduced the concept of complex-valued metric space and obtain a common fixed point theorem for a pair of mappings satisfying contractive type condition.

II. Definitions And Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.

Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$$

$$z_1 \precsim z_2,\ z_2 \precsim z_3 \Rightarrow z_1 \precsim z_3.$$

Definition 2.1 ([1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:

(i) $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex-valued metric on X, and (X,d) is called a complex-valued metric space.

Example 2. Let $X = \mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, defined by

$d(z_1, z_2) = |x_1 - x_2| + i\,|y_1 - y_2|$, for all $z_1, z_2 \in \mathbb{C}$

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

Then $(\mathbb{C}, d)$ is a complex-valued metric space.

Definition 2.3 ([2]). A point $x \in X$ is called a commuting point of $f, g : X \to X$ if $fgx = gfx$.

Definition 2.4 ([2]). Maps $f, g : X \to X$ are said to be converse commuting if $fgx = gfx$ implies $fx = gx$.

Definition 2.5 ([6]). Two self-maps $f, g : X \to X$ are said to be occasionally converse commuting, if for some x in X $fgx = gfx$ implies $fx = gx$.

Following example shows that, every conversely commuting mapping is (OCC) but the converse need not be true.

Example 2.6. Let $X = \{z \in \square : \operatorname{Im}(z) \geq 0, \operatorname{Re}(z) = 0\}$ and the self-mapping f and g are defined by:

$$f(z) = \begin{cases} \dfrac{i}{n+1}, & \text{if } z = \dfrac{i}{n}, n \in \square \\ 0, & \text{otherwise} \end{cases}, \qquad g(z) = \begin{cases} \dfrac{i}{n+2}, & \text{if } z = \dfrac{i}{n}, n \in \square \\ 0, & \text{otherwise} \end{cases}$$

$$fg\left(\frac{i}{n}\right) = f\left(\frac{i}{n+2}\right) = \frac{i}{n+3}, \quad gf\left(\frac{i}{n}\right) = g\left(\frac{i}{n+1}\right) = \frac{i}{n+3}.$$

But $f\left(\dfrac{i}{n}\right) \neq g\left(\dfrac{i}{n}\right)$.

III. Main Result

Theorem 3.1. Let (X,d) be a complex valued-metric space and let f, g, h and k be four self-maps defined on X, such that the pairs (f,k) and (g,h) are occasionally converse commuting maps satisfying:

$$d(fx, gy) \P \ \lambda \max\{d(kx,hy), d(fx,kx), d(gy,hy), d(gy,kx), d(fx,hy)\} \tag{3.1}$$

for all $x, y \in X$, $\lambda \in (0,1)$, then f, g, h and k have a unique common fixed point in X.

Proof. Let OCC (f,k) denote the set of occasionally converse commuting points of f and k. Since the pairs (f,k) is occasionally converse commuting, by definition, there exists some $u \in \mathrm{OCC}(f,k)$; such that $fku = kfu$ implies $fu = ku$. Hence $d(fu, ku) = 0$.

It follows that

$$ffu = fku = kfu. \tag{3.2}$$

Similarly, the occasionally converse commuting points for the pair (g,h) implies that there exists $v \in \mathrm{OCC}(g,h)$ such that $ghv = hgv$ implies $gv = hv$. Hence $d(gv, hv) = 0$ and so

$$ggu = ghv = hgv. \tag{3.3}$$

First, we prove that $fu = gv$. If not, then using (3.1) for x=u, y=v.

$$\begin{aligned} d(fu, gv) &\P \ \lambda \max\{d(ku,hv), d(fu,ku), d(gv,hv), d(gv,ku), d(fu,hv)\} \\ &\P \ \lambda \max\{d(fu,gv), 0, 0, d(gv,fu), d(fu,gv)\} \\ &\P \ \lambda \max\{d(fu,gv)\} \end{aligned}$$

which implies that

$|d(fu,gv)| \leq \lambda |d(fu,gv)|$, a contradiction.

Therefore $fu = gv$.

Now, we claim that $ffu = fu$. If not, then considering (3.1) for $x = gu$, $y = v$, we have

$$\begin{aligned} d(ffu, gv) &\P \ \lambda \max\{d(kfu,hv), d(ffu,kfu), d(gv,hv), d(gv,kfu), d(ffu,hv)\} \\ &\P \ \lambda \max\{d(ffu,gv), d(ffu,kfu), d(gv,hv), d(gv,kfu), d(ffu,hv)\} \\ &\P \ \lambda \max\{d(ffu,gv), 0, 0, d(fu,ffu), d(ffu,fu)\} \\ &\P \ \lambda \max\{d(ffu,fu)\} \end{aligned}$$

which implies that

$|d(ffu,fu)| \leq \lambda |d(ffu,fu)|$, a contradiction.

Therefore $ffu = fu$. Similarly $ggv = gv$. Since $fu = gv$, we have

$$fu = gv = ffu = fku = kfu = ggv = hgv = ghv. \tag{3.4}$$

Therefore $fu = w$ (say), is a common fixed point of f, g, h and k. For uniqueness, let $w' \neq w$ be another common fixed point of f, g, h and k, then by (3.1), we have

$$\begin{aligned} d(fw, gw') &\P \ \lambda \max\{d(kw,hw'), d(fw,kw), d(gw',hw'), d(gw',kw), d(fw,hw')\} \\ &\P \ \lambda \max\{d(w,w'), d(w,w), d(w',w'), d(w',w), d(w,w')\} \\ &\P \ \lambda \max\{d(w,w'), 0, 0, d(w,w'), d(w,w')\} \end{aligned}$$

which implies that

$|d(w,w')| \leq \lambda |d(w,w')|$, a contradiction.

Therefore, $w = fu$ is a unique common fixed point of f, g, h and k.

Corollary 3.2. Let (X,d) be a complex-valued metric space and let f, k be self-maps on X such that the pair (f,k) is occasionally converse commuting maps satisfying

$d(fx,fy) \precsim \lambda \max\{d(kx,ky),d(fx,kx),d(fy,ky),d(fy,kx),d(fx,ky)\}$

for all $x,y\in X$, $\lambda\in(0,1)$, then (f,k) have a unique common fixed point in X.

Example 3.3. Let $X=\{z\in\mathbb{C} : 0\le \mathrm{Im}(z)<1, \mathrm{Re}(z)=0\}$.

Let $d : X\times X\to\mathbb{C}$ be the metric, defined by

$d(z_1,z_2)=|x_1-x_2|+i|y_1-y_2|$, for all $z_1,z_2\in\mathbb{C}$

$$z_1=x_1+iy_1$$
$$z_2=x_2+iy_2$$

Define the maps f,g,h and $k : X\to X$ as follows:

$$f(z)=\begin{cases}\dfrac{i}{n+4}, & \text{if } z=\dfrac{i}{n}, n\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}, \qquad g(z)=\begin{cases}\dfrac{i}{n+3}, & \text{if } z=\dfrac{i}{n}, n\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}$$

$$h(z)=\begin{cases}\dfrac{i}{n+2}, & \text{if } z=\dfrac{i}{n}, n\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}, \qquad k(z)=\begin{cases}\dfrac{i}{n+1}, & \text{if } z=\dfrac{i}{n}, n\in\mathbb{N}\\ 0, & \text{otherwise}\end{cases}$$

There exists $u\in X-\left\{\dfrac{i}{n} : n\in\mathbb{N}\right\}$ such that $fku=kfu$ implies that $fu=ku$. Hence (f,k) is (OCC). Similarly, (g,h) is (OCC). The set of (OCC) of f and k, and g and h are given by

$$\mathrm{OCC}(f,k)=\mathrm{OCC}(g,h)=u\in X-\left\{\frac{i}{n} : n\in\mathbb{N}\right\}.$$

All the conditions of Theorem 3.1 are satisfied.

$fu=ku=gv=hv=0$ is the unique common fixed point of f,g,h and k.

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