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# Locally dually flat Finsler $(\alpha, \beta)$-metric

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***Abstract:*** *In Finsler space we see special $(\alpha,\beta)-$metrics such as Randers metric, Kropina metric and Matsumoto metric.,etc. Locally dually flat Finsler metrics arise from Information Geometry. Such metrics have special geometric properties and will play an important role in Finsler geometry. In this paper, we are going to study class of locally dually flat Finsler metrics which are defined as the sum of a Riemannian metric and $1-form$. In this paper, we study the special $(\alpha,\beta)-metric$ L satisfying*
*$L^2(\alpha,\beta) = 2\alpha^2 + \alpha\beta + 2\beta^2$, where $c^i$ are constants, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a differential $1 - form$.*
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## I. Introduction

Finsler spaces are the most natural generalization of Riemannian space, Finsler space is considered as space in which the line element is a function of positive homogeneity. Which was initiated by P.Finsler.

The concept of The concept of $(\alpha,\beta)$ −metric was introduced in 1972 by M. Matsumoto and studied by M. Hachiguchi(1975), Y. Ichijyo(1975), S. Kikuchi(1979), C.Shibata(1984). The examples of the $(\alpha,\beta)$ −metric are Randers metric, Kropina metric and Matsumoto metric. Z. Shen extended the notion of dually flatness to Finsler metrics.

For a Finsler metric $F = F(x,y)$ on a manifold M , the geodesics $c = c(t)$ of $F$ in local co-ordinates $(x^i)$ are characterized by $\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0$ (1.1)
where $(x^i(t))$ are the coordinates of $c(t)$ and $G^i = G^i(x,y)$ are defined by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^k x^l} y^k - [F^2]_{x^l}\} \quad (1.2)$$

where $g_{ij} = \frac{1}{2}[L^2]_{y^i y^j}(x,y)$ and $(g^{ij}) := (g_{ij})^{-1}$

The local functions $G^i = G^i(x,y)$ define a global vector field on . $G$ is called the spray of $F$ and $G^i$ are called the spray coefficients.

A Finsler metric $F = F(x,y)$is said to be locally dually flat if at every point there is a Co-ordinate system $(x^i)$ in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^i} \quad (1.3)$$

where $H = H(x,y)$ is a local scalar function on the tangent bundle $TM$ of $M$.

## II. Preliminaries

A Finsler metric on a manifold $M$ is a $C^\infty$function $F : TM \to [0,\infty)$ satisfies the following properties:

- Regularity: $L$ is $C^\infty$ on $TM \setminus \{0\}$;
- Positively homogeneity: $L(x,\lambda y) = \lambda L(x,y)$, for $\lambda > 0$;
- Strong convexity: The fundamental tensor $g_{ij}(x,y)$is positive for all $(x,y) \in TM \setminus \{0\}$;

where, $g_{ij} = \frac{1}{2}[L^2]_{y^i y^j}(x,y)$.

The concept of $(\alpha,\beta) - metrics$ was introduced in 1972 by M. Matsumoto and studied by many authors like ([4],[5],[9],[6],[10],[2]).The Finsler space $F^n = (M,L)$
is said to have an $(\alpha,\beta) - metric$ if $L$ is a positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$.

Dually flat Finsler metrics on an open subset in $R^n$can be characterized by a simple PDE.

**Lemma 2.1.** ([8]) A Finsler metric $L = L(x, y)$ on an open subset $u \subset R^n$ is dually flat if and only if it satisfies the following conditions:

$$[L^2]_{x^k x^l} y^k - [L^2]_{x^l} = 0 \qquad (2.1)$$

In this case, $H = H(x, y)$ in (1.1) is given by $H = -\frac{1}{6} L^2{}_{x^m} y^m$

There is another notion of locally projectively flat Finsler metrics. A Finsler metric $L = L(x, y)$ is locally projectively flat if at every point there is a coordinate system $(x^i)$ in which all geodesics are straight lines, or equivalently. The spray coefficients are in the following form

$$G^i = P y^i \qquad (2.2)$$

where $P = P(x, y)$ is a local scalar function.

Projectively flat metrics on an open subset in $R^n$ can be characterized by a simple PDE.

**Lemma 2.2.** ([3]) A Finsler metric $L = L(x, y)$ on an open subset $u \subset R^n$ is projectively flat if and only if it satisfies the following equations:

$$L_{x^l x^k} y^k - L_{x^l} = 0 \qquad (2.3)$$

In this case, Local function $P = P(x, y)$ in (2.2) is given by $P = \frac{L_{x^m} y^m}{2L}$

A Finsler metric is said to be dually flat and projectively flat on an open subset $u \subset R^n$ if the spray coefficients $G^i$ satisfy (1.1) and (2.2) in $u$. Therefore there are Finsler metrics on an open subset in $R^n$ which are dually flat and projectively flat.

## III. LOCALLY DUALLY FLAT SPECIAL (□ , □)-METRIC

In this section, we are going to study the locally dually flat special $(\alpha, \beta) - metric\ L$ satis-fying $L^2(\alpha, \beta) = 2\alpha^2 + \alpha\beta + \beta^2$, where $C_i$ are constants, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a differential 1-form. We prove the following theorem.

**Theorem 3.1.** Let $L$ be a special $(\alpha, \beta) - metric$ on a manifold $M$ satisfying $L^2(\alpha, \beta) = 2\alpha^2 + \alpha\beta + 2\beta^2$, where $C_i$ are constants, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a differential $1 - form$. The special $(\alpha, \beta) - metric\ L$ is locally dually flat if and only if an adapted coordinate system, the following conditions are satisfied by $\alpha$ and $\beta$.

$$r_{00} = 2\left(1 - \frac{2}{3c_1}\right)\theta\beta - \left(1 + \frac{2}{3c_1}\right)\tau c_1 c_2 \beta^2 + [(\tau c_1 c_2 b^2 - b_m y^m)]\alpha^2 \qquad (3.1)$$

$$s_{k0} = \frac{\beta\theta_k - \theta b_k}{3} \qquad (3.2)$$

$$G_\alpha^m = \frac{1}{3c_1}(2\theta + \tau c_2 c_3 \beta) y^m - \frac{1}{3}(\tau c_2 c_3 b^m - \theta^m)\alpha^2 \qquad (3.3)$$

where $\tau = \tau(x)$ is a scalar function and $\theta = \theta_k y_k$ is a$1 - form$ on $M$and $\theta^m = a^{im}\theta_i$

**Proof:** It is straight forward to verify the sufficient condition. Now we prove the necessary condition.

Assume that the special $(\alpha, \beta) - metric\ L$ satisfying $L^2(\alpha, \beta) = 2\alpha^2 + \alpha\beta + 2\beta^2$, where $c_i$ are constants, is locally dually flat on an open subset $u \subset R^n$. First we have the following identities:

$$\alpha_{x^k} = \frac{y_m}{\alpha}\frac{\partial G_\alpha^m}{\partial y^k}, \qquad \beta_{x^k} = b_{m/k} y^m + b_m \frac{\partial G_\alpha^m}{\partial y^k}, \qquad s_{y^k} \frac{\alpha b_k - s y_k}{\alpha^2} \qquad (3.4)$$

where $s = \beta/\alpha$ and $y^k = a_{jk} y^j$. By a direct computation, we obtain

$$[L^2]_{x^k} = (4y_m + \alpha b_m + s y_m + 4\beta b_m)\frac{\partial G_\alpha^m}{\partial y^k} + (\alpha + 4\beta) b_{m/k} y^m \qquad (3.5)$$

$$[L^2]_{x^l x^k} y^l = 2\left[4a_{mk} + \frac{b_m y^k}{\alpha} + \frac{\beta a_{mk}}{\alpha} + y_m \frac{\alpha b_k - s y^k}{\alpha^2} + 4 b_m b_k\right] G_\alpha^m \qquad (3.6)$$

Plugging (3.5) and (3.6) into (2.1), we obtain

$$\frac{2}{\alpha^3} A_1 G_\alpha^m - B_1 \frac{\partial G_\alpha^m}{\partial y^k} + C_1 r_{00} + D_1\left(b_{\frac{k}{0}} - 2 b_{\frac{0}{k}}\right) = 0 \qquad (3.7)$$

Where,

$$A_1 = 4a_{mk}\alpha^2 + b_m y_k \alpha^2 + a_{mk}\alpha^2\beta + y_m(\alpha^2 b_k - \beta y_k) + 4 b_m b_k \alpha^2$$

$$B_1 = 4y_m + \alpha b_m + \frac{\beta}{\alpha} y_m + \beta b_m$$

$C_1 = \frac{y_k}{\alpha} + 4b_k$
$D_1 = \alpha + 4\beta$
Rewriting (3.7) as a polynomial in α, we have
$$A_2\alpha^4 + B_2\alpha^3 + C_2\alpha^2 + D_2 = 0 \qquad (3.8)$$
where,
$A_2 = \left[\frac{1}{2}(3s_{k0} - r_{k0}) - \frac{1}{2}b_m\frac{\partial G^m_\alpha}{\partial y^k}\right]$
$B_2 = 4(a_{mk} + b_m b_k)G^m_\alpha - 2(y_m + b_m\beta)\frac{\partial G^m_\alpha}{\partial y^k} + 2b_k r_{00} + 2\beta(s_{k0} - r_{k0})$
$C_2 = [(a_{mk}\beta + b_m y^k + y_m b_k)G^m_\alpha - \left(\frac{1}{2}y_m\beta\right)\frac{\partial G^m_\alpha}{\partial y^k} + \frac{y_k r_{00}}{2}]$
$D_2 = y_m y_k \beta G^m_\alpha$

From (3.8), we know that the coefficients of $\alpha$ are zero. Hence the coefficients of$\alpha^3$ must be zero too. That is
$$B_2 = 0 \qquad (3.9)$$
Thus, we have
$$A_2\alpha^4 + B_2\alpha^2 - D_2 = 0 \qquad (3.10)$$
Note that $y_m\frac{\partial G^m_\alpha}{\partial y^k} = \frac{\partial(y_m G^m_\alpha)}{\partial y^k} - a_{mk}G^m_\alpha$ (3.11)
$$b_m\frac{\partial G^m_\alpha}{\partial y^k} = \frac{\partial(b_m G^m_\alpha)}{\partial y^k} \qquad (3.12)$$
Contracting (3.9) with $b^k$and by use of (3.11), (3.12), we obtain
$$2\frac{\partial G^m_\alpha}{\partial y^k}b^k + 2\beta\frac{\partial(b_m G^m_\alpha)}{\partial y^k}b^k = (6 + 4b^2)b_m G^m_\alpha + 2b^2 r_{00} + 2\beta(3s_0 - r_0) \qquad (3.13)$$
Contracting (3.10) with $b^k$and by use of (3.11), (3.12), we obtain
$$\frac{\beta\alpha^2}{2}\frac{\partial G^m_\alpha}{\partial y^k}b^k = \frac{(3s_0 - r_0)\alpha^4}{2} + \left(b^2 y_m G^m_\alpha + \frac{5}{2}\beta b_m G^m_\alpha + \frac{\beta}{2}r_{00}\right)\alpha^2 - \beta^2 y_m G^m_\alpha \qquad (3.14)$$
(3.13) $\times\frac{\alpha^4}{2}$ − (3.14) × $2\beta$ yields
$$A_3\alpha^2(\alpha^2 - \beta^2) = B_3(b^2\alpha^2 - \beta^2) \qquad (3.15)$$
Where,
$A_3 = [\frac{\partial G^m_\alpha}{\partial y^k}b^k - 3b_m G^m_\alpha]$
$B_3 = ((2b_m G^m_\alpha + r_{00})\alpha^2 - 2\beta y_m G^m_\alpha)$
Where,
$(b^2\alpha^2 - \beta^2)$ and $(2\alpha^2 - 2\beta^2)$ and $\alpha^2$ are all irreducible polynomials of $(y^i)$, and $(2\alpha^2 - 2\beta^2)$ and $\alpha^2$are relatively prime polynomials of $(y^i)$, we know that there is a function $\tau = \tau(x)$ on $M$ such that
$$(2b_m G^m_\alpha + r_{00})\alpha^2 - 2\beta y_m G^m_\alpha = r\alpha^2(\alpha^2 - \beta^2) \qquad (3.16)$$
$$\frac{\partial G^m_\alpha}{\partial y^k}b^k - 3b_m G^m_\alpha = \tau(b^2\alpha^2 - \beta^2) \qquad (3.17)$$
(3.16) can be reduced to
$2\beta y_m G^m_\alpha = (2b_m G^m_\alpha + r_{00} - \tau(\alpha^2 - \beta^2))\alpha^2$
Since $\alpha^2$does not contain the factor , we have $y_m G^m_\alpha = \theta_k\alpha^2$ (3.18)
$$b_m G^m_\alpha = \beta\theta - \frac{1}{2}r_{00} + \frac{\tau\alpha^2}{2} - \frac{\tau\beta^2}{2} \qquad (3.19)$$
where $\theta = \theta_k y^k$ is a $1 - form$ on $M$ . Then we obtain the following
$$\frac{\partial y_m G^m_\alpha}{\partial y^k} = \theta_k\alpha^2 + 2\theta y_k \qquad (3.20)$$
$$\frac{\partial b_m G^m_\alpha}{\partial y^k} = \theta_k\beta + b_k\theta - \tau_{k0} + \tau(y_k - \beta b_k) \qquad (3.21)$$
By using (3.19)-(3.21), (3.9) and (3.10) become
$$4\beta\alpha^2(2s_{k0} + \theta b_k - \beta\theta_k)(\tau b_k - \theta_k) + 6a_{mk}G^m_\alpha - 2y_k(2\theta + \tau\beta) = 0 \qquad (3.22)$$
$$[(3s_{k0} + b_k\theta - \beta\theta_k) + (\tau b_k - \theta_k)\beta]\alpha^2 - (2\theta + \tau\beta)\beta y_k + 3\beta a_{mk}G^m_\alpha = 0 \qquad (3.23)$$

subtracting (3.23) from the product of (3.22)and $\beta$ yields.
$$3s_{k0} + \theta b_k - \beta\theta_k = 0 \qquad (3.24)$$
This yields the required condition (3.4)contracting (3.23) with $a^{lk}$
we obtain $2\beta(3s^l_0 + \theta b^l - \beta\theta^l) + 2(\tau b^l - \theta^l)\alpha^2 + 6G^i_\alpha - 2(2\theta + \tau\beta)y^l = 0$ (3.25)
contracting (3.24) with $a^{lk}$, we obtain
$$3s^l_0 + \theta b^l - \beta\theta^l = 0 \qquad (3.26)$$

Then from (3.25), we obtain the required condition(3.3)substituting (3.22) into (3.20) we obtain the required condition(3.2)The converse part and the sufficient condition is obtained.

## IV. Conclusion

To study the geometric properties of a Finsler metric, one also considers non–Riemannian quantities. Hence different types of (α, β)-metrics will satisfy the locally dually flat properties.

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