IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 6 Ver. III (Nov - Dec. 2014), PP 26-31
www.iosrjournals.org

Fixed Point Theorem in Complete Metric Space

D. P. Shukla[1] and Ruchira singh[2]
Deptt. Of Mathematics/ computer science Govt. P.G. Science college Rewa (M.P.)

***Abstract**: In this paper, we introduce a generalisation of the Banach contraction mapping for fixed point theorem in complete metric space. The results presented in this paper substantially improve and extend the result due to Dutta and Choudhary we support our result by example.*

I. Introduction

It is well known the Banach contraction mapping theorem in one of the pivotal results of functional analysis. A mapping T: X $\rightarrow$ X where (x,d) is a metric space, is said to be a contraction if there exit $0 \leq k \leq 1$ such that

$$d(Tx,Ty) \leq k\, d(x,y), \quad for\ all\ x,y \in X$$

If the metric space (x,d) is complete then the mapping satisfying (1.1) has a unique fixed point which established by Banach (1922).

II. Perliminaries and definitions

Definition 2.1 (altering distance function [6]). A function $\psi : [0,\infty) \rightarrow [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(a) $\psi(0) = 0$,
(b) ψ is continuous and monotonically non-decreasing.

Theorem 2.2 (see [6]). Let (X,d) be a complete metric space, let ψ be an altering distance function, and let $f : X \rightarrow X$ be a self – mapping which satisfies the following inequality:

$$\psi(d(fx, fy)) \leq c\,\psi(d(x.y)) \tag{2.2}$$

for all $x, y \in X$ and for some $0 < c < 1$. Then f has a unique fixed point.

In fact Khan et al. proved a more general theorem [6, Theorem 2] of which the above result is a corollary.

Altering distance has been used in metric fixed point theory in a number of papers. Some of the works utilizing the concept of altering distance function are noted in [8-11].

In [12], 2-variable and in [13] 3-variable altering distance functions have been introduced as generalizations of the concept of altering distance function. It has also been extended in the context of multivalued [14] and fuzzy mappings [15]. The concept of altering distance function has also been introduced in Menger spaces [16].

Another generalization of the contraction principle was suggested by Alber and Guerre-Delebriere [7] in Hilbert Spaces. Rhoades [17] has shown that the result which Alber and Guerre-Delabriere have proved in [7] is also valid in complete metric spaces. We state the result of Rhoades in the following.

Definition 2.3 **(weakly contractive mapping)**. A mapping $T : X \rightarrow X$, where (X,d) is a metric space, is said to be weakly contractive if

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)), \tag{2.3}$$

Where $x, y \in X$ and $\phi : [0,\infty) \rightarrow [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if t = 0.

If one takes $\phi(t) = kt$ where $0 < k < 1$, then (2.3) reduces to (2.1).

Theorem 2.4 (see [17]). If $T : X \rightarrow X$ is a weakly contractive mapping, where (X, d) is a complete metric space, then T has a unique fixed point,

In fact, Alber and Guerre-Delabriere assumed an additional condition on ϕ which is $\lim_{t\to\infty} \emptyset(t)$. But Rhoades [17] obtained the result noted in Theorem 1.4 without using this particular assumption.

It may be observed that though the function ϕ has been defined in the same way as the altering distance function, the way it has been used in Theorem 1.4 is completely different from the use of altering distance function.

Weakly contractive mappings have been dealt with in a number of papers. Some of these works are noted in [17-20].

The purpose of this paper is to introduce a generalization of Banach contraction mapping principle which includes the generalizations noted in Theorems 2.2 and 2.4.

III. Main results

Theorem 3.1. Let (X.d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \phi(d(x,y)) - \phi^{'}(d(Tx,x)), \tag{3.1}$$

Where $\psi, \phi, \phi^{'} : [0,\infty) \to [0,\infty)$ are all continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t) = \phi^{'}(t)$ if and only if t = 0.

Then T has a unique fixed point.

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1,} \quad n = 1,2,\cdots$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), we obtain

$$\psi(d(x_n, x_{n+1})) \le \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)) - \phi^{'}(d(x_n, x_{n-1})), \tag{3.2}$$

Which implies

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n) \quad \text{(Using monotone property of } \psi \text{ -function).} \tag{3.3}$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing and consequently there exists $r \ge 0$ such that

$$d(x_n, x_{n+1}) \to r \qquad as \quad n \to \infty. \tag{3.4}$$

Letting $n \to \infty$ in (2.2) we obtain

$$\psi(r) \le \psi(r) - \phi(r) - \phi^{'}(r), \tag{3.5}$$

Which is a contradiction unless r = 0.

Hence

$$d(x_n, x_{n+1}) \to 0 \qquad as \quad n \to \infty. \tag{3.6}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\in > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with n (k) > m (K) > K such that

$$d(x_{m(k)}, x_{n(k)}) \ge \varepsilon . \tag{3.7}$$

Further, corresponding to m (K), we can choose n (K) in such a way that it is the smallest integer with n (K) > m (K) and satisfying (2.7).

Then

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon . \tag{3.8}$$

Then we have

$$\in \le d(x_{m(k)}, x_{n(k)}) \le d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) < \in + d(x_{n(k)-1}, x_{n(k)}). \qquad (3.9)$$

Setting $k \to \infty$ and using (3.6),

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon . \qquad (3.10)$$

Again,

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &\le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}), \\ d(x_{n(k)-1}, x_{m(k)-1}) &\le d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}). \end{aligned} \qquad (3.11)$$

Setting $k \to \infty$ in the above two inequalities and using (3.6), (3.10), we get

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon . \qquad (3.12)$$

.

Setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (3.1) and using (3.7), we obtain

$$\psi(\in) \le \psi(d(x_{m(k)}, x_{n(k)})) \le \psi(d(x_{m(k)-1}, x_{n(k)-1})) - \phi(d(x_{m(k)-1}, x_{n(k)-1})) - \phi^{'}(d(x_{m(k)}, x_{m(k)-1})) \qquad (3.13)$$

Setting $k \to \infty$, utilizing (3.6), (3.10) and (3.12), we obtain

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon) \qquad (3.14)$$

Which is a contradiction if $\varepsilon > 0$.
This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete metric space X.
Let

$$x_n \to z \quad \text{as} \quad n \to \infty . \qquad (3.15)$$

Substituting $x = x_{n-1}$ and y = z in (2.1), we obtain

$$\psi(d(x_n, Tz)) \le \psi(d(x_{n-1}, z)) - \phi(d(x_{n-1}, z)) - \phi^{'}(d(x_n, x_{n-1})) \qquad (3.16)$$

Setting $n \to \infty$, using (3.15) and continuity of ϕ, $\phi^{'}$ and ψ, we have

$$\psi(d(z, Tz)) \le \psi(0) - \phi(0) - \phi^{'}(0) = 0, \qquad (3.17)$$

Which implies $\psi(d(z, Tz)) = 0$, that is,

$$d(z, Tz) = 0 \quad \text{or} \quad z = Tz . \qquad (3.18)$$

To prove the uniqueness of the fixed point, let us suppose that z_1 and z_2 are two fixed points of T.

Putting $x = z_1$ and $y = z_2$ in (3.1),

$$\begin{aligned} &\psi(d(Tz_1, Tz_2)) \le \psi(d(z_1, z_2)) - \phi(d(z_1, z_2)) - \phi^{'}(d(Tz_1, z_1)) \\ \text{or } &\psi(d(z_1, z_2)) \le \psi(d(z_1, z_2)) - \phi(d(z_1, z_2)) - \phi^{'}(d(z_1, z_1)) \end{aligned} \qquad (3.19)$$

or $\phi(d(z_1, z_2)) \le 0$,

or equivalently $d(z_1, z_2) = 0$, that is, $z_1 = z_2$.
This proves the uniqueness of the fixed point.

Example 3.2. Let $X = [0,1] \cup \{2,3,4,\cdots\}$ and

$$d(x,y) = \begin{cases} |x,y|, & if\ x,y \in [0,1],\ x \ne y, \\ x+y, & if\ at\ least\ one\ of\ x\ or\ y \notin [0,1]\ and\ x \ne y, \\ 0, & if\ x = y. \end{cases} \tag{3.20}$$

Then (X,d) is a complete metric space [1].

Let $\psi : [0,\infty) \to [0,\infty)$ be defined as

$$\psi(t) = \begin{cases} t, & if\ 0 \le t \le 1, \\ t^2, & if\ t > 1, \end{cases} \tag{3.21}$$

Let $\phi' : [0,\infty) \to [0,\infty)$ be defined as

$$\phi'(t) = \begin{cases} \frac{1}{2}t^2, & if\ 0 \le t \le 1, \\ \frac{1}{2}, & if\ t > 1. \end{cases} \tag{3.22}$$

Let $\phi : [0,\infty) \to [0,\infty)$ be defined as

$$\phi(t) = \begin{cases} \frac{1}{2}t^2, & if\ 0 \le t \le 1, \\ \frac{1}{2}, & if\ t > 1. \end{cases} \tag{3.23}$$

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - \frac{1}{2}x^2, & if\ 0 \le x \le 1, \\ x-1, & if\ x \in \{2,3,\ldots\}. \end{cases} \tag{3.24}$$

Without loss of generality, we assume that x > y and discuss the following cases.
Case 1 $(x \in [0,1])$. Then

$$\begin{aligned} \psi(d(Tx,Ty)) &= \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right) \\ &= (x-y) - \frac{1}{2}(x-y)(x+y) \le (x-y) - \frac{1}{2}(x-y)^2 \\ &= d(x,y) - \frac{1}{2}(d(x,y))^2 \end{aligned} \tag{3.25}$$

$$= \psi(d(x,y)) - \frac{1}{2}(d(x,y))^2$$
$$= \psi(d(x,y)) - \phi(d(x,y)) \quad (\text{since } x - y \le x + y).$$

Case 2 $(x \in \{3,4,\dots\})$. Then

$$d(Tx,Ty) = d\left(x-1, y - \frac{1}{2}y^2\right) \quad if \ y \in [0,1]$$

Or $$d(Tx,Ty) = x - 1 + y - \frac{1}{2}y^2 \le x + y - 1, \tag{3.26}$$

$$d(Tx,Ty) = d\,(x-1, y-1) \quad if \ y \in \{2,3,4,\dots\}$$

or $$d(Tx,Ty) = x + y - 2 < x + y - 1.$$

Consequently,

$$\psi(d(Tx,Ty)) = (d(Tx,Ty))^2 \le (x+y-1)^2 < (x+y-1)(x+y+1)$$
$$= (x+y)^2 - 1 < (x+y)^2 - \frac{1}{2} \tag{3.27}$$
$$= \psi(d(x,y)) - \phi(d(x,y)).$$

Case 3 $(x = 2)$. Then $y \in [0,1]$, $Tx = 1, \ and \ d(Tx,Ty) = 1 - \left(y - \left(\tfrac{1}{2}\right)y^2\right) \le 1.$

So, we have $\psi(d(Tx,Ty)) \le \psi(1) = 1$.

Again $d(x,y) = 2 + y$.

So,

$$\psi(d(x,y)) - \phi(d(x,y)) = (2+y)^2 - \phi\left((2+y)^2\right)$$
$$= (2+y)^2 - \frac{1}{2} \tag{3.28}$$
$$= \frac{7}{2} + 4y + y^2 > 1$$
$$= \psi(d(Tx,Ty)).$$

Considering all the above cases, we conclude that inequality (3.1) remains valid for ϕ', ϕ, ψ, and T constructed as above and consequently by an application of Theorem 2.1, T has a unique fixed point.
It is seen that "0" is the unique fixed point of T.

References

[1]. D.W. Boyd and J. S. W. Wong, "On nonlinear contractions," Proceedings of the American Mathematical Society, vol. 20, no. 2, pp. 458-464, 1969.

[2]. A. D. Arvanitakis, " A proof of the generalized Banach contraction conjecture," Proceedings of the American Mathematical Society, vol. 131, no. 12, pp. 3647-3656, 2003.

[3]. J. Merryfield, B. Rothschild, and J. D. Stein Jr., " An application of Ramsey's theorem to the Banach contraction conjecture," Proceedings of the American Mathematical Society, vol. 130, no. 4, pp. 927-933, 2002.

[4]. T. Suzuki, " A generalized Banach contraction principle that characterizes metric completeness," Proceedings of the American Mathematical Society, vol. 136, no. 5, pp. 1861-1869, 2008.

[5]. O. Hadzic and E. Pap, Fixed point Theory in Probabilistic Metric Spaces, vol. 536 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[6]. M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points" Bulletin of the Australian Mathematical Society, vol. 30, no. 1, pp. 1-9, 1984.

[7]. Ya. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in New Results in Operator Theory and Its Applications, I. Gohberg and Y. Lyubich, Eds., vol. 98 of Operator Theory: Advances and Applications, pp. 7-22, Birkhäuser, Basel, Switzerland, 1997.

[8]. G. V. R. Babu, B. Lalitha, and M.L. Sandhya, "Common fixed point theorems involving two generalized altering distance functions in four variables," Proceedings of the Jamgjeon Mathematical Society, vol. 10, no. 1, pp. 83-93, 2007.

[9]. S. V. R. Naidu, "Some fixed point theorems in metric spaces by altering distances," Czechoslovak Mathematical Journal, vol. 53, no. 1, pp. 205-212, 2003.

[10]. K.P.R. Sastry and G. V. R. Babu, "Some fixed point theorems by altering distances between the points," Indian Journal of Pure and Applied Mathematics, vol. 30, no. 6, pp. 641-647, 1999.

[11]. K. P. R. Sastry, S. V. R. Naidu, G. V. R. Babu, and G. A.Naidu, "Generalization of common fixed point theorems for weakly commuting map by altering distances," Tamkang Journal of Mathematics, vol. 31, no. 3, pp. 243-250, 2000.

[12]. B. S. Choudhury and P. N. Dutta, " A unified fixed point result in metric spaces involving a two variable function," Filomat, no. 14, pp. 43-48, 2000.

[13]. B. S. Choudhury, "A common unique fixed point result in metric spaces involving generalized altering distances," Mathematical communications, vol. 10, no. 2, pp. 105-110, 2005.

[14]. B. S. Choudhury and A. Upadhyay, "On unique common fixed point for a sequence of multi-valued mappings on metric spaces," Bulletin of Pure & Applied Sciences. Section E, vol. 19, no. 2, pp. 529-533, 2000.

[15]. B. S. Choudhury and P. N. Dutta, " common fixed points for fuzzy mappings using generalized altering distances," Soochow Journal of Mathematics, vol. 31, no. 1, pp. 71-81, 2005.

[16]. B. S. Choudhury and K. Das, "A new contraction principle in Menger Spaces," Acta Mathematica Sinica, vol. 24, no. 8, pp. 1379-1386, 2008.