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Strong Equality of MAJORITY Domination Parameters

J. Joseline Manora[1]and B. John[2]
[1]*PG & Research Department of Mathematics, T.B.M.L College, Porayar, Nagai Dt., Tamilnadu, India.*
[2]*Department of Science and Humanities, E.G.S.PillayEngineering College, Nagappattinam.*

Abstract: *We study the concept of strong equality of majority domination parameters. Let P_1 and P_2 be properties of vertex subsets of a graph, and assume that every subset of $V(G)$ with property P_2 also has property P_1. Let $\psi_1(G)$ and $\psi_2(G)$, respectively, denote the minimum cardinalities of sets with properties P_1 and P_2, respectively. Then $\psi_1(G) \leq \psi_2(G)$. If $\psi_1(G) = \psi_2(G)$ and every $\psi_1(G)$-set is also a $\psi_2(G)$-set , then we say $\psi_1(G)$ strongly equals $\psi_2(G)$, written $\psi_1(G) \equiv \psi_2(G)$. We provide a constructive characterization of the trees T such that $\gamma_M(T) \equiv i_M(T)$, where $\gamma_M(T)$ and $i_M(T)$ are majority domination and independent majority domination numbers, respectively.*

Keywords: *Domination number, Majority domination number, Independent majority domination number, Strong equality.* ***2010 Mathematics Subject Classification: 05C69***

I. Introduction:

By a graph G, we mean a finite, simple and undirected. Let G be a graph with p vertices and q edges. For a vertex $v \in V(G)$, the open neighborhood of v, $N_G(v)$ is the set of vertices adjacent to v and the closed neighborhood $N_G[v] = N_G(v) \cup \{v\}$. Other graph theoretic terminology not defined here can be found in [6]. In [6], A set $S \subseteq V$ of vertices in a graph $G=(V,E)$ is a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set. The minimum cardinality of a minimal dominating set is called the domination number $\gamma(G)$ and the maximum cardinality of a minimal dominating set is called the upper domination number $\Gamma(G)$ in a graph G. A set $S \subseteq V$ of vertices in a graph G is called an independent set if no two vertices in S are adjacent. An independent set S is called a maximal independent set if any vertex set properly containing S is not independent. The minimum cardinality of a maximal independent set is called the lower independence number and also independent domination number and the maximum cardinality of a maximal independent set is called the independence number in a graph G and it is denoted by $i(G)$ and $\beta_o(G)$ respectively.

Definition 1.1[3]:

A subset $S \subseteq V(G)$ of vertices in a graph G is called majority dominating set if at least half of the vertices of $V(G)$ are either in S or adjacent to the vertices of S. i.e., $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$. A majority dominating set S is minimal if no proper subset of S is a majority dominating set of G. The majority domination number $\gamma_M(G)$ of a graph G is the minimum cardinality of a minimal majority dominating set in G. The upper majority domination number $\Gamma_M(G)$ is the maximum cardinality of a minimal majority dominating set of a graph G. This parameter has been studied by Swaminathan V and JoselineManora J.

Definition 1.2[2]:

A set S of vertices of a graph G is said to be a majority independent set if it induces a totally disconnected subgraph with $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|p_n[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. If any vertex set S' properly containing S is not majority independent then S is called maximal majority independent set.The maximum cardinality of a maximal majority independent set of G is called majority independence number of G and it is denoted by $\beta_M(G)$. Aβ_M-set is a maximum cardinality of a maximal majority independent set of G. This parameter is introduced by Swaminathan. V and JoselineManora. J.

Definition 1.3[1]:

A majority dominating set D of a graph $G=(V,E)$ is called an independent majority dominating (IMD) set if the induced subgraph<D> has no edges. The minimum cardinality of a maximal majority independent set is called lower majority independent set of G and it is also called independent majority domination number of G, denoted by $i_M(G)$.

If the degree of a vertex v satisfies $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, then the vertex $v \in V(G)$ is called a majority dominating vertex of G.

II. Strong equality of Majority domination Parameters.

Definition 2.1[5]:

Let P_1 and P_2 be properties of vertex subsets of a graph, and assume that every subset of $V(G)$ with property P_2 also has property P_1 . Let $\psi_1(G)$ and $\psi_2(G)$, respectively, denote the minimum cardinalities of sets with properties P_1 and P_2 , respectively. Then $\psi_1(G) \leq \psi_2(G)$ and every $\psi_1(G)$-set is also a $\psi_2(G)$-set, then we say $\psi_1(G)$ strongly equals $\psi_2(G)$,written $\psi_1(G) \equiv \psi_2(G)$.

Definition 2.2:

Let G be any graph with p vertices. Let $\gamma_M(G)$ and $i_M(G)$ be the majority domination number and independent majority domination number of a graph G. Then $\gamma_M(G)$ and $i_M(G)$ are strongly equal for G if $\gamma_M(G) = i_M(G)$ and every $\gamma_M(G)$-set is an $i_M(G)$-set. It is denoted by $\gamma_M(G) \equiv i_M(G)$.

Example 2.3:

Take $j=2$, $p=22j=44$. $D=\{u'_{1,1}, u'_{2,1}, u'_{3,1}, u'_{4,1}, u''_{1,1}, u''_{2,1}\}$. $\gamma_M(G_j) = |D| = 6$. Since all vertices in D are independent, $i_M(G_j) = |D| = 6$. $\therefore \gamma_M(G_j) \equiv i_M(G_j) = 3j, j=2$. Where as $\gamma(G) = i(G) = 8j = 16$, $j=2$.

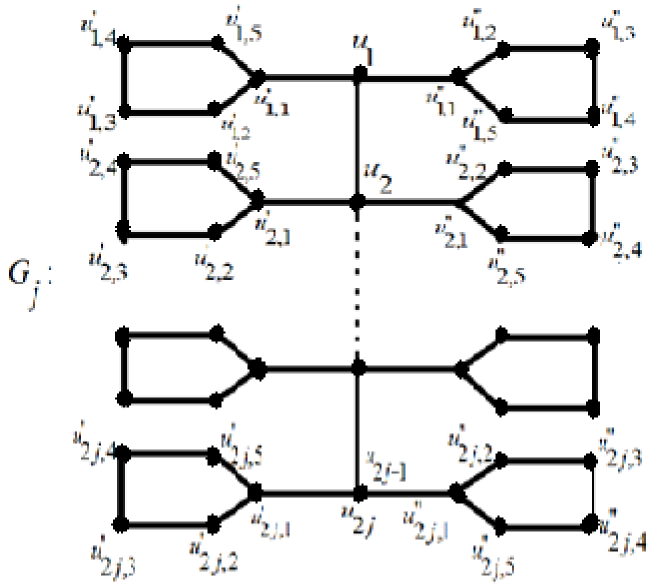

Fig (i)

Observations 2.4:

1. $\gamma_M(G_j) < \frac{|\gamma(G_j)|}{2} \Rightarrow 3j < \frac{8j}{2} = 4j$, where G_j is in Fig (i).
2. When $j=1$, $\gamma_M(G_j) = 3 = i_M(G_j)$.

   When $j=2$, $\gamma_M(G_j) = 6 = i_M(G_j)$.

   When $j=3$, $\gamma_M(G_j) = 9 = i_M(G_j)$.

   In general, for G_j, $\gamma_M(G_j) \equiv i_M(G_j) = 3j$, $j = 1, 2, ...$

We can extend this graph by applying values to $j = 2, 3, 4, ...$ Then we obtain $\gamma_M(G_j) = 3j = i_M(G_j)$. Also, every γ_M -set is an i_M -set of G_j. Hence $\gamma_M(G_j) \equiv i_M(G_j)$.

Example 2.5:

The graph G is obtained from disjoint copies of p_5 by joining a central vertex of one p_5 to the central vertices of the remaining graphs p_5.

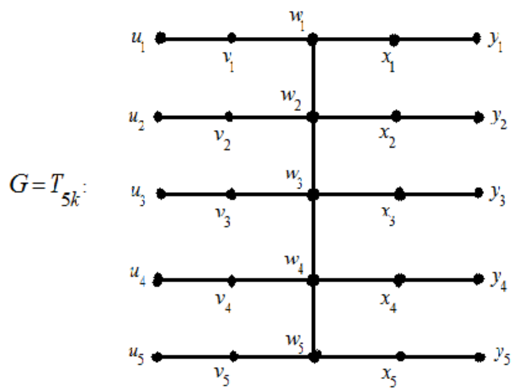

Fig (ii)

When $k=5,\ p=25$. $D_1=\{w_2,w_3,w_4,w_5\}$. D_1 dominates $\left\lceil \frac{p}{2} \right\rceil=13$ vertices. $\therefore \gamma_M(G)=|D_1|=4$. $\therefore$ This γ_M -set D_1 is not an i_M -set of G. $D_2=\{w_2,w_3,v_3,v_5\} \Rightarrow i_M(G)=|D_2|=4$. But D_2 is a γ_M -set which is also an i_M -set. Here, D_1 and D_2 are minimal majority dominating set for G. $\gamma_M(G)=i_M(G)=4$. Hence every γ_M -set is not an i_M -set for G. $\therefore \gamma_M(G)$ is not $\equiv i_M(G)$, if $G=T_{5k}, k=5$. Ingeneral, for any value of k, if $\gamma_M(G)$ is not $\equiv i_M(G)$, if $G=T_{5k}, k=5$.

Observations 2.6:

1. If $\gamma_M(G)=1$ then $\gamma_M(G)\equiv i_M(G)$.
2. If G has a full degree vertex then $\gamma_M(G)\equiv i_M(G)$.
3. For Corona graphs G, $\gamma_M(G)\equiv i_M(G)$, if $G=(C_p o K_1)$ and $G=(P_p o K_1)$.
4. $\gamma_M(G)\equiv i_M(G)$ if G= Caterpillar, with exactly one pendant.
5. $\gamma_M(G)\equiv i_M(G)$ if $G=mK_2$.
6. If G is Grotzsch graph, then $\gamma_M(G)\equiv i_M(G)$.
7. For Tutte graph G with $p=46,\ q=69$. $\gamma_M(G)=i_M(G)=6$.

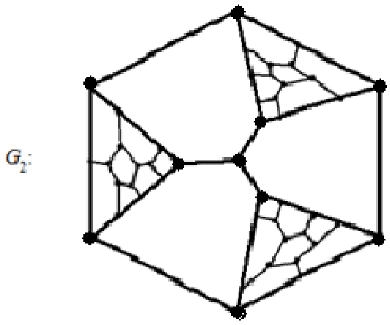

Fig (iii). Tutte Graph

Every $\gamma_M(G)$-set is an $i_M(G)$ -set. Hence $\gamma_M(G)\equiv i_M(G)$.

8. For a Grinberg graph G with $p=46,\ q=69$, then $\gamma_M(G)\equiv i_M(G)$.
9. For a Petersen graph P, $\gamma_M(P)$ *is not strongly equal to* $i_M(P)$.
10. For all Hajos graph H with p vertices,
   where $p=\frac{n(n+1)}{2}, n=3,4 \Rightarrow p=6,10$, then $\gamma_M(H)\equiv i_M(H)$.
   But if $n=5$ *and* $p=15$, then $\gamma_M(H)$ *is not strongly equal to* $i_M(H)$.

Proposition 2.7:

For the path P_p and cycle C_p,

1. $\gamma_M(P_{6k})\equiv i_M(P_{6k})=\gamma_M(C_{6k})\equiv i_M(C_{6k})=k,\ k=1,2,3,...$
2. $\gamma_M(P_{6k+3})\equiv i_M(P_{6k+3})=\gamma_M(C_{6k+3})\equiv i_M(C_{6k+3})=k+1,\ k=0,1,2,...$
3. $\gamma_M(P_{6k+4})\equiv i_M(P_{6k+4})=\gamma_M(C_{6k+4})\equiv i_M(C_{6k+4})=k+1,\ k=0,1,2,...$

4. $\gamma_M\left(P_{6k+5}\right)\equiv i_M\left(P_{6k+5}\right)=\gamma_M\left(C_{6k+5}\right)\equiv i_M\left(C_{6k+5}\right)=k+1,\ k=0,1,2,\dots$, but
5. $\gamma_M\left(P_{6k+1}\right)$ *is not* $\equiv i_M\left(P_{6k+1}\right)=\gamma_M\left(C_{6k+1}\right)$ *is not* $\equiv i_M\left(C_{6k+1}\right)=k+1,\ k=0,1,2,\dots$
6. $\gamma_M\left(P_{6k+2}\right)$ *is not* $\equiv i_M\left(P_{6k+1}\right)=\gamma_M\left(C_{6k+2}\right)$ *is not* $\equiv i_M\left(C_{6k+1}\right)=k+1,\ k=0,1,2,\dots$

Proposition 2.8[4]:
For any graph G, $\gamma_M(G)=1$ if and only if G has a majority dominating vertex.

Proposition 2.9:
$\gamma_M(G)\equiv i_M(G)=1$ if and only if G has a majority dominating vertex.

III. Trees T with $\gamma_M(T)\equiv i_M(T)$

Our aim in this section is to give a constructive characterization for the trees T having $\gamma_M(T)\equiv i_M(T)$. For this purpose, we first prove two lemmas.

Lemma 3.1:
Let w be a vertex of a tree T_w such that every leaf of T_w, except possibly for w itself, is at distance two from w. Let S_w be the set of support vertices of T_w. Let y be a pendant vertex of a non-trivial tree T_y. Let T be obtained from $T_w \bigcup T_y$ by adding theedge wy. Then $\gamma_M(T)=\gamma_M(T_y)+1$.

Proof: Let $T=T_y \bigcup T_w$ and y be a pendant vertex of $d(y)=2,\ y\in T$. Let $\gamma_M(T_y)$ be a majority domination number of T_y. Since w is a majority dominating vertex of T_w, $\gamma_M(T_y)$-set can be extended to a majority dominating set of T by adding the vertex $w\in T_w$ $\therefore$ $\gamma_M(T)\le\gamma_M(T_y)+1$.

Claim: $\gamma_M(T)\ge\gamma_M(T_y)+1$. Let D be a γ_M-set of T. Then $D_y=D\bigcap V(T_y)$ and $D_w=D\bigcap V(T_w)$. Since T_W has a majority dominating vertex w. $|D_w|=|\{w\}|=1$.

Since D is a γ_M–set of T, D_y is a majority dominating set of T_y. Then $\gamma_M(T_y)\le|D_y|\le|D-D_w|\Rightarrow\gamma_M(T_y)\le|D|-1=\gamma_M(T)-1\Rightarrow\gamma_M(T_y)+1\le\gamma_M(T)$.

Hence, $\gamma_M(T)=\gamma_M(T_y)+1$.□

Lemma 3.2:
Let T_w, T_y, and T be defined as in the statement of Lemma (3.1). Then $\gamma_M(T)\equiv i_M(T)$ if and only if $\gamma_M(T_y)\equiv i_M(T_y)$.

Proof: Suppose $\gamma_M(T)\equiv i_M(T)$ ......(1). Let D_y be a $\gamma_M(T_y)$-set. Then $D_y\bigcup\{w\}$ is a majority dominating set of T of cardinality $\gamma_M(T_y)+1$. Then by lemma (3.1), $\gamma_M(T)=\gamma_M(T_y)+1$. Therefore $D_y\bigcup\{w\}$ is a $\gamma_M(T)$-set and by (1), it is a $i_M(T)$-set. In particular, D_y is an independent majority dominating set of T_y and so, $|D_y|=\gamma_M(T_y)\le i_M(T_y)\le|D_y|$. Hence $|D_y|=i_M(T_y)$ and D_y is a $i_M(T_y)$-set. Thus, every $\gamma_M(T_y)$-set is an $i_M(T_y)$-set. $\therefore$ $\gamma_M(T_y)\equiv i_M(T_y)$

Conversely, Let $\gamma_M(T_y)\equiv i_M(T_y)$ ......(2). To prove $\gamma_M(T)\equiv i_M(T)$. Let D be a $\gamma_M(T)$-set and $D_y=D\bigcap V(T_y)$ and $D_w=D\bigcap V(T_w)$. Suppose $w\notin D$, then $|D_w|=|S_w|$ and $|D_y|=|D-D_w|=|D|-|S_w|$. Then $|D_y|=\gamma_M(T)-|S_w|\Rightarrow\gamma_M(T)=\gamma_M(T_y)+|S_w|$, which is a contradiction to lemma(3.1), $\gamma_M(T)=\gamma_M(T_y)+1$. Hence $w\in D$. Then $D_w=\{w\}\in T_w$, since w is a majority dominating vertex of T_w. Since T has an edge wy, w is the onlyvertex that dominates y.

Since y is already dominated by $w \in D_w$, D_y does not contain y in T_y. But D_y is itself a majority dominating set of T_y of $|D - D_w|$. *ie.*, $|D_y| = |D - D_w| = |D| - 1 = \gamma_M(T) - 1$. By lemma (3.1), $|D_y| = \gamma_M(T_y)$, by (2), $|D_y| \equiv i_M(T_y) \Rightarrow D_y$ is an independent majority dominating set of T_y. Furthermore, $D_w = \{w\}$ is also an independent majority dominating set of T_w. Hence D is an $i_M(T)$ -set. Thus every $\gamma_M(T)$ -set is an $i_M(T)$ -set. $\therefore \gamma_M(T) \equiv i_M(T)$.□

Next, a construction for characterization of the trees T for which $\gamma_M(T) \equiv i_M(T)$ is provided by using the following operation.

Operation -A: Let w be a vertex of a tree T_w such that every leaf of T_w, except possibly for w itself, is at distance two from w. Let S_w be the set of support vertices of T_w. Let y be a pendant vertex of a non-trivial tree T_y. Let T be obtained from $T_w \cup T_y$ by adding the edge wy. Define the family as

$$\Im_1 = \{ T / T = K_1 \; or \; T \; is \; obtained \; from \; a \; non-trivial \; star \; by \; a \; finite \; sequence \; of \; operation \; A \}$$

Theorem 3.3:

For any tree T, $\gamma_M(T) \equiv i_M(T)$ if and only if $T \in \Im_1$

Proof: Let $T \in \Im_1$.If $T = K_1$ or if T is a non-trivial star, then $\gamma_M(T) = i_M(T) = 1$ and $\gamma_M(T) \equiv i_M(T)$.On the other hand,if T is constructed from a non-trivial star by a finite sequence of atleast one operation(A), then repeated applications oflemma (3.2),we get $\gamma_M(T) \equiv i_M(T)$, since a star has majority domination number strongly equal to its independent majority domination number. Conversely, let $\gamma_M(T) \equiv i_M(T)$. To prove $T \in \Im_1$. By induction on the order p of a tree T for which $\gamma_M(T) \equiv i_M(T)$. If $T = K_1$ or K_2, then $T \in \Im_1$. If $diam T = 2$ then T is a non-trivial star and so $T \in \Im_1$. When $diam T = 3,4,5,6$ which satisfy $\gamma_M(T) \equiv i_M(T)$ since $\gamma_M(T) = 1 = i_M(T)$. Then $T \in \Im_1$. Now, assume that $diam T \geq 7$ which satisfy $\gamma_M(T) \equiv i_M(T)$. We now root the tree at a leaf r of maximum eccentricity $diam T$. Let w be the vertex at distance $(diam T - 2)$ from r on a longest path starting at r.

Let T_w be the subtree of T rooted at w. Then the vertex cannot be adjacent to a leaf. If not, it will contradict our assumption that $\gamma_M(T) \equiv i_M(T)$. Hence every leaf of T_w, except possibly for w itself, is at distance two from w. Let y denote the parent of w on T and let T_y denote the component of $T - wy$ containing y. Since $diam T \geq 7$, T_y is a non-trivial tree. By lemma (3.2), if $\gamma_M(T) \equiv i_M(T)$ then $\gamma_M(T_y) \equiv i_M(T_y)$.

Now, since T_y is a tree of order less than p satisfying $\gamma_M(T_y) \equiv i_M(T_y)$, we can apply the induction hypothesis, to T_y to show that $T_y \in \Im_1$. Since T is obtained form T_y by a operation A, we have $T \in \Im_1$. Hence the theorem.□

Theorem 3.4: Let D_i be the set of all γ_M -sets of G. Then

(i). $\gamma_M(G) \equiv i_M(G)$ if and only if induced subgraph $\langle D \rangle$ has only isolates, for every γ_M -set $D \in D_i$.

(ii). $\gamma_M(G) \; is \; not \equiv i_M(G)$ if and only if the induced subgraph $\langle D \rangle$ is not totally disconnected for any γ_M - set $D \in D_i$.

Proof: Let D_i be the set of all γ_M -set D of a graph G.

(i). Suppose $\gamma_M(G) \equiv i_M(G)$. Then $\gamma_M(G) \leq i_M(G)$ and every γ_M -set D of a graph G is an independent majority dominating set of G. The induced subgraph $\langle D \rangle$ has only isolates for every γ_M -set

$D \in D_i$. Conversely, for every γ_M-set D, the induced subgraph $\langle D \rangle$ has only isolates. Then D is an independent set of $G \Rightarrow$ every γ_M-set D is an i_M-set of G. $\therefore\ i_M(G) \le \gamma_M(G)$. For any graph G, $\gamma_M(G) \le i_M(G)$. Hence $\gamma_M(G) \equiv i_M(G)$.

(ii). Suppose $\gamma_M(G)$ *is not* $\equiv i_M(G)$. Then for any graph G, $\gamma_M(G) \le i_M(G)$ but not every γ_M-set D is an i_M-set of G. Then the γ_M-set D is not independent for any one $D \in D_i$. Hence, the induced subgraph $\langle D \rangle$ is not totally disconnected for any $D \in D_i$. Conversely, if $\langle D \rangle$ is not totally disconnected for atleast one $D \in D_i$ then D is not an independent γ_M-set. It does not satisfy the fact that every γ_M-set is an i_M-set of G. $\therefore\ i_M(G) \le \gamma_M(G)$ is not true. Thus, $\gamma_M(G)$ *is not* $\equiv i_M(G)$.□

IV. Strong Equality of $\gamma(G)$ and $\gamma_M(G)$ and of $i(G)$ and $i_M(G)$.

Observations 4.1:

1. For any graph G, $\gamma_M(G) \le \gamma(G)$.
2. If $\gamma(G)=1$ then $\gamma_M(G)=1$.
3. If G has a full degree vertex then every γ_M-set is a γ-set.

Proposition 4.2:

For any graph G, $\gamma(G) \equiv \gamma_M(G)$ if and only if G has a full degree vertex.

Proof: Let $\gamma(G) \equiv \gamma_M(G)$. Suppose G has no full degree vertex. Then $\gamma(G) \ge 2$. G may have a majority dominating vertex v with $d(v) \ge \left\lceil \frac{p}{2} \right\rceil - 1$. Then $\gamma_M(G)=1$ but $r>1$. Therefore every γ_M-set is not a γ-set $\Rightarrow \gamma(G)$ is not strongly equal to $\gamma_M(G)$, a contradiction. Hence G has a full degree vertex. Conversely, if G has a full degree vertex, then $r=1$. Then $\gamma_M(G)=1$. Since $\gamma(G)=\gamma_M(G)=1$, every γ_M-set is also a γ-set. Hence $\gamma(G) \equiv \gamma_M(G)$.□

Proposition 4.3:

For any graph G, $i(G) \equiv i_M(G)$ if and only if G has a full degree vertex.

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