IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 5 Ver. VI (Sep-Oct. 2014), PP 22-26
www.iosrjournals.org

Unique Common End Point Result for Set-Valued Mappings Satisfying Generalized (ξ,η) -Weak Contraction Condition.

[1,2] Kavita B. Bajpai, [2] Manjusha P. Gandhi
[1] *Karmavir Dadasaheb Kannamwar College of Engineering, Nagpur, India*
[2] *Yeshwantrao Chavan College of Engineering, Wanadongri, Nagpur, India*

Abstract: *In this paper a unique common end point result is proved for pair of set valued mappings satisfying a generalized (ξ,η) weak contractive condition in complete metric space. This theorem is the extension of some results existing in the literature. An example has been provided to validate the main result of this paper.*
Keywords: *Fixed point, end point, Complete metric space , Set valued mapping, Generalized contractive condition.*

I. Introduction

The Banach fixed point theorem is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces and provides a constructive method to find those fixed points.

Alber Y. et al [2] defined weakly contractive maps on a Hilbert space and established a fixed point theorem for such a map. Latter Rhoades B.[11] obtained a fixed point theorem in a complete metric space using the notion of weakly contractive maps,. Beg I. et al [3] obtained a common fixed point theorem extending weak contractive condition for two maps . In this direction , Zhang Q.et al [12] introduced the concept of a generalized ϕ- weak contraction condition and obtained a common fixed point for two maps. Doric D. [5] proved a common fixed point theorem for generalized (ψ,ϕ) - weak contractions.

First time Nadler S.[9] extended the Banach fixed point theorem from the single valued maps to the set-valued contractive maps in complete metric space. Further many fixed point theorems for multi-valued mappings were established by Ciric L. [4] , Dube L. [6] , Klim D. et al [8] , Namdeo R. et al [10] , Fisher B. [7] etc.

In the present paper we obtain a unique common end point result for two set-valued mappings without use of continuity of any map involved therein. Also it do not require any commutativity condition to prove the existence on common end point of two mappings. This theorem improves the earlier result by Abbas M. & Doric D. [1].

Now, we give preliminaries and basic definitions which are used through-out the paper.

Definition 1.1: Let (X,d) be a metric space. A sequence $\{x_n\}$ of points of X is said to be a Cauchy sequence in (X,d) if it has the property that given $\in > 0$ there is an integer N such that $d(x_n,x_m) < \in$ when ever $n,m \geq N$.

Definition 1.2: A sequence $\{x_n\}$ of points of X is said to be convergent to a point x if for given $\in > 0$, there is an integer N such that $d(x_n,x) < \in$, for all $n \geq N$.

Definition 1.3: The metric space (X,d) is said to be complete if every Cauchy sequence in X converges in X.

Definition 1.4: Let (X,d) be a metric space and let $B(X)$ be the class of all nonempty bounded subsets of X . We define the functions $\delta : B(X)\times B(X) \to R^+$ and $D : B(X)\times B(X) \to R^+$ as follows:

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$$
$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$$

where R^+ denotes the set of all positive real numbers. For $\delta(\{a\},B)$ and $\delta(\{a\},\{b\})$, we write $\delta(a,B)$ and $d(a,b)$ respectively. Clearly , $\delta(A,B) = \delta(B,A)$. We appeal to the fact that $\delta(A,B) = 0$ if and only if $A = B = \{x\}$, for $A, B \in B(X)$ and

$$0 \leq \delta(A,B) \leq \delta(A,C) + \delta(C,B) \text{, for all } A,B,C \in B(X).$$

A point $x \in X$ is called a fixed point of T if $x \in Tx$. If there exists a point $x \in X$ such that $Tx = \{x\}$, then x is called the end point of the mapping T.

II. Main Result

The following is the theorem for two set valued mappings:

Theorem 2.1: Let (X,d) be a complete metric space and let $f,g : X \to B(X)$ be two set valued mappings satisfying generalized (ξ,η) - weak contraction as

$$\xi(\delta(fx,gy)) \le \xi(\mu(x,y)) - \eta(\mu(x,y)) \quad \text{-------------- (2.1.1)}$$

where

$$\mu(x,y) = \max.\left\{d(x,y),\delta(x,fx),\delta(y,gy),\delta(x,fy),\delta(y,fy),\frac{1}{2}[D(x,fx)+D(y,gy)]\right\} \text{------(2.1.2)}$$

$\xi : [0,\infty) \to [0,\infty)$ is a continuous monotone non-decreasing function with $\xi(t) = 0$ if and only if $t = 0$ and $\eta : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\eta(t) = 0$ if and only if $t = 0$. Then there exists a unique end point $u \in X$ such that $\{u\} = fu = gu$.

Proof: Construct the convergent sequence $\{x_n\}$ in X and to prove that limit of that sequence is a unique common fixed point of f and g.

Let $x_0 \in X$ and n be some non-negative integer and

Let $x_{2n+1} \in fx_{2n} = P_{2n}$, $x_{2n+2} \in gx_{2n+1} = P_{2n+1}$ ----------------- (2.1.3)

Also let $\alpha_n = \delta(P_n, P_{n+1})$ and $\beta_n = d(x_n, x_{n+1})$ -------------------- (2.1.4)

Now, we show that the sequences α_n and β_n are convergent.

Suppose that $'n'$ is an odd number, substituting $x = x_{n+1}$, $y = x_n$ in (2.1.1) and using properties of ξ and η, we obtain ,

$$\begin{aligned}\xi(\delta(P_{n+1},P_n)) &= \xi(\delta(fx_{n+1},gx_n))\\ &\le \xi(\mu(x_{n+1},x_n)) - \eta(\mu(x_{n+1},x_n)) \quad \text{-------------- (2.1.5)}\\ &\le \xi(\mu(x_{n+1},x_n))\end{aligned}$$

which implies that $\delta(P_{n+1},P_n) \le \mu(x_{n+1},x_n)$ -------------- (2.1.6)

Now , from (2.1.2) and using the triangle inequality for δ , we have

$$\begin{aligned}\mu(x_{n+1},x_n) &= \max\left\{\begin{matrix} d(x_{n+1},x_n),\delta(x_{n+1},fx_{n+1}),\delta(x_n,gx_n),\delta(x_{n+1},fx_n),\delta(x_n,fx_n), \\ \frac{1}{2}[D(x_{n+1},fx_{n+1})+D(x_n,gx_n)] \end{matrix}\right\}\\ &\le \max\left\{\begin{matrix} \delta(P_n,P_{n-1}),\delta(P_n,P_{n+1}),\delta(P_{n-1},P_n),\delta(P_n,P_n),\delta(P_{n-1},P_n), \\ \frac{1}{2}[\delta(P_n,P_{n+1})+\delta(P_{n-1},P_n)] \end{matrix}\right\}\\ &= \max\{\delta(P_n,P_{n-1}),\delta(P_n,P_{n+1})\}\end{aligned}$$

If $\delta(P_n,P_{n+1}) > \delta(P_{n-1},P_n)$, then $\mu(x_n,x_{n+1}) \le \delta(P_{n+1},P_n)$ ----------------- (2.1.7)

From (2.1.6) and (2.1.7) it follows that $\mu(x_n,x_{n+1}) = \delta(P_{n+1},P_n) > \delta(P_{n-1},P_n) \ge 0$ ------ (2.1.8)

It further implies that ,

$$\begin{aligned}\xi(\delta(P_n,P_{n+1})) &\le \xi(\mu(x_n,x_{n+1})) - \eta(\mu(x_n,x_{n+1}))\\ &< \xi(\mu(x_{n+1},x_n)) \quad \text{---------------- (2.1.9)}\\ &= \xi(\delta(P_n,P_{n+1}))\end{aligned}$$

which is a contradiction , so we have , $\delta(P_n,P_{n+1}) \le \mu(x_n,x_{n+1}) \le \delta(P_{n-1},P_n)$ ---------- (2.1.10)

Similarly, (2.1.9) can be obtained also in the case when $'n'$ is an even number.

Therefore the sequence $\{\alpha_n\}$ defined in (2.1.4) is monotone non-increasing and bounded.

Let $\alpha_n \to \alpha$ when $n \to \infty$.

$\therefore$ From (2.1.10) , we have , $\lim_{n\to\infty} \delta(P_n, P_{n+1}) = \lim_{n\to\infty} \mu(x_n, x_{n+1}) = \alpha > 0$ ------------- (2.1.11)

Taking $n \to \infty$ in the inequality

$\xi(\delta(P_{2n}, P_{2n+1})) \le \xi(\mu(x_{2n}, x_{2n+1})) - \eta(\mu(x_{2n}, x_{2n+1}))$ --------------- (2.1.12)

We get , $\xi(\alpha) \le \xi(\alpha) - \eta(\alpha)$, which is a contradiction unless $\alpha = 0$.

Hence $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \delta(P_n, P_{n+1}) = 0$ --------------- (2.1.13)

From (2.1.13) and (2.1.3) , it follows that , $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ ---------- (2.1.14)

Now , show that $\{x_n\}$ is a Cauchy sequence.

i.e. for each $\in > 0$ there exists $n_0 \in N$, such that for all

$m, n \ge n_0 \Rightarrow \delta(P_{2m}, P_{2n}) < \in$ ---------------- (2.1.15)

By the method of contradiction we assume that $\{x_n\}$ is not a Cauchy sequence.

$\therefore$ There exists $\in > 0$ for which we can find non-negative integer sequence $\{m(k)\}$ and $\{n(k)\}$, such that $n(k)$ is the smallest element of the sequence $\{n(k)\}$ for which

$n(k) > m(k) > k$, $\delta(P_{2m(k)}, P_{2n(k)}) \ge \in$ -------------- (2.1.16)

This means that $\delta(P_{2m(k)}, P_{2n(k)-2}) < \in$ ---------------- (2.1.17)

From (2.1.16) and triangle inequality for δ , we have ,

$$\begin{aligned} \in &\le \delta(P_{2m(k)}, P_{2n(k)}) \\ &\le \delta(P_{2m(k)}, P_{2n(k)-2}) + \delta(P_{2n(k)-2}, P_{2n(k)-1}) + \delta(P_{2n(k)-1}, P_{2n(k)}) \\ &< \in + \delta(P_{2n(k)-2}, P_{2n(k)-1}) + \delta(P_{2n(k)-1}, P_{2n(k)}) \end{aligned} \quad \text{--------- (2.1.18)}$$

Taking limit as $k \to \infty$ and using (2.1.13) , we can conclude that ,

$\lim_{k\to\infty} \delta(P_{2m(k)}, P_{2n(k)}) = \in$ ---------------- (2.1.19)

Also from , $\left|\delta(P_{2m(k)}, P_{2n(k)+1}) - \delta(P_{2m(k)}, P_{2n(k)})\right| \le \delta(P_{2n(k)}, P_{2n(k)+1})$

$\left|\delta(P_{2m(k)-1}, P_{2n(k)}) - \delta(P_{2m(k)}, P_{2n(k)})\right| \le \delta(P_{2m(k)}, P_{2m(k)-1})$ ------------- (2.1.20)

Using (2.1.13) and (2.1.19) as $k \to \infty$, we get ,

$\lim_{k\to\infty} \delta(P_{2m(k)-1}, P_{2n(k)}) = \lim_{k\to\infty} \delta(P_{2m(k)}, P_{2n(k)+1}) = \in$ -------------- (2.1.21)

and from

$\left|\delta(P_{2m(k)-1}, P_{2n(k)+1}) - \delta(P_{2m(k)-1}, P_{2n(k)})\right| \le \delta(P_{2n(k)}, P_{2n(k)+1})$ -------------- (2.1.22)

Using (2.1.13) and (2.1.21) , we get $\lim_{k\to\infty} \delta(P_{2m(k)-1}, P_{2n(k)+1}) = \in$ -------------- (2.1.23)

Also, from the definition of (2.1.2) and from (2.1.13) , (2.1.21) , (2.1.23) , we have

$\lim_{k\to\infty} \mu(x_{2m(k)}, x_{2n(k)+1}) = \in$ ---------------- (2.1.24)

By substituting $x = x_{2m(k)}$, $y = x_{2n(k)+1}$ in (2.1.1) , we get

$$\begin{aligned} \xi(\delta(P_{2m(k)}, P_{2n(k)+1})) &= \xi(\delta(fx_{2m(k)}, gx_{2n(k)+1})) \\ &\le \xi(\mu(x_{2m(k)}, x_{2n(k)+1})) - \eta(\mu(x_{2m(k)}, x_{2n(k)+1})) \end{aligned}$$

Taking limit as $k \to \infty$ and using (2.1.21) , (2.1.24) , we will have

$\xi(\in) \le \xi(\in) - \eta(\in)$ --------------- (2.1.25)

which is a contradiction as $\in > 0$.

Hence $\{x_n\}$ is a Cauchy sequence.

Since X is a complete metric space , $\therefore$ there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Now , we prove that $'u'$ is end point of $'f'$

As the limit point $'u'$ is independent of the choice of $x_n \in P_n$, we also get

$$\lim_{n\to\infty} \delta(fx_{2n}, u) = \lim_{n\to\infty} \delta(gx_{2n+1}, u) = 0 \qquad \text{--------------- (2.1.26)}$$

From , $\mu(u, x_{2n+1}) = \max.\left\{ \begin{array}{l} d(u, x_{2n+1}), \delta(u, fu), \delta(x_{2n+1}, gx_{2n+1}), \delta(u, fx_{2n+1}), \delta(x_{2n+1}, fx_{2n+1}), \\ \frac{1}{2}[D(u, fu) + D(x_{2n+1}, gx_{2n+1})] \end{array} \right\}$

As $n \to \infty$, $\mu(u, x_{2n+1}) \to \delta(u, fu)$

Since $\xi(\delta(fu, gx_{2n+1})) \le \xi(\mu(u, x_{2n+1})) - \eta(\mu(u, x_{2n+1}))$ -------------- (2.1.27)

Taking limit as $n \to \infty$ and using (2.1.26) , we get

$$\xi(\delta(fu, u)) \le \xi(\delta(u, fu)) - \eta(\delta(u, fu)) \qquad \text{--------------- (2.1.28)}$$

which implies that $\xi(\delta(u, fu)) = 0$ i.e. $\delta(u, fu) = 0$ or $fu = \{u\}$.

Thus u is end point of f.

Now we prove that $'u'$ is also end point of g.

It can be easily proved that $\mu(u,u) = \delta(u, gu)$.

Using that $'u'$ is fixed point of f , we have ,

$$\begin{aligned} \xi(\delta(u, gu)) &= \xi(\delta(fu, gu)) \\ &\le \xi(\mu(u,u)) - \eta(\mu(u,u)) \qquad \text{--------------------- (2.1.29)} \\ &= \xi(\delta(u, gu)) - \eta(\delta(u, gu)) \end{aligned}$$

and using an argument similar to the above , it can be concluded that $\delta(u, gu) = 0$

i.e. $\{u\} = gu$.

Lastly to prove that $'u'$ is unique end point of f and g .

If there exists another fixed point $v \in X$, then one can easily show that $\mu(u, v) = d(u, v)$

and from
$$\begin{aligned} \xi(d(u, v)) &= \xi(\delta(fu, gv)) \\ &\le \xi(\mu(u, v)) - \eta(\mu(u, v)) \\ &= \xi(d(u, v)) - \eta(d(u, v)) \end{aligned}$$

which implies that $d(u, v) = 0$ i.e. $u = v$ .This completes the proof.

Corollary 2.2: Let (X, d) be a complete metric space and let $f : X \to B(X)$ be a set valued mapping satisfying generalized (ξ, η) - weak contraction as

$$\xi(\delta(fx, fy)) \le \xi(\mu(x, y)) - \eta(\mu(x, y)) \qquad \text{------------- (2.2.1) where}$$

$$\mu(x, y) = \max.\left\{ d(x, y), \delta(x, fx), \delta(y, fy), \delta(x, fy), \delta(y, fy), \frac{1}{2}[D(x, fx) + D(y, fy)] \right\} \qquad \text{------(2.2.2)}$$

$\xi : [0, \infty) \to [0, \infty)$ is a continuous monotone non-decreasing function with $\xi(t) = 0$ if and only if $t = 0$ and $\eta : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\eta(t) = 0$ if and only if $t = 0$. Then there exists a unique end point $u \in X$ such that $\{u\} = fu$.

Proof: Substitute $f = g$ in theorem 2.1 we get the required result.

Corollary 2.3: Let (X, d) be a complete metric space and let $f, g : X \to X$ be two self mappings satisfying generalized (ξ, η) - weak contraction as

$$\xi(d(fx, gy)) \le \xi(\mu(x, y)) - \eta(\mu(x, y)) \qquad \text{-------------- (2.3.1) where}$$

$$\mu(x,y)=\max.\left\{d(x,y),d(x,fx),d(y,gy),d(x,fy),d(y,fy),\frac{1}{2}\left[d(x,fx)+d(y,gy)\right]\right\}\text{------(2.3.2)}$$

$\xi,\eta:[0,\infty)\to[0,\infty)$, both are continuous monotone non-decreasing function with $\xi(t)=\eta(t)=0$ if and only if $t=0$. Then there exists a unique fixed point $u\in X$ such that $u=fu=gu$.

Proof: The proof of this Corollary is same as theorem 2.1.

Example 2.4: Let $X=\{1,2,3,4,5,6,7\}$ be a subspace of the real line with usual metric $'d'$ defined as $d(x,y)=|x-y|$. Let $f,g:X\to B(X)$ be defined as

$$f(x)=\begin{cases}\{7\} & ,x\in\{1,2,3\}\\ \{5\} & ,x\in\{4,5,6\}\\ \{2,3\} & ,x=7\end{cases}\quad\text{and}\quad g(x)=\begin{cases}\{6\} & ,x\in\{1,2,3\}\\ \{5\} & ,x\in\{4,5,6\}\\ \{4\} & ,x=7\end{cases}$$

Also $\xi,\eta:[0,\infty)\to[0,\infty)$ be defined as $\xi(t)=4t$ and $\eta(t)=\frac{2t}{3}$, then the mappings f, g satisfy the property of generalized (ξ,η) - weak contraction (2.1.1). Also f and g have a common unique end point $f(5)=g(5)=\{5\}$.

III. Conclusion

Unique common end point theorems have been proved for pair of set valued mappings and single mapping satisfying a generalized (ξ,η) weak contractive condition in complete metric space with example. Also a fixed point theorem for a pair of single valued mappings has been proved.

References

[1]. Abbas M. and D. Doric , A common end point theorem for Set-Valued generalized (ψ,ϕ) weak contraction , Hindawi Publishing Corporation , Fixed Point Theory and Applications Volume 2010, Article ID 509658, 8 pages

[2]. Alber Y.L. and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in New Results, in Operator Theory and Its Applications, I. Gohberg and Y. Lyubich, Eds., vol.98 of OperatorTheory: Advances and Applications, pp. 7–22, Birkh¨auser, Basel, Switzerland, 1997.

[3]. Beg I. and Abbas M., "Coincidence point and invariant approximation for mappings Satisfying generalized weak contractive condition," Fixed Point Theory and Applications, vol. 2006, Article ID 74503,7 pages, 2006.

[4]. Ciric, L.B., Fixed points for generalized multivalued contractions, Math. Vesnik, 9(24),1972, 265–272.

[5]. Doric D., "Common fixed point for generalized (ψ, ϕ)-weak contractions," Applied Mathematics Letters,vol. 22, no. 12, pp. 1896–1900, 2009.

[6]. Dube, L.S. A theorem on common fixed points of multivalued mapping, Ann. Soc. Sci, Bruxelles Sir, 1, 89, 1975, 463–468.

[7]. Fisher, B., Related fixed point theorem on two metric spaces. Math. Semi. Notes, 10, 1982, 17–26.

[8]. Klim, D. and Wardowski, D., Fixed point theorem for set valued contractions in complete metric spaces, J. Math. Anal. Appl., 334, 1, 2007, 132–139.

[9]. Nadler, S.B. Jr., Multi-valued contraction mapping. Pacific J. Math., vol. 30, 1969, 475– 488.

[10]. Namdeo, R.K. and Fisher, B., Related fixed point theorems for two pairs of set valued mappings on two complete and compact metric spaces, Indian Journal of Mathematics, vol.46, nos 2–3, 2004, 161–171.

[11]. Rhoades B. E., "Some theorems on weakly contractive maps," Nonlinear Analysis: Theory, Methods & Applications, vol. 47, no. 4, pp. 2683–2693, 2001.

[12]. Zhang Q. and Song Y., "Fixed point theory for generalized ϕ-weak contractions," Applied Mathematics Letters, vol. 22, no. 1, pp. 75–78, 2009.