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Application of Homotopy and Variational Iteration Methods to the Atmospheric Internal Waves Model

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Abstract: *Atmospheric internal waves can be represented by a nonlinear system of partial differential equation (PDE) under shallow-fluid assumption. In this paper, we exploited the homotopy analysis method (HAM) and variational iteration method (VIM) to obtain an approximate analytical solutions of the system. The results of both methods are then compared with numerical method. It is shown that both HAM and VIM are efficient in approximating the numerical solutions.*

Keywords: *Atmospheric internal waves, homotopy analysis method, nonlinear PDE system, variational iteration method.*

I. Introduction

Internal waves, also called internal gravity waves, is a natural phenomenon whose existence cannot be seen because it propagates in the interior of a fluid, rather than on its surface. Other than below of the ocean surface, there are also internal waves in the upper atmosphere. The propagation of internal waves can be recognized by changes in temperature which occurs through the fluid in three dimensions. So, the wavelength consists of three spatial components. Internal waves may also transfer their momentum and energy to the flow with which they interact, such as winds or other waves [1]. Atmospheric internal waves can be visualized by wave clouds. In Australia, internal waves in the atmosphere result in Morning Glory clouds. Many researchers have used propagation and interaction of internal waves in the atmosphere to mathematical models and numerical simulations [1,2,3,4].

Internal waves in the atmosphere can be represented by a mathematical equation under the shallow fluid assumption. The name “shallow fluid” refers to the fact that the depth of the fluid layer is small compared to the horizontal scale of the perturbation [4]. Shallow fluid assumption has been applied to climate modeling [5], Kelvin waves [6], Rossby waves [7] and tsunami models [8]. Basically, the mathematical model of internal waves in the atmosphere is represented by a system of partial differential equations system (PDE). In many cases, these equations cannot be solved analytically. So, they must be converted to a form that can provide an approximate solution. Liao in 1992 was introduced Homotopy Analysis Method (HAM) as an analytical approach [9]. The HAM is rather general and valid for nonlinear ordinary and partial differential equations in many types [10]. It has been applied in a variety of nonlinear problems such as Klein-Gordon equation [11], generalized Huxley equation [12], Zakharov-Kuznetsov equation [13], and a single species population model [14]. The advantage of HAM method is presence of auxiliary parameter that allows us to fine-tune the region and rate of convergence of a solution [15].

Another analytical approach that can be used in solving the nonlinear system of PDE is Variational Iteration Method (VIM). It was first introduced by He in 1997 [16]. VIM is a powerful tool which is capable of solving linear/nonlinear partial differential equation. The VIM has been successfully applied to nonlinear thermoelasticity [17], Sawada-Kotera equation [18], nonlinear Whitham-Broer-Kaup equation [19], KdV-Burgers-Kuramoto equation [20], and reaction-diffussion-convection problems [22]. The VIM has many advantages, such as it avoids linerization and perturbation in order to find solutions of a given nonlinear equations. In addition, the VIM provides explicit and numerical solutions with high accuracy [22].

In this paper, we focused on solving the mathematical model of atmospheric internal waves by implementing HAM and VIM as analytical approaches. Solutions by both methods were then compared with the solution by numerical method (NUM) of mathematical models of internal waves in the atmospheric problem.

II. Atmospheric Internal Waves Models

Models of internal waves in the atmosphere are represented by a system of nonlinear PDE. This model was developed from the basic equations of fluid under shallow-fluid assumption. The fundamental equations of fluid motion in differential from are derived by conservation of mass and momentum. Shallow fluid refers to the fact that the depth of the fluid layer is the small compared with the wavelength. Here, the atmosphere is assumed to be fluid homogeneous (condition of fluid means that density does not vary in space), autobarotropic, and hydrostatic. The system is

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-fv+g\frac{\partial h}{\partial x}=0\,,\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+fu+gH=0\,,\\
&\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+vH+h\frac{\partial u}{\partial x}=0,\\
&H=-\frac{f}{g}\overline{U},
\end{aligned} \tag{1}$$

Where x is a space coordinat, t is time, the independent variables u and v are the cartesian velocity, h is depth of a fluid, f represent the Coriolis parameter, g is acceleration of gravity, H represent mean depth of a fluid, and $\overline{U}$ is the specified, constant mean geostrophic speed on which the u perturbation is superimposed [4].

III. Analysis of Method

3.1 Homotopy Analysis Method (HAM)

The principles of the Homotopy Analysis Method (HAM) are given in [9]. In this section we ilustrate of the concept of HAM based on [10-14]. We considered the following nonlinear equation in a general form

$$\mathcal{N}[u(x,t)]=0, \tag{2}$$

where $\mathcal{N}$ is a nonlinear operator, $u(x,t)$ is an unknown function, x and t denote independent variables. Furthermore, we defined a linear operator $\mathcal{L}$ which satisfies

$$\mathcal{L}[f(x,t)]=0,\ \text{when}\ f(x,t)=0. \tag{3}$$

Let $u_0(x,t)$ denotes an initial guess of the exact solution $u(x,t)$. We constructed homotopy $\phi(x,t;q)=\Omega\times[0,1]\rightarrow\mathbb{R}$ which satisfies

$$\mathcal{H}[\phi(x,t;q),u_0(x,t),\hbar,,q]=(1-q)\mathcal{L}[\phi(x,t;q)-u_0(x,t)]-q\hbar\mathcal{N}[\phi(x,t;q)], \tag{4}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq0$ is an auxiliary parameter. By the means of generalizing the traditional HAM, Liao constructed the zero-deformation equation

$$(1-q)\mathcal{L}[\phi(x,t;q)-u_0(x,t)]=q\hbar\mathcal{N}[\phi(x,t;q)]. \tag{5}$$

Setting $q=0$, the zero-order deformation equation (5) becomes

$$\mathcal{L}[\phi(x,t;0)-u_0(x,t)]=0, \tag{6}$$

which gives, using the equation (2)

$$\phi(x,t;0)=u_0(x,t), \tag{7}$$

when $q=1$, since $\hbar\neq0$, the zero-order deformation equation (5) becomes

$$\mathcal{N}[\phi(x,t;1)]=0, \tag{8}$$

which is exactly the same as the original equation (2), provided

$$\phi(x,t;1)=u(x,t). \tag{9}$$

Thus according to (7) and (9), as q increase from 0 to 1, the solution $\phi(x,t;q)$ deforms continously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$ of the original equation (2). Liao [6] expanded $\phi(x,t;q)$ in term of a power series of q as follows :

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t)q^m, \tag{10}$$

with

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{11}$$

Assume that auxiliary linear operator $\mathcal{L}$, the initial aproximation $u_0(x,t)$, the auxiliary parameter $\hbar$, and the auxiliary function $\mathcal{B}(x,t)$ are properly chosen such that the series (10) is converges at $q=1$.
Then we have the aproximate solution of equation (2), i.e.

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t). \tag{12}$$

Define the vectors,

$$\vec{u}_{m-1}=\big(u_0(x,t),u_1(x,t),u_2(x,t),\dots,u_m(x,t)\big), \tag{13}$$

Differentiating the zeroth-order deformation equation (5) m times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by m!, we have the m-order deformation equation.

$$\mathcal{L}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\hbar\mathcal{R}_m[\vec{u}_{m-1}], \tag{14}$$

where,

$$\mathcal{R}_m[\vec{u}_{m-1}]=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;q))]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{15}$$

and

$$\chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases}. \tag{16}$$

The right hand side of equation (14) depends only on the terms $\vec{u}_{m-1}$. Thus, we easy by obtain the series of $u_m, m = 1,2,3,...$ by solving the linear high-order deformation equation (13) using symbolic computation software such as Maple, Matlab or Mathematica.

3.2. Variational Iteration Method (VIM)

In this section we ilustrate the basic ideas of variational iteration method (VIM) based on [15-20]. We consider the following partial deferential equation,

$$\mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] = \mathcal{F}(x,t), \tag{17}$$

where $\mathcal{L}$ is a linear differential operator, $\mathcal{N}$ is a nonlinear operator and $\mathcal{F}$ is an inhomogeneous terms. According to the VIM, we constructed a correction functional as follows [16-20],

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\zeta)\{\mathcal{L}[u_k(x,\zeta)] + \mathcal{N}[\tilde{u}_k(x,\zeta)] - \mathcal{F}(x,\zeta)\}\, d\zeta, \tag{18}$$

where λ is a general Lagrange multiplier, whose optimal value can be identified by using the stationary conditions of the variational theory. The second terms on the the right-hand side (18) is called correction and $\tilde{u}_k$ is considered as a restricted variation, i.e $\delta\tilde{u}_k = 0$. The subscript k for $k = 0,1,2 ...,$ indicates the k^{th}-order approximation. As k tends to infinity, then iteration leads to the exact solution of (17). Consequently, the solution

$$u(x,t) = \lim_{k\to\infty} u_k(x,t). \tag{19}$$

IV. Aplication of HAM and VIM Methods

4.1 Aplication of Homotopy Analysis Method

In this section, the homotopy analysis method is applied to solve the problem of internal waves in the atmospheric, where system (1) will be solved by generalizing the described homotopy analysis method. By means of the homotopy analysis method, the linear operator can be defined as below,

$$\mathcal{L}_i[\phi_i(x,t,q)] = \frac{\partial\phi_i(x,t,q)}{\partial t}, i = 1,2,3. \tag{20}$$

According to system (1), nonlinear operators $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_3$ can be defined as follows:

$$\begin{aligned}
\mathcal{N}_1[\phi_1,\phi_2,\phi_3] &= \frac{\partial\phi_1}{\partial t} + \phi_1\frac{\partial\phi_1}{\partial x} - f\phi_2 + g\frac{\partial\phi_3}{\partial x},\\
\mathcal{N}_2[\phi_1,\phi_2,\phi_3] &= \frac{\partial\phi_2}{\partial t} + \phi_1\frac{\partial\phi_2}{\partial x} + f\phi_1 + gH,\\
\mathcal{N}_3[\phi_1,\phi_2,\phi_3] &= \frac{\partial\phi_3}{\partial t} + \phi_1\frac{\partial\phi_3}{\partial x} + \phi_2 H + \phi_3\frac{\partial\phi_1}{\partial x}.
\end{aligned} \tag{21}$$

We construct the zeroth-order deformation equation

$$\begin{aligned}
(1-q)\mathcal{L}_1[\phi_1(x,t;q) - u_0(x,t)] &= q\hbar_1\mathcal{N}_1[\phi_1,\phi_2,\phi_3],\\
(1-q)\mathcal{L}_2[\phi_2(x,t;q) - v_0(x,t)] &= q\hbar_2\mathcal{N}_2[\phi_1,\phi_2,\phi_3],\\
(1-q)\mathcal{L}_3[\phi_3(x,t;q) - h_0(x,t)] &= q\hbar_3\mathcal{N}_3[\phi_1,\phi_2,\phi_3].
\end{aligned} \tag{22}$$

According to equation (22), when $q = 0$ we can write

$$\begin{aligned}
\phi_1(x,t,0) &= u_0(x,t) = u(x,0),\\
\phi_2(x,t,0) &= v_0(x,t) = v(x,0),\\
\phi_3(x,t,0) &= h_0(x,t) = h(x,0),
\end{aligned} \tag{23}$$

and when $q = 1$, we have

$$\begin{aligned}
\phi_1(x,t,0) &= u(x,t),\\
\phi_2(x,t,0) &= v(x,t),\\
\phi_3(x,t,0) &= h(x,t).
\end{aligned} \tag{24}$$

Thus, we obtain the m^{th}-order deformation equation

$$\begin{aligned}
\mathcal{L}_1[u_m(x,t) - \chi_m u_{m-1}(x,t)] &= \hbar_1\mathcal{R}_{1,m}(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}),\\
\mathcal{L}_2[v_m(x,t) - \chi_m v_{m-1}(x,t)] &= \hbar_2\mathcal{R}_{2,m}(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}),\\
\mathcal{L}_3[h_m(x,t) - \chi_m h_{m-1}(x,t)] &= \hbar_3\mathcal{R}_{3,m}(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}).
\end{aligned} \tag{25}$$

Now, the solution of the m^{th}-order deformation equation (25) for $m \ge 1$ becomes,

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar_1\int_0^t \mathcal{R}_{1,m}(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1})\, ds \tag{26}$$

$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar_2 \int_0^t \mathcal{R}_{2,m}\left(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}\right) ds$$

$$h_m(x,t) = \chi_m h_{m-1}(x,t) + h_3 \int_0^t \mathcal{R}_{3,m}\left(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}\right) ds$$

Where,

$$\mathcal{R}_{1,m}\left(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}\right) = \frac{\partial u_{m-1}}{\partial t} + \sum_{n=0}^{m-1} u_n \frac{\partial u_{m-1-n}}{\partial x} - f v_{m-1} + g\frac{\partial h_{m-1}}{\partial x},$$

$$\mathcal{R}_{2,m}\left(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}\right) = \frac{\partial v_{m-1}}{\partial t} + \sum_{n=0}^{m-1} u_n \frac{\partial v_{m-1}}{\partial x} + f u_n + \frac{\partial^{m-1}}{\partial q^{m-1}} gH, \quad (27)$$

$$\mathcal{R}_{3,m}\left(\vec{u}_{m-1},\vec{v}_{m-1},\vec{h}_{m-1}\right) = \frac{\partial h_{m-1}}{\partial t} + \sum_{n=0}^{m-1} u_n \frac{\partial h_{m-1-n}}{\partial x} + h_n \frac{\partial u_{m-1-n}}{\partial x} + v_{m-1} H.$$

According to equation (12), the results of system (1) can be obtained by solving the following series:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t),$$

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t), \quad (28)$$

$$h(x,t) = h_0(x,t) + \sum_{m=1}^{+\infty} h_m(x,t).$$

4.2 Aplication of Variational Iteration Method

In this section, we implement VIM for obtaining the analytical approximate solution of system (1). By means of the variational iteration method refers to system (1), we construct correction functionals as follow

$$u_{k+1}(x,t) = u_k + \int_0^t \lambda_1(\zeta)\left((u_k)_\zeta + \tilde{u}_k(\tilde{u}_k)_x - f\tilde{v}_k + g\tilde{u}_k\left(\tilde{h}_k\right)_x\right) d\zeta,$$

$$v_{k+1}(x,t) = v_k + \int_0^t \lambda_2(\zeta)\left((v_k)_\zeta + \tilde{u}_k(\tilde{v}_k)_x + f\tilde{u}_k + gH\right) d\zeta, \quad (29)$$

$$h_{k+1}(x,t) = h_k + \int_0^t \lambda_3(\zeta)\left((h_k)_\zeta + \tilde{u}_k\left(\tilde{h}_k\right)_x + H\tilde{v}_k + \tilde{h}_k(u_k)_x\right) d\zeta,$$

where λ_1, λ_2 and $\lambda_3$are general Lagrange multipliers which their optimal values can be found by using variational theory. Now, taking variation with respect to indepent variables $\tilde{u}_k, \tilde{v}_k$ and $\tilde{h}_k$ we have

$$\delta u_{k+1}(x,t) = \delta u_k + \delta \int_0^t \lambda_1(\zeta)\left((u_k)_\zeta + \tilde{u}_k(\tilde{u}_k)_x - f\tilde{v}_k + g\tilde{u}_k\left(\tilde{h}_k\right)_x\right) d\zeta,$$

$$\delta v_{k+1}(x,t) = \delta v_k + \delta \int_0^t \lambda_2(\zeta)\left((v_k)_\zeta + \tilde{u}_k(\tilde{v}_k)_x + f\tilde{u}_k + gH\right) d\zeta, \quad (30)$$

$$\delta h_{k+1}(x,t) = \delta v_k + \delta \int_0^t \lambda_3(\zeta)\left((h_k)_\zeta + \tilde{u}_k\left(\tilde{h}_k\right)_x + H\tilde{v}_k + \tilde{h}_k(u_k)_x\right) d\zeta.$$

To find the optimal value of λ_1, λ_2 and λ_3, we employ $\delta\tilde{u}_k = 0, \delta\tilde{v}_k = 0, \delta\tilde{h}_k = 0$, from which we have

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \lambda_1\left((u_k)_\zeta\right) d\zeta,$$

$$\delta v_{k+1}(x,t) = \delta v_k(x,t) + \delta \int_0^t \lambda_2\left((v_k)_\zeta\right) d\zeta, \quad (31)$$

$$\delta h_{k+1}(x,t) = \delta v_k(x,t) + \delta \int_0^t \lambda_3\big((h_k)_\zeta\big)\, d\zeta,$$

The considered stationary conditions are then obtained in following from:

$$\begin{aligned} 1+\lambda_1(\zeta) &= 0, \qquad \lambda'_1(\zeta) = 0 \\ 1+\lambda_2(\zeta) &= 0, \qquad \lambda'_2(\zeta) = 0 \\ 1+\lambda_3(\zeta) &= 0, \qquad \lambda'_3(\zeta) = 0 \end{aligned} \tag{32}$$

Thus, the Lagrange multipliers are defined as follow:

$$\lambda_1(\zeta) = -1, \qquad \lambda_2(\zeta) = -1\,, \qquad \lambda_3(\zeta) = -1. \tag{33}$$

Substituting Lagrange multipliers in (33) into the correction functional in equation (28) gives the iteration formula,

$$\begin{aligned} u_{k+1}(x,t) &= u_k - \int_0^t \big((u_k)_\zeta + u_k(u_k)_x - fv_k + g(h_k)_x\big)\, d\zeta, \\ v_{k+1}(x,t) &= v_k - \int_0^t \big((v_k)_\zeta + u_k(v_k)_x + fu_k + gH\big)\, d\zeta, \\ h_{k+1}(x,t) &= h_k - \int_0^t \big((h_k)_\zeta + u_k(h_k)_x + Hv_k + h_k(u_k)_x\big)\, d\zeta. \end{aligned} \tag{34}$$

Using the initial conditions $u_0(x,t), v_0(x,t)$ and $h_0(x,t)$ into (34) we obtain the following successive approximations:

$$\begin{aligned} u(x,t) &= \lim_{k\to\infty} u_k(x,t), \\ v(x,t) &= \lim_{k\to\infty} v_k(x,t), \\ h(x,t) &= \lim_{k\to\infty} h_k(x,t). \end{aligned} \tag{35}$$

V. Results and Discussions

In this section we compare solutions obtained by HAM and VIM with one obtained by numerical methods (NUM) toward system (1) through graphical representation. Suppose the following parameters are given for numerical simulation: coriolis parameter $f = 2\Omega\sin\alpha$, where $\Omega = 7.29 \times 10^{-5}$ rad/s and $\alpha = \frac{\pi}{3}$, constant of gravity $g = 9.8$ m/s^2 and constant of pressure gradient of desired magnitude $H = -\frac{f}{g}\overline{U}$, where $\overline{U} = 2.5$ m/s is specified. To get solutions of HAM, VIM and NUM, we start the procedures with the given initial aproximation:

$$\begin{aligned} u(x,0) &= e^x \operatorname{sech}^2 x\,, \\ v(x,0) &= 2\,x \operatorname{sech}^2 2x, \\ h(x,0) &= x^2 \operatorname{sech}^2 2x. \end{aligned} \tag{36}$$

By means of the solution of the mth-order deformation equation (26) and the initial conditions (36) we obtain a number of terms as parts of series solution as follow:

$$\begin{aligned} u_0(x,t) &= e^x \operatorname{sech}^2 x \\ u_1(x,t) &= \hbar(e^x \operatorname{sech}^2 x\,(e^x \operatorname{sech}^2 x - 2\cdot e^x \operatorname{sech}^2 x \tanh x)t + 19.5997474\, xt \operatorname{sech}^2 2x \\ &\quad - 39.2\, x^2 t \operatorname{sech}^2 2x \tanh 2x) \\ &\vdots \\ v_0(x,t) &= 2\,x \operatorname{sech}^2(2x) \\ v_1(x,t) &= \hbar(-0.0003156662596t + e^x \operatorname{sech}^2 x\,(2 \operatorname{sech}^2 2x - 8x \operatorname{sech}^2 2x \tanh 2x)t \\ &\quad + 0.0001266509\, e^x t \operatorname{sech}^2 2x) \\ &\vdots \\ h_0(x,t) &= x^2 \operatorname{sech}^2 2x \\ h_1(x,t) &= \hbar(\, e^x \operatorname{sech}^2 x\,(2x \operatorname{sech}^2 2x - 4x^2 \operatorname{sech}^2 2x \tanh 2x)t \\ &\quad + x^2 \operatorname{sech}^2 2x\,(e^x \operatorname{sech}^2 x - 2\, e^x \operatorname{sech}^2 x \tanh x)t \\ &\quad - 0.00006442168564\, xt \operatorname{sech}^2 2x) \\ &\vdots \end{aligned} \tag{37}$$

The rest of the components of the iteration formulas by HAM can easily be obtained by symbolic computation software. Thus, we obtain the following aproximate solution in term of a series up to 5th-order:

$$\begin{aligned} u(x,t) &\approx u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t), \\ v(x,t) &\approx v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + v_4(x,t) + v_5(x,t), \\ h(x,t) &\approx h_0(x,t) + h_1(x,t) + h_2(x,t) + h_3(x,t) + h_4(x,t) + h_5(x,t). \end{aligned} \tag{38}$$

Furthermore, by means of the iteration formula equation (34) in the VIM and the initial condition equation (36), we obtain the approximations by symbolic computation software.

In Fig. 1, Fig. 2 and Fig. 3, we draw graphical solutions of $u(x,t), v(x,t), h(x,t)$, where for HAM, we use auxiliary parameter $\hbar = -0.825$ and a series up to 5^{th}-order, and for VIM, we use iteration until 4^{th}-order. By those figures, we can see that NUM, HAM and VIM solutions are similar.

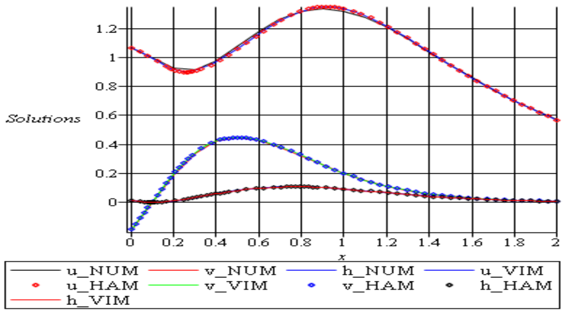


Figure 1 NUM, HAM and VIM solutions $u(x,t), v(x,t), h(x,t)$, for $t = 0.1$ and $0 \leq x \leq 2$.

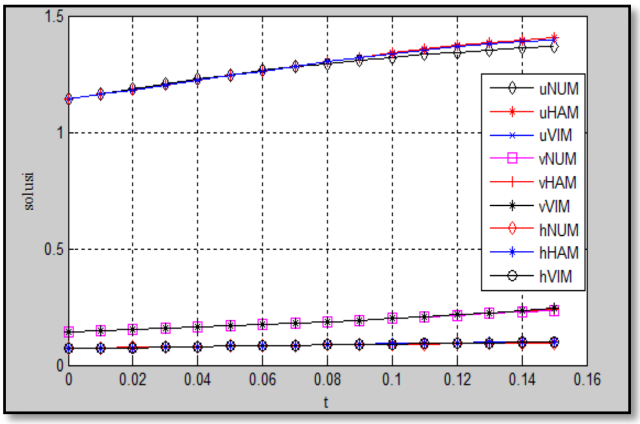
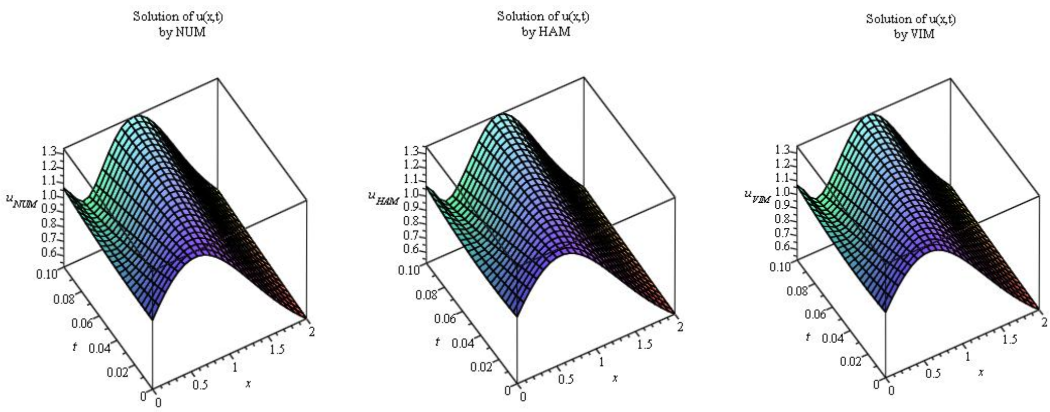
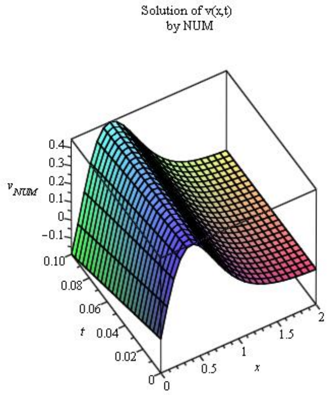
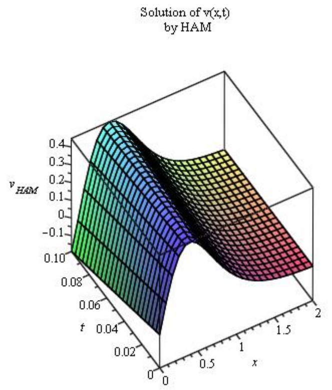


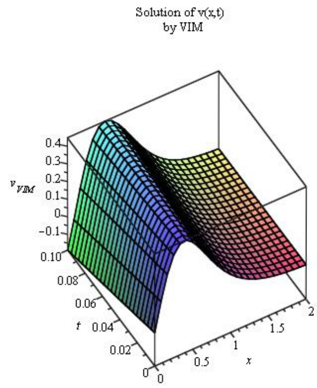
Figure 2 NUM, HAM and VIM solutions $u(x,t), v(x,t), h(x,t)$, for $x = 1$ and $0 \leq t \leq 0.15$.



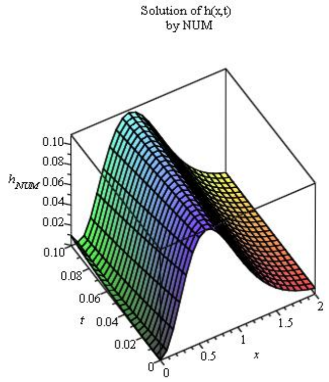
a. Solution u_{NUM} b. Solution u_{HAM} c. Solution u_{VIM}

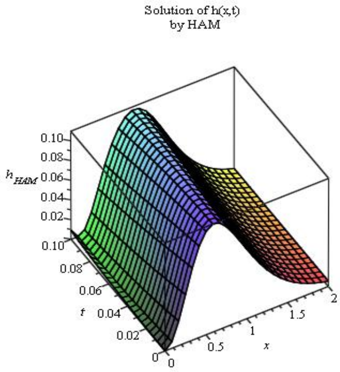


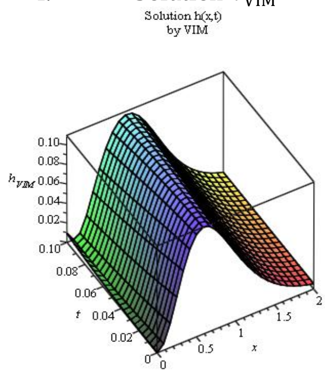




d. Solution v_{NUM} e. Solution v_{HAM} f. Solution v_{VIM}







g. Solution h_{NUM} h. Solution h_{HAM} i. Solution h_{VIM}

Figure 3 NUM, HAM and VIM solutions $u(x,t), v(x,t), h(x,t)$, for $0 \leq x \leq 2$ and $0 \leq t \leq 0.1$.

VI. Conclusion

Homotopy analysis method and variational iteration method has been succesfully applied in finding the approximate solution of the atmospheric internal waves model. Solutions by those methods are then compared with one of numerical method. It is shown that all solutions are in excellent agreement. It means that both HAM and VIM are efficient in approximating the numerical solution.

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