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Some Properties of Annihilator Graph of a Commutative Ring

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Abstract: *Let R be a commutative ring with unity. Let Z(R) be the set of all zero-divisors of R. For* $x \in Z(R)$, *let* $ann_R(x) = \{y \in R \mid yx = 0\}$. *We define the annihilator graph of R, denoted by* $ANN_G(R)$, *as the undirected graph whose set of vertices is* $Z(R)^* = Z(R) - \{0\}$, *and two distinct vertices x and y are adjacent if and only if* $ann_R(xy) \neq ann_R(x) \cap ann_R(y)$. *In this paper, we study the ring-theoretic properties of R and the graph-theoretic properties of* $ANN_G(R)$. *For a commutative ring R, we show that* $ANN_G(R)$ *is connected, the diameter of* $ANN_G(R)$ *is at most two and the girth of* $ANN_G(R)$ *is at most four provided that* $ANN_G(R)$ *has a cycle. For a reduced commutative ring R, we study some characteristics of the annihilator graph ANNG(R) related to minimal prime ideals of R. Moreover, for a reduced commutative ring R, we establish some equivalent conditions which describe when* $ANN_G(R)$ *is a complete graph or a complete bipartite graph or a star graph.*

Keywords: *Annihilator graph, diameter, girth, zero-divisor graph.*
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1. Introduction

Let R be a commutative ring with unity, and Z(R) be its set of all zero-divisors. For every $X \subseteq R$, we denote $X - \{0\}$ by X^*. The concept of a *zero-divisor graph* of a commutative ring R was first introduced by I. Beck in [6], where all the elements of the ring R were taken as the vertices of the graph. In [2], D. F. Anderson and P. S. Livingston modified this concept by taking the zero-divisor graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of a commutative ring R and two distinct vertices x and y are adjacent if and only if $xy = 0$. In [2], for a commutative ring R it was shown that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$ and that $gr(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle (This was proved for commutative artinian rings in [2]) In general, if $\Gamma(R)$ contains a cycle it was shown that $gr(\Gamma(R)) \leq 4$ in [11] and [7] and a simple proof is given in [4]. Thus $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $gr(\Gamma(R)) \in \{3, 4, \infty\}$. For $x \in Z(R)$, let $ann_R(x) = \{y \in R \mid yx = 0\}$. In [5], A. Badawi defined and studied the *annihilator graph* $AG(R)$ of a commutative ring R, where the set of vertices of $AG(R)$ is $Z(R)^* = Z(R) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. In [5], it was shown that $diam(AG(R)) \in \{0, 1, 2\}$ and $gr(AG(R)) \in \{3, 4, \infty\}$. In this paper, we give the definition of the *annihilator graph* in another way. In this paper, we define the annihilator graph of R, denoted by $ANN_G(R)$, as the undirected graph whose set of vertices is $Z(R)^* = Z(R) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cap ann_R(y)$. It follows that each edge (path) of the zero-divisor graph $\Gamma(R)$ and the annihilator graph $AG(R)$ ($AG(R)$ was defined by A. Badawi in [5]) is an edge (path) of $ANN_G(R)$, but converse may not be true. We show that $ANN_G(R)$ is connected with diameter at most two. If $ANN_G(R)$ contains a cycle, we show that girth of $ANN_G(R)$ is at most four. For a reduced commutative ring R, we show that the annihilator graph $ANN_G(R)$ is identical to the zero-divisor graph $\Gamma(R)$ if and only if R has exactly two minimal prime ideals. Then for a reduced commutative ring R, we show that the annihilator graph $ANN_G(R)$ is identical to the zero-divisor graph $\Gamma(R)$, as well as to the annihilator graph $AG(R)$ ($AG(R)$ was defined by A. Badawi in [5]) if and only if R has exactly two minimal prime ideals. Moreover, for a reduced commutative ring R, we establish some equivalent conditions which describe when $ANN_G(R)$ is a complete graph or a complete bipartite graph or a star graph.

For the sake of completeness, we state some definitions and notations used throughout this paper. Let G be an undirected graph. We denote the *vertex set* and the *edge set* of G by $V(G)$ and $E(G)$. We say that G is *connected* if there exists a path between any two distinct vertices. A *subgraph* of G is a graph having all of its points and lines in G. A *spanning subgraph* is a subgraph containing all the vertices of G. The *distance* between two vertices x and y of G, denoted by $d(x, y)$, is the length of a shortest path connecting them ($d(x, x) = 0$ and if such a path does not exist, then $d(x, y) = \infty$). The *diameter* of G is $diam(G) = sup\{d(x, y) \mid x$ and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G (if G contains no cycle, then $gr(G) = \infty$). We denote by C^n the graph consisting of a *cycle* with n vertices. A graph G is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n (we allow n to be an infinite cardinal). A *complete bipartite* graph is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is singleton, we call G is a *star graph*. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (we allow m and n to be an infinite cardinal); hence a star graph is a $K^{1,n}$.

Throughout this paper, R is a commutative ring with unity, $Z(R)$ is the set of all zero-divisors of R, $Nil(R)$ is the set of all nilpotent elements of R, $U(R)$ is the group of units, $T(R)$ is the total quotient ring of R and $Min(R)$ is the set of all minimal prime ideals of R. For every $X \subseteq R$, we denote $X - \{0\}$ by X^*. We call R is *reduced* if $Nil(R) = \{0\}$. The distance between two distinct vertices x and y of $\Gamma(R)$ will be denoted by $d_{\Gamma(R)}(x, y)$. For any two graphs G and H, if G is identical to H, then we write $G = H$; otherwise, we write $G \neq H$. As usual, the ring of integers and the ring of integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. Any undefined notation or terminology is standard as in [8] or [9].

2. Some basic properties of $ANN_G(R)$

This section provides the study of some basic properties of the annihilator graph $ANN_G(R)$. If $|Z(R)^*| = 1$ for a commutative ring R, then R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/\langle X^2\rangle$. In this case $ANN_G(R) = \Gamma(R)$, $ANN_G(R) = AG(R)$ and thus $ANN_G(R) = AG(R) = \Gamma(R)$. Hence throughout this article, we consider commutative rings with more than one nonzero zero-divisors.

Lemma 2.1. *Let R be a commutative ring.*

(1) *Let x and y be distinct elements of $Z(R)^*$. Then $x — y$ is not an edge of $ANN_G(R)$ if and only if $ann_R(x) = ann_R(xy) = ann_R(y)$.*

(2) *If $x — y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then $x — y$ is an edge of $ANN_G(R)$. In particular, if P is a path in $\Gamma(R)$, then P is a path in $ANN_G(R)$.*

(3) *If $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$, then $x — y$ is an edge of $ANN_G(R)$.*

(4) *If $x — y$ is not an edge of $ANN_G(R)$ for some distinct $x, y \in Z(R)^*$, then there is a $w \in Z(R)^* - \{x, y\}$ such that $x — w — y$ is a path in $\Gamma(R)$, and hence $x — w — y$ is also a path in $ANN_G(R)$.*

(5) *If $x — y$ is an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$, then $x — y$ is an edge of $ANN_G(R)$. In particular, if P is a path in $AG(R)$, then P is a path in $ANN_G(R)$.*

(6) *If $ANN_G(R) = \Gamma(R)$, then $ANN_G(R) = AG(R)$.*

Proof. (1) Suppose that $x — y$ is not an edge of $ANN_G(R)$. Then $ann_R(xy) = ann_R(x) \cap ann_R(y)$ by definition. Thus $ann_R(xy) \subseteq ann_R(x)$ and $ann_R(xy) \subseteq ann_R(y)$. But $ann_R(x) \subseteq ann_R(xy)$ and $ann_R(y) \subseteq ann_R(xy)$. Hence $ann_R(x) = ann_R(xy) = ann_R(y)$. Conversely, suppose that $ann_R(x) = ann_R(xy) = ann_R(y)$. Then $ann_R(xy) = ann_R(x) \cap ann_R(y)$. Hence $x — y$ is not an edge of $ANN_G(R)$ by definition.

(2) Suppose that $x — y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then $xy = 0$ and $ann_R(xy) = ann_R(0) = R$. Since $x \neq 0, y \neq 0$ we have $ann_R(x) \neq R$ and $ann_R(y) \neq R$. Therefore $ann_R(xy) \neq ann_R(x)$ and $ann_R(xy) \neq ann_R(y)$. Hence $x — y$ is an edge of $ANN_G(R)$ by (1). In particular, suppose that $P: x_0 — x_1 — x_2 — \ldots — x_{n-1}$ is a path of length n in $\Gamma(R)$. Then $x_i — x_{i+1}$ is an edge of $\Gamma(R)$ for all i $(0 \leq i < n - 1)$. This implies $x_i — x_{i+1}$ is an edge of $ANN_G(R)$ for all i $(0 \leq i < n - 1)$. Hence $P: x_0 — x_1 — x_2 — \ldots — x_{n-1}$ is a path of length n in $ANN_G(R)$.

(3) Suppose that $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$. So assume $x — a — b — y$ is a shortest path connecting x and y in $\Gamma(R)$, where $a, b \in Z(R)^*$ and $a \neq b$. This implies $xa = 0$, $ab = 0$, $by = 0$, $xb \neq 0$ and $ay \neq 0$. Now $xa = 0 \Rightarrow xya = 0 \Rightarrow a \in ann_R(xy)$ and $by = 0 \Rightarrow xyb = 0 \Rightarrow b \in ann_R(xy)$. Thus $\{a, b\} \subseteq ann_R(xy)$ such that $a \notin ann_R(y)$ and $b \notin ann_R(x)$. Therefore $ann_R(xy) \neq ann_R(x)$ and $ann_R(xy) \neq ann_R(y)$. Hence $x — y$ is an edge of $ANN_G(R)$ by (1).

(4) Suppose that $x — y$ is not an edge of $ANN_G(R)$ for some distinct $x, y \in Z(R)^*$. Then $ann_R(x) = ann_R(y) = ann_R(xy)$ by (1). Also $x — y$ is not an edge of $\Gamma(R)$ by (2) and hence $xy \neq 0$.

Therefore there is a $w \in ann_R(x) = ann_R(y)$ such that $w \neq 0$. If $w \in \{x, y\}$, then $xy = 0$, a contradiction. Thus $w \in Z(R)^* - \{x, y\}$ such that $x — w — y$ is a path in $\Gamma(R)$. Hence $x — w — y$ is a path in $ANN_G(R)$ by (2).

(5) Suppose that $x — y$ is an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$. Then $ann_R(xy) \neq ann_R(x)$ and $ann_R(xy) \neq ann_R(y)$ by [5, Lemma 2.1 (1)]. Hence $x — y$ is an edge of $ANN_G(R)$ by (1). In particular, suppose that $P : x_0 — x_1 — x_2 — \ldots — x_{n-1}$ is a path of length n in $AG(R)$. Then $x_i — x_{i+1}$ is an edge of $AG(R)$ for all i $(0 \leq i < n - 1)$ This implies $x_i — x_{i+1}$ is an edge of $ANN_G(R)$ for all i $(0 \leq i < n - 1)$. Hence $P : x_0 — x_1 — x_2 — \ldots — x_{n-1}$ is a path of length n in $ANN_G(R)$.

(6) Let $ANN_G(R) = \Gamma(R)$. If possible, suppose that $ANN_G(R) \neq AG(R)$. Then there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $AG(R)$. So $x — y$ is not an edge of $\Gamma(R)$ by [5, Lemma 2.1 (2)] , and hence $ANN_G(R) \neq \Gamma(R)$, a contradiction. Thus $ANN_G(R) = AG(R)$. □

Remark 2.1. (1) The converse of the Lemma 2.1 (2) may not be true. In $\mathbb{Z}_8$, $2 — 6$ is an edge of $ANN_G(\mathbb{Z}_8)$, but $2 — 6$ is not an edge of $\Gamma(\mathbb{Z}_8)$.

(2) The converse of the Lemma 2.1 (5) may not be true. In $\mathbb{Z}_{12}$, $2 — 4$ is an edge of $ANN_G(\mathbb{Z}_{12})$, but $2 — 4$ is not an edge of $AG(\mathbb{Z}_{12})$.

(3) Every edge of $\Gamma(R)$ is an edge of $ANN_G(R)$ by Lemma 2.1 (2) and $V(ANN_G(R)) = V(\Gamma(R))$. So $\Gamma(R)$ is a spanning subgraph of $ANN_G(R)$. Again every edge of $AG(R)$ is an edge of $ANN_G(R)$ by Lemma 2.1 (5) and $V(ANN_G(R)) = V(AG(R))$. So $AG(R)$ is also a spanning subgraph of $ANN_G(R)$.

In light of Lemma 2.1 (4), we have the Theorem 2.1

Theorem 2.1. *Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $ANN_G(R)$ is connected and $diam(ANN_G(R)) \leq 2$.*

Proof. Let x and y be two distinct elements of $Z(R)^*$. If $x — y$ is an edge of $ANN_G(R)$, then $d(x, y) = 1$. Suppose that $x — y$ is not an edge of $ANN_G(R)$. Then there is a $w \in Z(R)^* - \{x, y\}$ such that $x — w — y$ is a path in $\Gamma(R)$, and hence $x — w — y$ is also a path in $ANN_G(R)$ by Lemma 2.1 (4). Thus $d(x, y) = 2$. Hence $ANN_G(R)$ is connected and $diam(ANN_G(R)) \leq 2$. □

Lemma 2.2. *Let R be a commutative ring. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. If there is a $w \in ann_R(xy) - \{x, y\}$ such that $wx \neq 0$ or $wy \neq 0$, then $x — w — y$ is a path in $ANN_G(R)$ that is not a path in $\Gamma(R)$ and $ANN_G(R)$ contains a cycle C of length* 3 *such that at least two edges of C are not the edges of $\Gamma(R)$.*

Proof. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then $xy \neq 0$. Assume there is a $w \in ann_R(xy) - \{x, y\}$ such that $wx \neq 0$ or $wy \neq 0$. Then we have $y \in ann_R(xw) - \{ann_R(x) \cap ann_R(w)\}$ and $x \in ann_R(yw) - \{ann_R(y) \cap ann_R(w)\}$. Therefore $ann_R(xw) \neq ann_R(x) \cap ann_R(w)$ and $ann_R(yw) \neq ann_R(y) \cap ann_R(w)$. So $x — w$ and $y — w$ are the two edges of $ANN_G(R)$. Thus $x — w — y$ is a path in $ANN_G(R)$. Since $xw \neq 0$ or $wy \neq 0$, we have $x — w — y$ is not a path in $\Gamma(R)$. Hence $C : x — w — y — x$ is a cycle of length 3 in $ANN_G(R)$ and at least two edges C are not the edges of $\Gamma(R)$. □

Theorem 2.2. *Let R be a commutative ring. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. If $x^2y \neq 0$ and $xy^2 \neq 0$, then there is a $w \in Z(R)^* - \{x, y\}$ such that $x — w — y$ is a path in $ANN_G(R)$ that is not a path in $\Gamma(R)$ and $ANN_G(R)$ contains a cycle C of length* 3 *such that at least two edges of C are not the edges of $\Gamma(R)$.*

Proof. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then $xy \neq 0$ and there is a $w \in ann_R(xy) - \{ann_R(x) \cap ann_R(y)\}$ such that $w \neq 0$. This implies $w \in Z(R)^*$ such that $wx \neq 0$ or $wy \neq 0$. If $w \in \{x, y\}$, then either $x^2y = 0$ or $xy^2 = 0$, a contradiction. Therefore $w \in ann_R(xy) - \{x, y\}$ such that $wx \neq 0$ or $wy \neq 0$. Thus $x — w — y$ is a path in

$ANN_G(R)$ that is not a path in $\Gamma(R)$ and $ANN_G(R)$ contains a cycle C of length 3 such that at least two edges of C are not the edges of $\Gamma(R)$ by Lemma 2.2. □

Corollary 2.2.1. *Let R be a reduced commutative ring. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then there is a $w \in ann_R(xy) - \{x, y\}$ such that $x — w — y$ is a path in $ANN_G(R)$ that is not a path in $\Gamma(R)$ and $ANN_G(R)$ contains a cycle C of length 3 such that at least two edges of C are not the edges of $\Gamma(R)$.*

Proof. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Since R is reduced, we have $(xy)^2 \neq 0$. This implies $x^2y \neq 0$ and $xy^2 \neq 0$. Thus the claim is now clear by Theorem 2.2. □

Corollary 2.2.2. *Let R be a reduced commutative ring and suppose that $ANN_G(R) \neq \Gamma(R)$. Then $gr(ANN_G(R)) = 3$. Moreover, there is a cycle C of length 3 in $ANN_G(R)$ such that at least two edges of C are not the edges of $\Gamma(R)$.*

Proof. Since $ANN_G(R) \neq \Gamma(R)$, there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. Since R is reduced, we have $(xy)^2 \neq 0$. This implies $x^2y \neq 0$ and $xy^2 \neq 0$. Thus the claim is now clear by Theorem 2.2. □

Theorem 2.3. *Let R be a commutative ring. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$. Then there is a $w \in Z(R)^* - \{x, y\}$ such that $x — w — y$ is a path in $ANN_G(R)$ and $ANN_G(R)$ contains a cycle C of length 3 such that exactly one edge of C is not an edge of $AG(R)$.*

Proof. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$. Then $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$ by [5, Lemma 2.1 (3)], and there is a $w \in Z(R)^* - \{x, y\}$ such that $x — w — y$ is a path in $\Gamma(R)$ [5, Lemma 2.1 (6)]. Thus $x — w — y$ is a path in $ANN_G(R)$ by Lemma 2.1 (2). Hence $C : x — w — y — x$ is a cycle of length 3 in $ANN_G(R)$. We have $x — w — y$ is a path in $AG(R)$ by [5, Lemma 2.1 (2)] and thus exactly one edge of C is not an edge of $AG(R)$. □

Corollary 2.3.1. *Let R be a commutative ring. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$. Then there is a $w \in ann_R(xy) - \{x, y\}$ such that $x — w — y$ is a path in $ANN_G(R)$ and $ANN_G(R)$ contains a cycle C of length 3 such that exactly one edge of C is not an edge of $AG(R)$.*

Proof. It follows directly from Theorem 2.3. □

Corollary 2.3.2. *Let R be a commutative ring and suppose that $ANN_G(R) \neq AG(R)$. Then $gr(ANN_G(R)) = 3$. Moreover, there is a cycle C of length 3 in ANNG(R) such that exactly one edge of C is not an edge of $AG(R)$.*

Proof. Since $ANN_G(R) \neq AG(R)$, there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $AG(R)$. Thus the claim is now clear by Theorem 2.3. □

Theorem 2.4. *Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $gr(ANN_G(R)) \neq 3$ if and only if $gr(ANN_G(R)) \in \{4, \infty\}$.*

Proof. If $gr(ANN_G(R)) \neq 3$, then $ANN_G(R) = AG(R)$ by Corollary 2.3.2. Then we have the following two cases.

Case 1: If $ANN_G(R) = AG(R) = \Gamma(R)$, then $gr(ANN_G(R)) = gr(AG(R)) = gr(\Gamma(R))$. We know that $gr(AG(R)) = gr(\Gamma(R)) \in \{3, 4, \infty\}$. Thus $gr(ANN_G(R)) \in \{3, 4, \infty\}$. Since $gr(ANN_G(R)) \neq 3$, we have $gr(ANN_G(R)) \in \{4, \infty\}$.

Case 2: If $ANN_G(R) = AG(R) \neq \Gamma(R)$, then $gr(AG(R)) \in \{3, 4\}$ by [5, Corollary 2.11]. Thus $gr(ANN_G(R)) \in \{3, 4\}$. Since $gr(ANN_G(R)) \neq 3$, we have $gr(ANN_G(R)) = 4$.

Thus combining both the cases, we have $gr(ANN_G(R)) \in \{4, \infty\}$.

Conversely, if $gr(ANN_G(R)) \in \{4, \infty\}$, then clearly $gr(ANN_G(R)) \neq 3$. □

Corollary 2.4.1. *Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $gr(ANN_G(R)) \in \{3, 4, \infty\}$.*

Proof. It is a direct implication of Theorem 2.4. □

Theorem 2.5. *Let R be a commutative ring and suppose that $ANN_G(R) \neq \Gamma(R)$. Then $gr(ANN_G(R)) \in \{3, 4\}$.*

Proof. Since $ANN_G(R) \neq \Gamma(R)$, there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. Since $\Gamma(R)$ is connected, we have $|Z(R)^*| \geq 3$. Again, since $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$, we have $d_{\Gamma(R)}(x, y) \in \{2, 3\}$.

Case 1: If $d_{\Gamma(R)}(x, y) = 2$, then there exists a path of length 2 from x to y in $ANN_G(R)$ by Lemma 2.1(2). Since $x — y$ is an edge of $ANN_G(R)$, we have $ANN_G(R)$ contains a cycle of length 3. Hence $gr(ANN_G(R)) = 3$.

Case 2: If $d_{\Gamma(R)}(x, y) = 3$, then there exists a path of length 3 from x to y in $ANN_G(R)$ by Lemma 2.1(2). Since $x — y$ is an edge of $ANN_G(R)$, we have $ANN_G(R)$ contains a cycle of length 4. In this case, $|Z(R)^*| \geq 5$ by [2, Example 2.1 (b)]. Hence $gr(ANN_G(R)) \in \{3, 4\}$.

Thus combining both the cases, we have $gr(ANN_G(R)) \in \{3, 4\}$. □

Theorem 2.6. *Let R be a commutative ring and suppose that $ANN_G(R) \neq \Gamma(R)$ with $gr(ANN_G(R)) \neq 3$. Then there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ and there is no path of length 2 from x to y in $\Gamma(R)$.*

Proof. Since $ANN_G(R) \neq \Gamma(R)$, there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. If possible, suppose that $x — w — y$ is a path of length 2 in $\Gamma(R)$. Then $x — w — y$ is a path of length 2 in $ANN_G(R)$ by Lemma 2.1 (2). Therefore $x — w — y — x$ is a cycle of length 3 in $ANN_G(R)$ and hence $gr(ANN_G(R)) = 3$, a contradiction. Thus there is no path of length 2 from x to y in $\Gamma(R)$. □

3. When is $ANN_G(R)$ identical to $\Gamma(R)$ and $AG(R)$?

Let R be a commutative ring with unity such that $|Z(R)^*| \geq 2$. Then $diam(\Gamma(R)) \leq 3$ by [2, Theorem 2.3]. Hence if $ANN_G(R) = \Gamma(R)$, then $diam(\Gamma(R)) \leq 2$ by Theorem 2.1.

Lemma 3.1. *[2, the proof of Theorem 2.8] Let R be a reduced commutative ring that is not an integral domain. Then $\Gamma(R)$ is complete if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Lemma 3.2. *[10, Theorem 2.6(3)] Let R be a commutative ring. Then diam $(\Gamma(R)) = 2$ if and only if either (i) R is reduced with exactly two minimal primes and at least three nonzero zero-divisors, or (ii) Z(R) is an ideal whose square is not {0} and each pair of distinct zero-divisors has a nonzero annihilator.*

In this section we study the case when R is a reduced commutative ring.

Lemma 3.3. *[5, Lemma 3.2] Let R be a reduced commutative ring that is not an integral domain and let $z \in Z(R)^*$. Then*

(1) $ann_R(z) = ann_R(z^n)$ for each positive integer $n \geq 2$;

(2) *If $c + z \in Z(R)$ for some $c \in ann_R(z) - \{0\}$, then $ann_R(z+c)$ is properly contained in $ann_R(z)$ (i.e., $ann_R(c+z) \subset ann_R(z)$). In particular, if Z(R) is an ideal of R and $c \in ann_R(z) - \{0\}$, then $ann_R(z+c)$ is properly contained in $ann_R(z)$.*

Theorem 3.1. *Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent:*

(1) *$ANN_G(R)$ is complete*;
(2) *AG(R) is complete*;
(3) *$\Gamma(R)$ is complete*;
(4) *R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$*.

Proof. (1) ⇒ (2): Let $x \in Z(R)^*$. If possible, suppose that $x^2 \neq x$. Since R is reduced, we have $x^3 \neq 0$. Now $ann_R(x) = ann_R(x^2)$ and $ann_R(x) = ann_R(x^3)$ by Lemma 3.3(1). Therefore $ann_R(x) = ann_R(x^3) = ann_R(x^2)$ and hence $x — x^2$ is not an edge of $ANN_G(R)$ by Lemma 2.1 (1), a contradiction. Thus $x^2 = x$ for each $x \in Z(R)$. Since R is reduced, we have $|Z(R)^*| \geq 2$. Let x and y be any two distinct elements of $Z(R)^*$. We have to show that $x — y$ is an edge of $AG(R)$. Suppose that $x — y$ is not an edge of $AG(R)$. Therefore $ann_R(xy) = ann_R(x)$ or $ann_R(xy) = ann_R(y)$ by [5, Lemma

2.1(1)]. Without loss of generality assume that $ann_R(xy) = ann_R(x)$. Then we have either $xy = x$ or $xy \neq x$. Clearly $xy \neq 0$. Let $xy = x$. Since $x^2 = x$, we have $xy = x^2$. This implies $x(y-x) = 0$. Also we have $x(1-x) = 0$ and $y(y-x) = y - x \neq 0$. Now $ann_R(x)$ and $ann_R(y-x)$ are two ideals of *R*. Then $ann_R(x) + ann_R(y-x)$ is also an ideal of *R*. Now $1 - x \in ann_R(x)$ and $x \in ann_R(y-x)$. This implies $(1 - x) + x = 1 \in ann_R(x) + ann_R(y-x)$. Therefore $R = ann_R(x) + ann_R(y-x)$. Then $y \in R = ann_R(x) + ann_R(y-x)$. Since $y = y + 0 = 0 + y$, we have $y \in ann_R(x)$ or $y \in ann_R(y-x)$, a contradiction. Next, let $xy \neq x$. Then $ann_R(x(xy)) = ann_R(x^2y) = ann_R(xy) = ann_R(x)$. Thus $x — xy$ is not an edge of $ANN_G(R)$ by Lemma 2.1 (1), a contradiction. Hence $x — y$ is an edge of $AG(R)$.

(2) ⇔ (3): It is clear by [5, Theorem 3.3].

(3) ⇔ (4): It is clear by Lemma 3.1.

(4) ⇒ (1): It follows directly since *R* is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Remark 3.1. If *R* is a reduced commutative ring, then it has at least two minimal prime ideals. So for a reduced commutative ring *R*, we have $|Min(R)| \geq 2$. If $Z(R)$ is an ideal of *R*, then $Min(R)$ may be infinite, as $Z(R) = \cup \{I \mid I \in Min(R)\}$. Example of a reduced commutative ring *R* with infinitely many minimal prime ideals such that $Z(R)$ is an ideal of *R* is found in [1, Example 3.13] and [10, Section 5 (Examples)].

Theorem 3.2. *Let R be a reduced commutative ring that is not an integral domain and suppose that Z(R) is an ideal of R. Then* $\Gamma(R) \neq ANN_G(R) \neq AG(R)$ *and* $gr(ANN_G(R)) = 3$.

Proof. Let $z \in Z(R)^*$ and $c \in ann_R(z) - \{0\}$. We have $c \neq z$, as *R* is reduced. Since $Z(R)$ is an ideal of *R*, we have $c + z \in Z(R)^* - \{c, z\}$. Since $(c + z)z = cz + z^2 = z^2 \neq 0$, we have $(c + z) — z$ is not an edge of $\Gamma(R)$. Now $ann_R((c+z)z) = ann_R(z^2) = ann_R(z)$ by Lemma 3.3(1). But $ann_R(c+z) \subset ann_R(z) = ann_R((c+z)z)$ by Lemma 3.3 (2). Since $ann_R((c+z)z) = ann_R(z)$, we have $(c + z) — z$ is not an edge of $AG(R)$ by [5, Lemma 2.1 (1)]. Again since $ann_R((c+z)z) \neq ann_R(c+z)$, we have $(c + z) — z$ is an edge of $ANN_G(R)$ by Lemma 2.1(1). Thus $\Gamma(R) \neq ANN_G(R) \neq AG(R)$ and hence $gr(ANN_G(R)) = 3$ by Corollary 2.2.2 or Corollary 2.3.2. □

Theorem 3.3. *Let R be a reduced commutative ring and* $|Min(R)| \geq 3$ *(Min(R) may be infinite). Then* $ANN_G(R) \neq \Gamma(R)$ *and* $gr(ANN_G(R)) = 3$.

Proof. If $Z(R)$ is an ideal of *R*, then $ANN_G(R) \neq \Gamma(R)$ by Theorem 3.2. Hence we assume that $Z(R)$ is not an ideal of *R*. Since $|Min(R)| \geq 3$, we have $diam(\Gamma(R)) = 3$ by Lemma 3.2. Thus $ANN_G(R) \neq \Gamma(R)$ by Theorem 2.1. Since *R* is reduced and $ANN_G(R) \neq \Gamma(R)$, we have $gr(ANN_G(R)) = 3$ by Corollary 2.2.2. □

Theorem 3.4. *Let R be a reduced commutative ring that is not an integral domain. Then* $ANN_G(R) = \Gamma(R)$ *if and only if* $|Min(R)| = 2$.

Proof. Assume that $ANN_G(R) = \Gamma(R)$. Since *R* is reduced commutative ring that is not an integral domain, we have $|Min(R)| = 2$ by Theorem 3.3. Conversely, suppose that $|Min(R)| = 2$. Let *P* and *Q* be the two minimal prime ideals of *R*. Since *R* is reduced, we have $Z(R) = P \cup Q$ and $P \cap Q = \{0\}$. Let $x, y \in Z(R)^*$. Suppose that $x, y \in P$. So neither $x \in Q$ nor $y \in Q$ and thus $xy \neq 0$. Since $PQ \subseteq P \cap Q = \{0\}$, we have $ann_R(xy) = ann_R(x) = ann_R(y) = Q$. Hence $x — y$ is not an edge of $ANN_G(R)$ by Lemma 2.1(1). Similarly, if $x, y \in Q$, then also $x — y$ is not an edge of $ANN_G(R)$. If $x \in P$ and $y \in Q$, then $xy = 0$ and hence $x — y$ is an edge of $ANN_G(R)$. Thus each edge of $ANN_G(R)$ is an edge of $\Gamma(R)$. Hence $ANN_G(R) = \Gamma(R)$. □

In light of Theorem 3.4, Lemma 2.1(6) and [5, Theorem 3.6], we have the Theorem 3.5.

Theorem 3.5. *Let R be a reduced commutative ring that is not an integral domain. Then* $ANN_G(R) = AG(R) = \Gamma(R)$ *if and only if* $|Min(R)| = 2$.

Theorem 3.6. *Let R be a reduced commutative ring. Then the following statements are equivalent:*

(1) $gr(ANN_G(R)) = 4$;

(2) $ANN_G(R) = AG(R) = \Gamma(R)$ *and* $gr(AG(R)) = gr(\Gamma(R)) = 4$;

(3) $gr(AG(R)) = gr(\Gamma(R)) = 4$;

(4) *$T(R)$ is ring-isomorphic to $K_1 \times K_2$, where each K_i is a field with* $|K_i| \geq 3$;

(5) $|Min(R)| = 2$ *and each minimal prime ideal of R has at least three distinct elements;*

(6) $AG(R) = \Gamma(R) = K^{m,n}$ *with* $m, n \geq 2$;

(7) $ANN_G(R) = K^{m,n}$ *with* $m, n \geq 2$.

Proof. (1) $\Rightarrow$ (2): Since $gr(ANN_G(R)) = 4$, we have $ANN_G(R) = \Gamma(R)$ by Corollary 2.2.2 and $ANN_G(R) = AG(R)$ by Corollary 2.3.2. Thus $ANN_G(R) = AG(R) = \Gamma(R)$ and hence $gr(AG(R)) = gr(\Gamma(R)) = 4$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Leftrightarrow$ (4): It is clear by [3, Theorem 2.2] and [5, Theorem 3.7].

(4) $\Leftrightarrow$ (5): It is clear by [5, Theorem 3.7].

(5) $\Leftrightarrow$ (6): If $|Min(R)| = 2$ and each minimal prime ideal of R has at least three distinct elements, then $AG(R) = \Gamma(R)$ by [5, Theorem 3.6] and hence $AG(R) = \Gamma(R) = K^{m,n}$ with $m, n \geq 2$ by [5, Theorem 3.7]. Conversely, if $AG(R) = \Gamma(R) = K^{m,n}$ with $m, n \geq 2$, then $|Min(R)| = 2$ and each minimal prime ideal of R has at least three distinct elements by [5, Theorem 3.7].

(6) $\Rightarrow$ (7): Since (6) implies $|Min(R)| = 2$, we have $ANN_G(R) = \Gamma(R)$ by Theorem 3.4 and $ANN_G(R) = AG(R) = \Gamma(R)$ by Theorem 3.5. But $AG(R) = \Gamma(R) = K^{m,n}$ with $m, n \geq 2$. Hence $ANN_G(R) = K^{m,n}$ with $m, n \geq 2$.

(7) $\Rightarrow$ (1): Since $ANN_G(R) = K^{m,n}$ with $m, n \geq 2$, we have $gr(ANN_G(R)) = 4$. □

Theorem 3.7. *Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent:*

(1) $gr(ANN_G(R)) = \infty$;

(2) $ANN_G(R) = AG(R) = \Gamma(R)$ *and* $gr(AG(R)) = gr(\Gamma(R)) = \infty$;

(3) $gr(AG(R)) = gr(\Gamma(R)) = \infty$;

(4) *$T(R)$ is ring-isomorphic to $\mathbb{Z}_2 \times K$, where K is a field;*

(5) $|Min(R)| = 2$ *and at least one minimal prime ideal of R has exactly two distinct elements;*

(6) $AG(R) = \Gamma(R) = K^{1,n}$ *for some* $n \geq 1$;

(7) $ANN_G(R) = K^{1,n}$ *for some* $n \geq 1$.

Proof. (1) $\Rightarrow$ (2): Since $gr(ANN_G(R)) = \infty$, we have $ANN_G(R) = \Gamma(R)$ by Corollary 2.2.2 and $ANN_G(R) = AG(R)$ by Corollary 2.3.2. Thus $ANN_G(R) = AG(R) = \Gamma(R)$ and hence $gr(AG(R)) = gr(\Gamma(R)) = \infty$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Leftrightarrow$ (4): It is clear by [3, Theorem 2.4] and [5, Theorem 3.8].

(4) $\Leftrightarrow$ (5): It is clear by [5, Theorem 3.8].

(5) $\Leftrightarrow$ (6): If $|Min(R)| = 2$ and at least one minimal prime ideal of R has exactly two distinct elements, then $AG(R) = \Gamma(R)$ by [5, Theorem 3.6] and hence $AG(R) = \Gamma(R) = K^{1,n}$ for some $n \geq 1$ by [5, Theorem 3.8]. Conversely, if $AG(R) = \Gamma(R) = K^{1,n}$ for some $n \geq 1$, then $|Min(R)| = 2$ and at least one minimal prime ideal of R has exactly two distinct elements by [5, Theorem 3.8].

(6) $\Rightarrow$ (7): Since (6) implies $|Min(R)| = 2$, we have $ANN_G(R) = \Gamma(R)$ by Theorem 3.4 and $ANN_G(R) = AG(R) = \Gamma(R)$ by Theorem 3.5. But $AG(R) = \Gamma(R) = K^{1,n}$ for some $n \geq 1$. Hence $ANN_G(R) = K^{1,n}$ for some $n \geq 1$.

(7) $\Rightarrow$ (1): Since $ANN_G(R) = K^{1,n}$ for some $n \geq 1$, we have $gr(ANN_G(R)) = \infty$. □

In light of Theorem 3.6 and Theorem 3.7, we have the Theorem 3.8.

Theorem 3.8. *Let R be a reduced commutative ring. Then* $ANN_G(R) = AG(R) = \Gamma(R)$ *if and only if* $gr(ANN_G(R)) = gr(AG(R)) = gr(\Gamma(R)) \in \{4, \infty\}$.

4. Conclusion

In view of Theorem 2.2 and Corollary 2.2.1, the following is an example of a nonreduced commutative ring R, where $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, but there is a path in $ANN_G(R)$ of length 2 from x to y that is also a path in $\Gamma(R)$.

Example 4.1. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. Then $(0, 1) — (2, 1)$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. But $(0, 1) — (2, 0) — (2, 1)$ is a path of length 2 from $(0, 1)$ to $(2, 1)$ in $ANN_G(R)$ that is also a path in $\Gamma(R)$.

In view of Theorem 2.2 and Corollary 2.2.1, the following is an example of a nonreduced commutative ring R, where $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, but every path in $ANN_G(R)$ of length 2 from x to y is also a path in $\Gamma(R)$.

Example 4.2. Let $R = \mathbb{Z}_2[X] / \langle X^3 \rangle$. Then $X + \langle X^3 \rangle — X + X^2 + \langle X^3 \rangle$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. Now $X + \langle X^3 \rangle — X^2 + \langle X^3 \rangle — X + X^2 + \langle X^3 \rangle$ is the only path in $ANN_G(R)$ of length 2 from $X + \langle X^3 \rangle$ to $X + X^2 + \langle X^3 \rangle$ and it is also a path in $\Gamma(R)$. Here $ANN_G(R) = K^3$, $\Gamma(R) = K^{1,2}$, $gr(\Gamma(R)) = \infty$, $gr(ANN_G(R)) = 3$, $diam(\Gamma(R)) = 2$ and $diam(ANN_G(R)) = 1$.

In view of Theorem 2.2 and Corollary 2.2.2, the following is an example of a nonreduced commutative ring R, where $ANN_G(R) \neq \Gamma(R)$, but $gr(ANN_G(R)) = 3$.

Example 4.3. Let $R = \mathbb{Z}_8$. Then $ANN_G(R) = K^3$ and $\Gamma(R) = K^{1,2}$. So $ANN_G(R) \neq \Gamma(R)$, but $gr(ANN_G(R)) = 3$.

In view of Theorem 2.3 and Corollary 2.3.2, the following are the examples of nonreduced and reduced commutative ring R, where $ANN_G(R) = AG(R)$ with $gr(ANN_G(R)) = 3$, 4 or ∞.

Example 4.4. Let $R = \mathbb{Z}_8$. Then R is nonreduced and $ANN_G(R) = AG(R) = K^3$ with $gr(ANN_G(R)) = 3$. Let $R = \mathbb{Z}_9$. Then R is nonreduced and $ANN_G(R) = AG(R) = K^{1,1}$ with $gr(ANN_G(R)) = \infty$. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then R is reduced and $ANN_G(R) = AG(R) = C^4$ with $gr(ANN_G(R)) = 4$. Let $R = \mathbb{Z}_6$. Then R is reduced and $ANN_G(R) = AG(R) = K^{1,2}$ with $gr(ANN_G(R)) = \infty$.

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