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On Weakly Ricci φ-Symmetric ε-Trans-Sasakian Manifolds

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Abstract*:In this paper we introduce the notion of weakly Ricci φ-symmetricε-trans-Sasakian manifolds and study characteristic properties of locally φ-Ricci symmetric and φ-recurrent spaces. Finally, local symmetry of a generalized recurrent weakly symmetricε-trans-Sasakian manifolds is discussed.*

Keyword: *ε-trans-Sasakian manifolds, Weakly symmetric ε-trans-Sasakian manifolds.*

I. Introduction

In Riemannian geometry, we study manifolds with metric which is positive definite. Since manifolds with indefinite metric have significant use in physics, it is interesting to study such manifolds equipped with different structure. In [1], A. Benjancu and K.L. Duggal introduced the notion ofε-Sasakian manifolds with indefinite metric. In [2], XuXufeng and Chao Xixaoli proved that ε-Sasakian manifolds is a hypersurface of an indefinite Kählerian manifold. Further R. Kumar, R. Rani and R. Nagaich studied ε-Sasakian manifolds in [3]. Since Sasakian manifolds with indefinite metric play significant role in physics [4], so it is important to study them. Recently in 2009, U.C. De and AvijitSarkar [5] introduced and studied the notion of ε- Kenmotsu manifolds with indefinite metric. Oubina [6], studied a new class of almost contact metric manifolds known as trans-Sasakian manifolds which generalizes both α-Sasakian and β-Kenmotsu manifolds. In [7], C. Gherghe introduced nearly trans-Sasakian manifolds.

In [17], Prasad, Shukla and Tripathi have studied some special type of trans-Sasakian manifolds. Ralph R., Gomez [8], studied about the weakly symmetric spaces. In 2010, S.S. Shukla and D.D. Shingh [9] have introduced the notion of ε-trans-Sasakian manifolds and studied its basic results and using these results studied some properties. Earlier to this in 1969 Takahashi [10] had introduced the notion of almost contact manifold equipped with pseudo Riemannian metric. In particular, he studied the Sasakian manifolds equipped with Riemannian metricg. This indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as ε-almost contact metric manifolds and ε-Sasakian manifolds respectively. Recently [11] and [12], we have observe that there does not exist a light like surface in the ε-Sasakian manifolds.

On the other hand in almost para contact manifold defined by Motsumoto [13], the semi-Riemannian manifolds has index 1 and the structure vector fieldξ is always a time like. This motivated the Tripathi and other [14] to introduce ε-almost para contact structure vector fields ξ is space like or time like according as $\varepsilon = 1$ or$\varepsilon = -1$.

The paper is organized as follows. In section 2, we give various preliminary results of ε-trans-Sasakian manifolds which is needed for the next section. In section 3, characterization of locallyφ-Ricci-symmetric and φ-recurrent spaces are discussed. It is established that if the weakly Ricciφ-symmetric ε-trans-Sasakian manifolds of non zeroξ-sectional curvature is locallyφ-Ricci-symmetric, then the sum of the associated 1-form $A, B$and$C$ is zero everywhere. In section 4, it is proved that if the weakly Ricci φ-symmetric ε-trans-Sasakian manifold of non zeroξ-sectional curvature is φ-Ricci-recurrent then the associated 1-forrmB and Care in opposite directions. Finally in section 5, local symmetry of a generalized recurrent weakly symmetric ε-trans-Sasakian manifolds is discussed.

II. Preliminaries

In this section, we defineε-trans-Sasakian manifolds and we give an example of ε-trans-Sasakian manifolds. Here we also gives some basic results of ε-trans-Sasakian manifolds**.**

2.1Definition: LetMbe a $(2n+1)$ dimensional almost contact metric manifold equipped with almost contact metric structure (φ, ξ, η, g), whereφ is $(1,1)$ tensor field,ξ is a vector field, $\eta$is 1-form and $g$is indefinite metric such that

$$\varphi^2 = -I + \eta \otimes \xi \quad (2.1)$$
$$\eta(\xi) = 1, \quad \varphi\xi = 0 \quad (2.2)$$
$$g(\xi,\xi) = \varepsilon, \; \eta(X) = \varepsilon g(X,\xi) \quad (2.3)$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y) \quad (2.4)$$

for all vector fields X, YonM, where $\varepsilon = \pm 1. \varepsilon = 1or\varepsilon = -1$ according asξ is space like or light like vector fields and rank φis2n. if

$$d\eta(X,Y) = g(X,\varphi Y)\text{forall}X,Y \in \Gamma TM \quad (2.5)$$

ThenM (φ,ξ,η,g) is called anε-almost contact metric manifold.

2.2 Definition: An ε-almost contact metric manifold is called an ε-trans-Sasakian manifold if

$$(\nabla_X\varphi)Y = \alpha\{g(X,Y)\xi - \varepsilon\eta(Y)X\} + \beta\{g(\varphi X,Y)\xi - \varepsilon\eta(Y)\varphi X\} \quad (2.6)$$

for anyX, Y $\in \Gamma TM$. Where ∇ is Levi-Civita connection of semi-Riemannian metric gand αandβ are smooth functions onM.

From equation (2.1), (2.2), (2.3), (2.4) and (2.6), we get

$$\nabla_X\xi = \varepsilon\{-\alpha\varphi X + \beta(X - \eta(X)\xi)\} \quad (2.7)$$

$$(\nabla_X\eta)Y = -\alpha g(\varphi X,Y) + \beta\{g(X,Y)\xi - \varepsilon\eta(X)\eta(Y)\} \quad (2.8)$$

$$\nabla_\xi\varphi = 0 \quad (2.9)$$

Now, we define ξ-sectional curvature of an ε-trans-Sasakian manifold.

2.3Dfinition: Theξ-sectional curvature of anε-trans-Sasakian manifold for a unit vector fieldX orthogonal toξ is defined by

$$K(\xi,X) = R(\xi,X,\xi,X) \quad (2.10)$$

Let us define the tensor hby$2h = £_\xi\varphi$. Where £ is the Lie differentiation operator.

Now we shall give an example of ε-trans-Sasakian manifold.

2.1 Example: Let us consider a 3-dimentional manifold= $\{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where $(x,y,z)$are the standard co-ordinate in$\mathbb{R}^3$.

Let $e_1 = (e^z(\partial/(\partial y)) + y(\partial/(\partial z)))$ $e_2 = (e^z\,\partial/(\partial y))$ $e_3 = (\partial/(\partial z))$

which are linearly independent vector fields at each point ofM. Define a semi-Riemannian metric gon Mas

$g(e^1,e^3) = g(e^2,e^3) = g(e^1,e^2) = 0$, $g(e^1,e^1) = g(e^2,e^2) = g(e^3,e^3) = \varepsilon$where$\varepsilon = \pm 1$.Let ηbe the 1-form defined by$\eta(Z) = \varepsilon g(Z,e^3)$for any$Z \in \Gamma TM$. Let φ be a tensor of type (1,1) defined by

$$\varphi e_1 = e_2, \qquad \varphi e_2 = -e_1, \qquad \varphi e_3 = 0$$

Then by using linearity of φandg, we have

$$\eta(e_3) = 1,\ \varphi^2 Z = -Z + \eta(Z)\xi$$

$$g(\varphi Z,\varphi U) = g(Z,U) - \varepsilon\eta(Z)\eta(U) \textit{for any } Z,U \in \Gamma TM.$$

If we take$e_3 = \xi$ and$M\ (\varphi,\xi,\eta,g)$is called anε-almost contact metric structure.

Let ∇ be the Levi-Civita connection with respect to gand Rbe the curvature of type (1,3), then we have

$$[e_1,e_2] = \varepsilon(ye^z e_2 - e^{2z}e_3),$$

$$[e_1,e_2] = -\varepsilon e_1$$

$$[e_1,e_2] = \varepsilon e_2$$

By using Koszul's formula for Levi-Civita connection with respect tog, we have

$$\nabla_{e_1}e^3 = -\varepsilon e^1 + \left(\frac{1}{2}\right)\varepsilon e^{2z}e^2, \qquad \nabla_{e_2}e_3 = -\varepsilon e_2 - (1/2)\varepsilon e^{2z}e_1, \quad \nabla_{e_3}e_3 = 0$$

$$\nabla_{e_1}e_2 = -(1/2)\varepsilon e^{2z}e_3, \quad \nabla_{e_2}e_2 = -\varepsilon e_3 + \varepsilon ye^{2z}e_1, \quad \nabla_{e_3}e_2 = -(1/2)\varepsilon e^{2z}e_1$$

$$\nabla_{e_1}e_1 = \varepsilon e_3\nabla_{e_1}e_3 = -\varepsilon ye^z e_2 + (1/2)\varepsilon e^{2z}e_3, \nabla_{e_3}e_1 = (1/2)\varepsilon e^{2z}e_2$$

Now for $\xi = e_3$,above results satisfy

$$\nabla_X\xi = \varepsilon\{-\alpha\varphi X + \beta(X - \eta(X)\xi)\}$$

With$\alpha = (1/2)e^{2z}$and$\beta = -1$. Consequently M (φ,ξ,η,g) is a 3-dimensionalε-trans-Sasakian manifold.

Now, we give some results onε-trans-Sasakian manifoldM of dimension$(2n+1)$.

$$R(X,Y)\xi = (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\varphi X - \eta(X)\varphi Y + \varepsilon\{\eta(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y\} \quad (2.11)$$

$$R(\xi,Y)X = (\alpha^2 - \beta^2)\{\varepsilon g(X,Y)\xi - \eta(X)Y\} + 2\alpha\beta\{\varepsilon g(\varphi X,Y)\xi + \eta(X)\varphi Y\} + \varepsilon(X\alpha)\varphi Y + \varepsilon g(\varphi X,Y)(\text{grad}\alpha) - \varepsilon g(\varphi X,Y)(\text{grad}\alpha) + \varepsilon(X\beta)\{Y - \eta(Y)\xi\} \quad (2.12)$$

$$R(\xi,Y)\xi = \{\alpha^2 - \beta^2 + \varepsilon(\xi\beta)\}\{Y + \eta(Y)\xi\} + \{2\alpha\beta + \varepsilon(\xi\alpha)\}\varphi Y \quad (2.13)$$

$$2\alpha\beta + \varepsilon(\xi\alpha) = 0 \quad (2.14)$$

$$S(X,\xi) = \{2n(\alpha^2 - \beta^2) - \varepsilon(\xi\beta)\}\eta(X) - \varepsilon(\varphi X)\alpha - \varepsilon(2n-1)(X\beta) \quad (2.15)$$

$$Q\xi = 2n\varepsilon\{(\alpha^2 - \beta^2) - \varepsilon(\xi\beta)\}\xi + \varphi(\text{grad}\alpha) - (2n-1)(\text{grad}\beta)] \quad (2.16)$$

$$\text{if } (2n-1)(\text{grad}\beta) - \varphi(\text{grad}\alpha) = (2n-1)(\xi\beta)\xi$$

Then (2.15) and (2.16) respectively reduce to

$$S(\xi,\xi) = \{2n(\alpha^2 - \beta^2) - \varepsilon(\xi\beta)\} \quad (2.17)$$

$$Q\xi = 2n\varepsilon\{(\alpha^2 - \beta^2) - \varepsilon(\xi\beta)\}\xi \quad (2.18)$$

2.1 Theorem: ξ-sectional curvature ofε-trans-Sasakian manifold Mis non vanishing. If

$$\alpha^2-\beta^2+\varepsilon(\xi\beta)\neq 0$$

Proof.From(2.12)we get

$$R(\xi,X,\xi,X)=\{\alpha^2-\beta^2+\varepsilon(\xi\beta)\}g(\varphi^2X,X)+\{2\alpha\beta+\varepsilon(\xi\alpha)\}g(\varphi X,X)\quad(2.19)$$

Using (2.14), theξ-sectional curvature is given by

$$K(\xi,X)=\alpha^2-\beta^2+\varepsilon(\xi\beta)\quad(2.20)$$

If$\alpha^2-\beta^2+\varepsilon(\xi\beta)\neq 0$, then Mis non vanishing ξ-sectional curvature.

III. Weakly Ricci φ-Symmetric And Locally Ricci φ-Symmetric Spaces

In this section, we introduce the notion of weakly Ricci φ-Symmetricε-trans-Sasakian manifolds Mand in such a space characterization of locally Ricci φ-symmetric property is discussed.

3.1 Definition: Aε-trans-Sasakian manifolds M $(n>1)$ is said to be weakly Ricci φ-symmetric if the non zero Ricci curvature tensor Qof type (1,1) satisfies the condition

$$\varphi^2(\nabla_XQ)(Y)=A(X)Q(Y)+B(Y)Q(X)+g(QX,Y)\rho\quad(3.1)$$

Where the vector fieldsX andY on M, ρis a vector field such that $g(\rho,V)=C(V)$, AandB are associated vector fields (not simultaneously zero) andφ is a tensor field of type (1,1) onM.

3.2 Definition: A weakly Ricci φ-symmetric ε-trans-Sasakian manifolds M $(n>1)$ is said to be locally Ricci φ-symmetric if$\varphi^2(\nabla Q)=0$.

3.1 Theorem: If a weakly Ricci φ-symmetricε-trans-Sasakian manifolds M $(n>1)$ of non vanishing ξ-sectional curvature is locally Ricci φ-symmetric if

$$A(\xi)+B(\xi)+C(\xi)=0,\ \text{provided}\ \alpha^2-\beta^2+\varepsilon(\xi\beta)\neq 0$$

Proof. From equation (3.1) we get

$$g(\varphi^2(\nabla_XQ)(Y),V)=A(X)S(Y,V)+B(Y)S(X,V)+C(V)S(X,Y)\quad(3.2)$$

Suppose weakly Ricciφ-symmetricε-trans-Sasakian manifolds M $(n>1)$ is locally Ricci φ-symmetric. Then from (3.2) and using definition, we have

$$A(X)S(Y,V)+B(Y)S(X,V)+S(X,Y)C(V)=0\quad(3.3)$$

Put $X=Y=V=\xi$in (3.3), we find

$$A(\xi)+B(\xi)+C(\xi)=0,\ \text{provided}\ \alpha^2-\beta^2+\varepsilon(\xi\beta)\neq 0.\quad(3.4)$$

This completes the proof.

3.2Theorem: If a weakly Ricciφ-symmetricε-trans-Sasakian manifolds M $(n>1)$ of non vanishing ξ-sectional curvature is locally Ricciφ-symmetric, then the sum of the associated 1-form A, Band Cis zero everywhere.

Proof. Set $Y=V=\xi$in (3.3), we get

$$A(X)S(\xi,\xi)=-\{B(\xi)+C(\xi)\}S(X,\xi)$$

Similarly, we have

$$B(Y)S(\xi,\xi)=-\{A(\xi)+C(\xi)\}S(Y,\xi)$$
$$C(V)S(\xi,\xi)=-\{A(\xi)+B(\xi)\}S(V,\xi)$$

Where $S(\xi,\xi)\neq 0$ on Mis given by (2.17) and $S(X,\xi)$ is given by (2.15). Adding above equation by taking $X=Y=V$and using (3.4), we get

$$A(X)+B(X)+C(X)=0\quad(3.5)$$

For any vector field Xon Mso that

$$A+B+C=0\ \textit{provided}\ \alpha^2-\beta^2+\varepsilon(\xi\beta)\neq 0\quad(3.6)$$

This completes the proof.

IV. Recurrent Spaces

In this section, we study about the recurrent spaces ofε-trans-Sasakian manifoldsM.

4.1 Definition:A weakly Ricci φ-symmetricε-trans-Sasakian manifolds M $(n>1)$ is said to be Ricci φ-recurrent if it satisfies condition

$$\varphi^2(\nabla_XQ)(Y)=A(X)Q(Y)\quad(4.1)$$

whereA is the non zero associated 1-form and X, Y are any vector fields on M.

4.1 Theorem: If a weakly Ricci φ-symmetric ε-trans-Sasakian manifolds M $(n>1)$ of non vanishing ξ-sectional curvature is Ricci φ-recurrent, then the 1-forms Band Care in the opposite directions.

Proof. Using equation (4.1), we get

$$g(\varphi^2(\nabla_X Q)(Y),V) = A(X)S(Y,V)$$

where S is the Ricci tensor of type (0,2) is given by

$$S(Y,V) = g(Q(Y),V)$$

If a weakly Ricci φ-symmetric ε-trans-Sasakian manifolds M $(n > 1)$ is Ricci φ-recurrent, then from (3.2) we have

$$B(Y)S(X,V) + C(V)S(X,Y) = 0 \qquad (4.2)$$

for any vector fields X, Y on M. Now put $X = Y = V = \xi$ in (4.2). Then , we have
$B(\xi) + C(\xi) = 0$ provided $\alpha^2 - \beta^2 + \varepsilon(\xi\beta) \neq 0$.
Further proceeding as in the proof of Theorem 3 and using the fact that

$$B(\xi) + C(\xi) = 0$$

Obviosly $B(X) + C(X) = 0$
For any vector field X on M, so that $B + C = 0$.
This completes the proof.

4.1 Corollary: If a weakly Ricci φ-symmetric ε-trans-Sasakian manifolds M $(n > 1)$ of type $(\alpha, 0)$ with α non zero constant is Ricci φ-recurrent, then both the associated 1-form B and C are in the opposite directions.
Proof. It is follows from Theorem 4.

4.2 Corollary: If a weakly Ricci φ-symmetric ε-trans-Sasakian manifolds M $(n > 1)$ of type $(0, \beta)$ with β non zero constant is Ricci φ-recurrent, then both the associated 1-form B and C are in the opposite directions.
Proof. It is also follows from Theorem 4.

V. Generalized Recurrent Spaces

In this section, we study about the locally symmetric generalized recurrent spaces of ε-trans-Sasakian manifolds M.

5.1 Definition: Anon flat Riemannian manifold M is said to be the generalized recurrent manifold if it curvature tensor R satisfies the condition

$$\nabla_X R)(Y,Z)V = A(X)R(Y,Z) + B(X)\{g(Z,V)Y - g(Y,V)Z\} \qquad (5.1)$$

Where A and B are associated 1-form and X, Y, Z, V are any vector fields on M.

5.1 Theorem: If a generalized recurrent weakly symmetric ε-trans-Sasakian manifolds M $(n > 1)$ of non vanishing ξ-sectional curvature is locally symmetric, then the 1-form A and B are related by

$$\left(\alpha^2 - \beta^2 + \varepsilon(\xi\beta)\right)A - 2n\varepsilon B = 0 \qquad (5.2)$$

Proof. Suppose a generalized recurrent weakly symmetric ε-trans-Sasakian manifolds M $(n > 1)$ of non vanishing ξ-sectional curvature is locally symmetric. Then $\nabla R = 0$
so that from (5.1), we get

$$A(X)R(Y,Z) + B(X)\{g(Z,V)Y - g(Y,V)Z\} = 0 \qquad (5.3)$$

Now (5.3) can be written as

$$A(X)R(Y,Z,V,U) + B(X)\{g(Z,V)g(Y,V) - g(Y,V)g(Z,V)\} \qquad (5.4)$$

Where $R(Y,Z,V,U) = g(R(X,Y)V,U)$.
Now contracting Y and U, in (5.4), we have

$$A(X)S(Z,V) + 2nB(X)g(Z,V) = 0 \qquad (5.5)$$

On putting $Z = V = \xi$ in (5.5), we get

$$\left(\alpha^2 - \beta^2 + \varepsilon(\xi\beta)\right)A(X) - 2n\varepsilon B(X) = 0 \qquad (5.6)$$

For any vector field X so that

$$\left(\alpha^2 - \beta^2 + \varepsilon(\xi\beta)\right)A - 2n\varepsilon B = 0 \qquad (5.7)$$

This completes the proof.

5.1 Corrolory: If a generalized recurrent weakly symmetric ε-trans-Sasakian manifolds M $(n > 1)$ with a non zero constant is locally symmetric, then the relation $A - 2n\varepsilon B = 0$ holds.

5.2 Theorem: If a generalized recurrent weakly symmetric ε-trans-Sasakian manifolds M $(n > 1)$ of vanishing ξ-sectional curvature is locally symmetric, iff 1-form A and B are zero.
Proof. For a Locally symmetric spaces, (5.3) holds. If the ξ-sectional curvature vanishes.
Then from (5.7), $B = 0$. Again from (5.3), it follows that $A = 0$. Second part is obvious from (5.1).
This completes the proof.

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