

Euclidean and Function Space Null-Controllability of Nonlinear Delay Systems with Restrained Controls

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Abstract: *The basic tool we use in our investigation is the nonlinear variation of constants formula for delay systems whilst we determine the controllability of nonlinear functional differential systems. This paper also touches on capture in nonlinear functional differential games of pursuit.*

Keywords: *Delay systems, controllability, variation of constants formula.*

I. Introduction

Hermes and Lasalle [1] investigated the controllability of the system:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x^0\end{aligned}\quad (1)$$

where $x \in E^n$, $A(\cdot)$ and $B(\cdot)$ are $n \times n$ - and $n \times m$ -matrix valued functions, respectively, with components summable over finite real interval and $u(\cdot)$ is vector valued function which is constrained to lie in a compact set \mathbb{U} of E^m . They showed that if:

$$\dot{x}(t) = A(t)x(t) \quad (2)$$

is uniformly asymptotically stable and if (1) is controllable with $L_2([\sigma, \infty), E^m)$ controls then (1) is null – controllable with controls in \mathbb{U} . Chukwu [2] examined the nonlinear retarded system:

$$\begin{aligned}\dot{x}(t) &= L(t, x_t) + B(t)u(t) + f(t, x_t, u(t)), \quad t \geq \sigma \\ x_\sigma &= \varphi \text{ in } [-r, 0]\end{aligned}\quad (3)$$

where $B(t)$ is a continuous $n \times m$ -matrix, and for each $t \in E$ the linear operator $\varphi \rightarrow L(t, \varphi)$, $\varphi \in W_2^{(1)}$ has the form:

$$L(t, \varphi) = \int_{-r}^0 d_s \eta(t, s) \varphi(s) \quad (4)$$

where the integral is in the Lebesgue-Stieljes sense and $(t, \theta) \rightarrow \eta(t, \theta)$, $(t, \theta) \in E \times E$ is a mapping with values in L_{nn} , the space of continuous $n \times n$ -matrices. It is assumed that $t \rightarrow \eta(t, \theta)$ is continuous for each fixed $\theta \in [-r, 0]$ and $\theta \rightarrow \eta(t, \theta)$ is of bounded variation on $[-r, 0]$ for each fixed $t \in E$. Moreover, $\eta(t, \theta) = 0$, $\theta \geq 0$, $\eta(t, \theta) = \eta(t, -r)$, $\theta \leq r$ and $\theta \rightarrow \eta(t, \theta)$ is left continuous on $[-r, 0]$ satisfying:

$$\text{Var}_{s \in E} \eta(t, s) \leq \rho(t), \quad t \in E \quad (5)$$

where $\rho(t)$ is locally integrable. The function:

$$f: [\sigma, \infty) \times W_2^{(1)} \times E^m \rightarrow E^n \quad (6)$$

is continuous and uniformly Lipschitzian in the last two arguments. He showed that if:

(i) the system:

$$\begin{aligned}\dot{x}(t) &= L(t, x_t) + B(t)u(t) \\ x_\sigma &= \varphi \in W_2^{(1)}([-r, 0], E^n)\end{aligned}\quad (7)$$

where $u \in \mathbb{U} \subseteq L_2^{\text{loc}}([\sigma, \infty), U)$, $U \subseteq E^m$, $0 \in \text{Int } U$ is complete; that is, zero is in the interior of the reachable set of (7), and

(ii) the free system:

$$\dot{x}(t) = L(t, x_t) \tag{8}$$

is uniformly asymptotically stable so that there exist some constants $k \geq 1, \alpha > 0$ such that every solution of (8) satisfies the boundedness condition:

$$\|x_t(\sigma, \varphi)\| \leq k\|\varphi\|e^{-\alpha(t-\sigma)}, t \geq \sigma \tag{9}$$

and if:

$$\begin{aligned} \text{(iii)} \quad & f(t, 0, 0) = 0 \\ & f(t, \varphi, 0) = f_1(t, \varphi) + f_2(t, \varphi) \end{aligned} \tag{10}$$

where

$$|f_1(t, \varphi)| \leq \pi(t)\|\varphi\|, \quad |f_2(t, \varphi)| \leq \varepsilon\|\varphi\| \tag{11}$$

with

$$\int_{\sigma}^{\infty} \pi(t)dt \leq \infty, \quad \varepsilon = \frac{\alpha}{2k}, \quad t \geq \sigma, \quad \varphi \in W_2^{(1)}$$

Chukwu [3] investigated controllability under unpredictable disturbances whose state space is the Sobolev space, $W_2^{(1)}([\sigma, \infty), E^n)$, with dynamical equation;

$$\dot{x}(t) = L(t, x_t) - p(t) + q(t), t \in [\sigma, \infty) \tag{12}$$

where $p(\cdot) \in L_2([\sigma, t], P), P \subseteq E^m, q(\cdot) \in L_2([\sigma, t], Q), Q \subseteq E^m, L(t, \varphi)$ satisfies the conditions in (3). Here q is the quarry control and p is the control strategy for the pursuer. The target is either a subset of $W_2^{(1)}$, a point in $W_2^{(1)}$, a subset of E^n , or a point of E^n . There is capture if the initial function can be forced to the target by the pursuer no matter what the quarry does. He showed that the associated linear retarded functional differential equation:

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) - u(t), \quad t \geq \sigma \\ x_{\sigma} &= \varphi \in W_2^{(1)}([-r, 0], E^n) \\ u(t) &\in \mathbb{U} \end{aligned} \tag{13}$$

$$P^*Q = \mathbb{U} = \{x: x + Q \subseteq P\} \tag{14}$$

is controllable if and only if there is capture for game (12).

Shanholt [4] examined the system:

$$\dot{y}(t) = f(t, y_t) + g(t, y_t) \tag{15}$$

where $f, g: \Gamma \rightarrow E^n$ are continuous in $\Gamma = (\tau, \infty) \times \Lambda, \Lambda$ is open in $C([-r, 0], E^n)$ and $f(t, \varphi)$ has continuous Frechet derivative $f'(t, \varphi)$ with respect to $\varphi \in \Lambda$. By examining the linear variational equation:

$$\dot{z}(t) = f'(t, x_t(\sigma, \varphi))z_t, t \in [\sigma, \infty) \tag{16}$$

with respect to $x(\sigma, \varphi)$, the solution of the nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \geq \sigma \\ x_{\sigma} &= \varphi \end{aligned} \tag{17}$$

he developed a nonlinear Alekseev like variation of constants formula for the nonlinear retarded system. He showed that the solution of the perturbed nonlinear retarded system (15) is given by:

$$y_t(\sigma, \varphi) = x_t(\sigma, \varphi) + \int_{\sigma}^t T(t, s) Y_0 g(s, y_s(\sigma, \varphi)) ds \quad (18)$$

where $x(\sigma, \varphi)$ is the solution of the nonlinear system:

$$\dot{x}(t) = f(t, x_t) \quad (19)$$

and Y_0 is the $n \times n$ - matrix function defined by the relation:

$$Y_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0 \\ I, & \theta = 0 \end{cases} \quad (20)$$

(I – identity matrix).

In this paper we use the variation of constants formula for nonlinear systems as developed in Shanholt [4] to investigate the controllability of a retarded system:

$$\dot{x}(t) = g(t, x_t) + B(t)u(t) \quad (21)$$

where $g(t, \varphi)$ is nonlinear. We employ the method developed in Chukwu [5] to establish function space null-controllability with constraints of the nonlinear retarded system. This paper is in four sections. The first section introduces the subject and the second section establishes notations and definitions. Section three establishes Euclidean and Function space null-controllability while section four studies both Euclidean and Function space capture in nonlinear retarded differential games of pursuit.

II. Notations and Definitions

Let E^n denote the n -dimensional Euclidean space. For $\sigma, t_1 \in E$, the symbol $L_2([\sigma, t_1], E^n)$ denotes the space of square integrable functions from the interval $[\sigma, t_1]$ into E^n . We shall denote the Sobolev space of all absolutely continuous functions $x: [\sigma, t_1] \rightarrow E^n$ whose derivative $\dot{x}(\cdot) \in L_2([\sigma, t_1], E^n)$ by $W_2^{(1)}([\sigma, t_1], E^n)$. Let $r > 0$ be given. For functions $x: [\sigma - r, t_1] \rightarrow E^n$ and $t \in [\sigma, t_1]$ we use x_t to denote the function on $[-r, 0]$ defined by:

$$x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0. \quad (22)$$

Consider the nonlinear delay functional differential system of the form:

$$\begin{aligned} \dot{x}(t) &= g(t, x_t), t \geq \sigma \\ x_{\sigma} &= \varphi \in W_2^{(1)}([-r, 0], E^n) \end{aligned} \quad (23)$$

where we assume as basic that $g: E \times W_2^{(1)} \rightarrow E^n$ is continuous and has continuous Frechet derivative with respect to $x_t(\sigma, \varphi)$ and where $g(t, \varphi)$ is locally Lipschitzian in φ in each compact set of $E \times W_2^{(1)}$. Let $\dot{x}(t)$ denote the right-hand derivative of $x(u)$ at $u = t$. For any $\sigma \in E$, and $\varphi \in W_2^{(1)}$ we say that $x(\sigma, \varphi)$ is a solution of (22) with initial function φ at σ if there exists an $a > 0$ such that $x(\sigma, \varphi) \in W_2^{(1)}([\sigma - r, \sigma + a], E^n)$, $x_{\sigma} = \varphi$ and $x(\sigma, \varphi)$ satisfies (22) for $t \in [\sigma, \sigma + a]$. From the above assumptions (22) has a unique solution $x(\sigma, \varphi)$ which depends continuously on σ, φ and on $t \in E$. We also assume as basic that:

$$g(t, 0) = 0 \text{ for } t \geq \sigma \quad (24)$$

and that the solution $x(\sigma, \varphi)$ of (22) satisfies $x(\sigma, \varphi) = 0$ if, and only if $\varphi \equiv 0$.

In the system:

$$\dot{x}(t) = g(t, x_t) + B(t)u(t) \quad (25)$$

let g satisfy the conditions of (23) and let $B(t)$ be a continuous function such that:

$$\|B(t)\| \leq \rho(t) \tag{26}$$

where $\rho(t)$ is a continuous function on $[0, \infty)$ satisfying:

$$\int_0^\infty \rho(t) dt < \infty \tag{27}$$

and where $\|\cdot\|$ denotes a matrix norm. Let $u \in \mathbb{U} = L_2([\sigma, \infty), C^m)$ where $C^m = \{u \in E^m: |u_j| \leq 1, j = 1, \dots, m\}$. Corresponding to each solution $x(\sigma, \varphi)$ of (23) we define a linear delay functional differential system:

$$\dot{z}(t) = f'(t, x_t(\sigma, \varphi))z_t \tag{28}$$

For each $t \in J(\sigma, x)$, the maximal interval of existence of $x(\sigma, \varphi)$. The system (28) is called the linear variational equation of (23) with respect to $x(\sigma, \varphi)$. For any fixed σ and $t \geq \sigma$, $z(\sigma, \cdot)(t)$ is a continuous linear operator from $W_2^{(1)}$ into E^n . Thus, throughout this work, to the linear system (28), we shall associate a family of linear operators $T(t, \sigma)(\cdot): W_2^{(1)} \rightarrow W_2^{(1)}$, $t \geq \sigma$, by defining for each $\varphi \in W_2^{(1)}$:

$$T(t, \sigma)\varphi = z_t(\sigma, \varphi). \tag{29}$$

For any initial function $\varphi \in W_2^{(1)}([-r, 0], E^n)$ we may define a solution of (28) with initial function φ at σ by $z(\sigma, \varphi)$. Therefore, if the $n \times n$ matrix function Y_0 is defined by:

$$Y_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0 \\ I, & \theta = 0 \end{cases} \tag{30}$$

where I is the identity matrix, then the operator $T(t, \sigma)$ may be defined on the columns of Y_0 . If $B(t), u(t)$ satisfy the conditions in (25), then $x(\sigma, \varphi, u)$ is the solution of (25) with initial function $\varphi \in W_2^{(1)}$ at $\sigma \in E$, if and only if $x(\sigma, \varphi, u)$ satisfies the integral equation:

$$x_t(\sigma, \varphi, u) = x_t(\sigma, \varphi) + \int_\sigma^t T(t, s)Y_0B(s)u(s)ds, \quad t \geq \sigma \tag{31}$$

where $x_t(\sigma, \varphi)$ is the solution of (23).

Definition 2.1

The system (23) is globally exponentially asymptotically stable if there exists an $L > 0, c > 0$ such that the solution $x(\sigma, \varphi)$ of (23) satisfies, $x_\sigma(\sigma, \varphi) = \varphi$ and:

$$\|x_t(\sigma, \varphi)\| \leq L e^{-c(t-\sigma)} \|\varphi\|, \quad t \in [\sigma, \infty). \tag{32}$$

Definition 2.2

The system (25) is proper on an interval $(\sigma, t]$ if $C^T T(t, s)IB(s) = 0$ a. e. $s \in [\sigma, t]$, for each $\varphi \in W_2^{(1)}$ implies $C = 0$, where C^T denotes the transpose of C in E^n . If (23) is proper on each interval $[\sigma, t]$, $t \geq \sigma$ we say the system is proper.

Definition 2.3

The system (25) is null-controllable with constraints if given any initial state $\varphi \in W_2^{(1)}$, there exists $t_1 \geq \sigma$ and an admissible control $u \in \mathbb{U}$ such that the solution $x(\sigma, \varphi, u)$ satisfies $x_\sigma(\sigma, \varphi, u) = \varphi$ and $x(t_1, \sigma, \varphi, u) = 0$.

Definition 2.4

The system (25) is null-controllable if given any initial state $\varphi \in W_2^{(1)}([-r, 0], E^n)$, there exist $t_1 \geq \sigma$ And an admissible control $u \in L_2([\sigma, \infty), E^m)$ such that the solution $x(\sigma, \varphi, u)$ satisfies $x_\sigma(\sigma, \varphi, u) = \varphi$ and $x(t_1, \sigma, \varphi, u) = 0$.

III. Controllability Results

The Euclidean space reachable set of (25) is the set:

$$\mathbb{R}(t, \sigma) = \left\{ \int_{\sigma}^t T(t, s)B(s)u(s)ds : u \in \mathbb{U} \right\} \quad (33)$$

Respectively, the function space reachable set is given by:

$$\mathbb{A}(t, \sigma) = \left\{ \int_{\sigma}^t T(t, s)Y_0B(s)u(s)ds : u \in \mathbb{U} \right\} \quad (34)$$

where $u \in \mathbb{U}$ is as given in (25) and Y_0 is as defined by (30), $t \geq \sigma$ and $\varphi \in W_2^{(1)}([-r, 0], E^n)$.

Theorem 3.1

The system (25) is proper on $[\sigma, t]$ if, and only if $0 \in \text{Int } \mathbb{R}(t, \sigma)$.

Proof:

Given that $\mathbb{U} = L_2([\sigma, \infty), C^m)$ where $C^m = \{u \in E^m : |u_j| \leq 1, j = 1, \dots, m\}$. \mathbb{U} is weak* compact. Let $Y(s) = T(t, s)B(s)$ and for the sequence $\{u_i\}_1^{\infty} \subseteq \mathbb{U}$ let

$$s_i = \int_{\sigma}^t Y(s)u_i(s)ds \in \mathbb{R}(t, \sigma), \quad i = 1, 2, \dots \quad (35)$$

We may extract a convergent subsequence, $\{u_k\}_1^{\infty}$ say, such that $u_k \rightarrow u \in \mathbb{U}$, and

$$\lim_{i \rightarrow \infty} \int_{\sigma}^t Y(s)u_k(s)ds \rightarrow \int_{\sigma}^t Y(s)u(s)ds \quad (36)$$

then

$$\lim_{i \rightarrow \infty} s_i = s \in \mathbb{R}(t, \sigma). \quad (37)$$

Hence $\mathbb{R}(t, \sigma)$ is closed. Furthermore, consider the controls $u_1, u_2 \in \mathbb{U}, 0 < \alpha < 1$, we find that

$$\begin{aligned} & \alpha \int_{\sigma}^t Y(s)u_1(s)ds + (1 - \alpha) \int_{\sigma}^t Y(s)u_2(s)ds \quad (38) \\ &= \int_{\sigma}^t Y(s)[\alpha u_1(s) + (1 - \alpha)u_2(s)]ds. \end{aligned}$$

But for each j satisfying $1 \leq j \leq m$,

$$|\alpha u_{1j}(s) + (1 - \alpha)u_{2j}(s)| \leq \alpha |u_{1j}(s)| + (1 - \alpha)|u_{2j}(s)| = 1. \quad (39)$$

It follows that $\alpha u_1(s) + (1 - \alpha)u_2(s) \in \mathbb{U}$ so that:

$$\int_{\sigma}^t Y(s)[\alpha u_1(s) + (1 - \alpha)u_2(s)]ds \in \mathbb{R}(t, \sigma) \quad (40)$$

Hence, $\mathbb{R}(t, \sigma)$ is convex, so $\mathbb{R}(t, \sigma)$ is closed and convex.

Consider a point x_1 on the boundary of $\mathbb{R}(t, \sigma)$. Since the reachable set is closed and convex, there is a support plane π through x_1 . That is $C^T(x - x_1) \leq 0$ for each $x \in \mathbb{R}(t, \sigma)$, where $C \neq 0$ is an outward normal to π . If u_1 is the corresponding control to x_1 then we have:

$$C^T \int_{\sigma}^t Y(s)u(s)ds \leq C^T \int_{\sigma}^t Y(s)u_1(s)ds, \quad \text{for each } u \in \mathbb{U}. \quad (41)$$

Since C^m is unit cube, this last inequality will hold for each $u \in C^m$ if, and only if

$$C^T \int_{\sigma}^t Y(s)u(s)ds \leq \int_{\sigma}^t |C^T Y(s)u_1(s)|ds = \int_{\sigma}^t |C^T Y(s)|ds. \quad (42)$$

And $u_1(s) = \text{sgn } C^T Y(s)$. Since we always have $0 \in \mathbb{R}(t, \sigma)$, if 0 is not in the interior, then it is on the boundary. This is equivalent to:

$$0 = \int_{\sigma}^t |C^T Y(s)| ds \text{ so that } C^T Y(s) = 0 \text{ a. e. } s \in [\sigma, t]. \tag{43}$$

Contradiction. Hence, the system is proper. This completes the proof.

Theorem 3.2: Suppose,

i) The solution $x(\sigma, \varphi)$ of (23) is globally exponentially asymptotically stable and

ii) The system (25) is proper,

then the system (25) is Euclidean space null-controllable with constraints.

Proof:

By the variation of constants formula, the solution of (25) is given as:

$$x(t, \sigma, \varphi, u) = x(t, \sigma, \varphi) + \int_{\sigma}^t T(t, s)B(s)u(s)ds: t \geq \sigma \tag{44}$$

$$x(t_1, \sigma, \varphi, u) = x(t_1, \sigma, \varphi) + \int_{\sigma}^{t_1} T(t_1, s)B(s)u(s)ds \tag{45}$$

and

$$x(t_1, \sigma, \varphi) = - \int_{\sigma}^{t_1} T(t_1, s)B(s)u(s)ds. \tag{46}$$

Now define H as follows:

$$H(t_1) = \left\{ x(t_1, \sigma, \varphi) + \int_{\sigma}^{t_1} T(t_1, s)B(s)u(s)ds: u \in \mathbb{U}, \varphi \in W_2^{(1)}, x_{\sigma} = \varphi \right\} \tag{47}$$

Clearly, $H(t_1)$ is a sum of two sets, the first of which is contained in the second part, the Euclidean space reachable set.

By definition the domain \mathcal{C} of null-controllability is the collection $\varphi \in W_2^{(1)}$ such that for any initial state φ there exists a t_1 and some $u \in \mathbb{U}$ such that the solution $x(\sigma, \varphi, u)$ of (25) satisfies $x_{\sigma} = \varphi$ and $x(t_1, \sigma, \varphi, u) = 0$. Hence \mathcal{C} is the subset of $W_2^{(1)}$ such that:

$$x(t_1, \sigma, \mathcal{C}) = \{x(t, \sigma, \varphi): \varphi \in \mathcal{C}, x(\sigma, \varphi) \text{ is a solution of (23)}\} \tag{48}$$

Since (25) is proper $0 \in \text{Int } \mathcal{C}$ for each t . Hence $0 \in \text{Int } \mathcal{C}$. Indeed, suppose not, since $0 \in \mathcal{C}$ there is a sequence $\{\varphi_m\}_1^{\infty} \subseteq W_2^{(1)}$ such that $\varphi_m \rightarrow 0$ as $m \rightarrow \infty$ and no φ_m is in \mathcal{C} so that $\varphi_m \neq 0$. Suppose $x(t, \sigma, \varphi_m) = x_m$. Then $x_m \neq 0$, and by continuity $x_m \rightarrow x(t, \sigma, 0) = 0$ as $m \rightarrow \infty$. Hence 0 is not in the interior of the reachable set, a contradiction. Hence 0 is in the interior of \mathcal{C} . Since the 0 function is in the interior of \mathcal{C} , there is an open ball $\mathcal{P} \subseteq W_2^{(1)}$ around 0 which is contained in \mathcal{C} .

In (25) set $u = 0$ on $[\sigma, \infty)$. Then the solution $x(\sigma, \varphi, u)$ of (25) with initial state φ is the solution of (23). By condition (i) of Theorem 3.2 every solution of (23) satisfies:

$$x_{\sigma} = \varphi \text{ and } \|x_t(\sigma, \varphi)\| \leq Le^{-c(t-\sigma)} \|\varphi\| \tag{49}$$

$\in \mathcal{P} \subseteq \mathcal{C}$. With $x_{t_1}(\sigma, \varphi) \in \mathcal{C}$ as an initial function there are a t_2 and control $u \in \mathbb{U}$ such that the solution satisfies:

$$x_{t_2}(\sigma, x_{t_1}) = x_{t_1} \text{ and } x_{t_2}(\sigma, x_{t_1}, u) = 0. \tag{50}$$

This completes the proof.

Definition 2.3

The system (25) is said to be complete on $[\sigma, t]$ if, and only if:

$$0 \in A(t, \sigma), t \geq \sigma + r. \tag{51}$$

The system (25) is complete if it is complete on each interval $[\sigma, t]$, $t \geq \sigma + r$. It is complete with constraints if $0 \in \text{Int } A(t, \sigma)$, $u \in \mathbb{U}$, $t \geq \sigma + r$ where \mathbb{U} is as defined in (25).

Let $g'(t, x_t)$ be the Frechet derivative of $g(t, x_t)$ with respect to $x_t(\sigma, \varphi)$, where $x(\sigma, \varphi)$ is the solution of (23). Consider the linear control system:

$$\dot{z}(t) = g'(t, x_t(\sigma, \varphi))z_t + B(t)u(t) \tag{52}$$

where $B(t)$ and $u(t)$ are as given in (25). Consider also the free system:

$$\dot{z}(t) = g'(t, x_t(\sigma, \varphi))z_t. \tag{53}$$

The solution of (52) is given by:

$$z_t(\sigma, \varphi, u) = z_t(\sigma, \varphi) + \int_{\sigma}^t T(t, s)Y_0 B(s)u(s)ds \tag{54}$$

for $t \geq \sigma$, $\varphi \in W_2^{(1)}$, Y_0 as in (30) and $T(t, \sigma)$ is the family of linear operators associated with (53); where the integral equation (54) is an integral equation in E^n and is to be interpreted as:

$$z_t(\theta) = [T(t, \sigma)\varphi](\theta) + \int_{\sigma}^t [T(t, s)Y_0](\theta)B(s)u(s)ds \tag{55}$$

for $t \geq \sigma$, $-r \leq \theta \leq 0$. For a derivation of (54) see Hale [6, pp 80-86]. Consequently, the function space reachable set of (52) is given by:

$$A(t, \sigma) = \left\{ \int_{\sigma}^t T(t, s)Y_0 B(s)u(s)ds : u \in L_2([\sigma, \infty), E^m) \right\}. \tag{56}$$

Theorem 3.4

The system (52) is complete on $[\sigma, t]$, $t \geq \sigma + r$ if, and only if (52) is controllable on $[\sigma, t]$, $t \geq \sigma + r$.

Proof:

We note that $A(t, \sigma)$ is a subspace of $W_2^{(1)}$. Assume (52) is controllable on $[\sigma, t]$. Then $A(t, \sigma) = W_2^{(1)}$, $t \geq \sigma + r$. So that $0 \in \text{Int } A(t, \sigma)$ for each $t \geq \sigma + r$.

Conversely, assume that (52) is complete on $[\sigma, t]$, $t \geq \sigma + r$. Then $0 \in \text{Int } A(t, \sigma)$. Since $A(t, \sigma)$ is a subspace this implies that:

$$A(t, \sigma) = W_2^{(1)}, t \geq \sigma + r \tag{57}$$

This completes the proof.

Theorem 3.5

The system (25) is complete if, and only if, the system (52) is complete.

Proof:

Assume the system (52) is complete. This is equivalent to $0 \in \text{Int } A(t, \sigma)$, $t \geq \sigma + r$. Since the function space reachable set of (25) is the same as the function space reachable set of (52), it follows that the system (25) is complete. This observation completes the proof.

Corollary 3.6

Let $B^+(t)$ denote the Moore-Penrose generalized inverse, as in Luenberger [7], of $B(t)$, $t \in E$. Suppose that the map $t \rightarrow B^+(t)$, $t \in E$ is essentially bounded on each interval $[t - r, t]$ and $\text{rank } B(t) = n$ on each interval $[t - r, t]$. Then (52) is complete on each interval $[\sigma, t]$, $t \geq \sigma + r$.

Proof:

By theorem 3.1 of Luenberger [7], (52) is controllable on each interval $[\sigma, t]$, $t \geq \sigma + r$ if, and only if $\text{rank } B = n$ on each interval $[t - r, t]$. By theorem 3.4 the system (52) is complete on $[\sigma, t]$, $t \geq \sigma + r$ if, and only if (52) is controllable on $[\sigma, t]$, $t \geq \sigma + r$. This completes the proof.

Remark 3.7

As a consequence of Corollary 3.6, (52) is complete if, and only if $\text{rank } B = n$, when B is a constant matrix

Theorem 3.8: Suppose

- (i) the solution $x(\sigma, \varphi)$ of (23) is globally exponentially asymptotically stable;
- (ii) the system (25) is complete;

then the system (25) is function space null-controllable with constraints.

Proof:

The proof is skipped as it is similar to that of theorem 3.2.

Example 3.8

We consider the homogeneous nonlinear differential difference equation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[x(t-r)]^2 := g(t, x_t) \\ x_\sigma &= \varphi \in W_2^{(1)} \end{aligned} \tag{58}$$

has a nontrivial solution if, and only if

$$\lambda - A - Be^{-\lambda r} = 0. \tag{59}$$

However, we are interested in the equation defined by the Frechet derivative given by

$$\dot{z}(t) = g'(t, x_t)z_t \tag{60}$$

And the controlled equation

$$\dot{z}(t) = g'(t, x_t)z_t + Cu(t) \tag{61}$$

We assume that there are constants $k > 0, \alpha > 0$ such that for all $\sigma \in E$

$$|T(t, \sigma)| \leq ke^{-\alpha(t-\sigma)}, t \geq \sigma \tag{62}$$

We apply Theorem 3.4 and Corollary 3.5 to conclude that (61) is complete. Since the solution operator of (60) apply satisfies relation (62), we apply Theorem 3.4 to show that (61) is Euclidean space null-controllable.

IV. Game of Pursuit Results:

We investigate a game whose state space is the Sobolev space, $W_2^{(1)}$, with dynamical equation:

$$\dot{x}(t) = g(t, x_t) - p(t) + q(t), t \in [\sigma, \infty) \tag{63}$$

where $p \in L_2([\sigma, t], P), P \subseteq E^m, q \in L_2([\sigma, \infty), Q), Q \subseteq E^m$. With the above assumptions, given $\sigma, t \in E$ and $\varphi \in W_2^{(1)}$ there exist an absolutely continuous function $x(\cdot, \sigma, \varphi, p, q): [\sigma - r, t_1] \rightarrow E^n$ which satisfies (63) a.e. on $[\sigma, t_1]$ with initial condition $x_\sigma = \varphi$. The solution to (63) is given by:

$$x_t(\sigma, \varphi, p, q) = x_t(\sigma, \varphi) - \int_\sigma^t T(t, s)Y_0[p(s) - q(s)]ds \tag{64}$$

where $t \geq \sigma, \varphi \in W_2^{(1)}$ and $x(\sigma, \varphi)$ is the solution of (23).

Definition 4.1

There is complete Euclidean space capture everywhere at time t_1 for game (63) if for each $\varphi \in W_2^{(1)}, x_1 \in E^n$ and for quarry control $q \in L_2$ there exist pursuer strategy $p \in L_2$ subject to the following

- i) foreach $t \in [\sigma, t_1], p(t)$ depends on $q(t), t_1, \varphi$ and $g(t, x_t)$;
- ii) the pair of controls so obtained is such that the solution of (63) satisfies $x(t_1) = x_1$ and $x_\sigma = \varphi$.

There is Euclidean space capture in zero at time t_1 or game (63) if the assumptions in Definition 4.1 are met with $x_1 = 0$. Associated with our game is a nonlinear control equation:

$$\dot{x}(t) = g(t, x_t) - u(t) \tag{65}$$

where $u \in L_2([-r, 0], U)$ and the control set is the Pontryagin difference of sets as given in Hajek [8],

$$U = P \underset{*}{-} Q = \{x: x + Q \subseteq P\}. \tag{66}$$

Theorem 4.2

Assume $0 \in Q$ and P compact. There is complete Euclidean space capture everywhere at time t_1 for game (63) if, and only if the associated nonlinear control system

$$\begin{aligned} \dot{x}(t) &= g(t, x_t) - u(t) \\ u(t) &\in U = (P + \ker T(t, s)) \underset{*}{-} Q \end{aligned} \tag{67}$$

is Euclidean space controllable at time t_1 . Furthermore, $\alpha(q, t) = u(t) + q$ modulo $\ker T(t, s)$ for all $q \in Q$ and $t \in [\sigma, t_1]$ can be used to determine a suitable strategy $u \in L_2([\sigma, t_1], U)$ in (67) and vice versa.

Remark 4.3

In (67) φ is the initial function that is forced to the target by α and q .

Proof of Theorem 4.2

Assume there is complete Euclidean space capture everywhere at time t_1 for game (63). There is a mapping $\alpha: Q \times [\sigma, t_1] \rightarrow P$ such that for any quarry control $q \in L_2$ the map $p(t) = \alpha(q(t), t)$ is a pursuer strategy; p and q steer arbitrary $\varphi \in W_2^{(1)}$ to any $x_1 \in E^n$ in time t_1 .

$$x(t_1, \sigma, \varphi, p, q) = x_1 = x(t_1, \sigma, \varphi) - \int_{\sigma}^{t_1} T(t_1, s) Y_0 [p(s) - q(s)] ds \tag{68}$$

or

$$x(t_1, \sigma, \varphi) = x_1 + \int_{\sigma}^{t_1} T(t_1, s) Y_0 [p(s) - q(s)] ds. \tag{69}$$

For quarry control $0 \in Q$,

$$x(t_1, \sigma, \varphi) = x_1 + \int_{\sigma}^{t_1} T(t_1, s) Y_0 u(s) ds \tag{70}$$

where $u(s) = \alpha(0, s)$.

Now take any point $q \in Q$ and time $t \in [\sigma, t_1]$, consider the piecewise constant quarry control such that it is zero on the interval $[\sigma, t]$ and q on the interval $[t, t_1]$. We apply these values in equation (69) to obtain

$$x(t_1, \sigma, \varphi) = x_1 + \int_{\sigma}^t T(t_1, s) Y_0 u(s) ds + \int_t^{t_1} T(t_1, s) Y_0 [\alpha(q, s) - q] ds. \tag{71}$$

Subtract (71) from (70) to obtain

$$\int_t^{t_1} T(t_1, s) Y_0 [u(s) + q - \alpha(q, s)] ds = 0 \tag{72}$$

$$T(t_1, s) Y_0 [u(s) + q - \alpha(q, s)] = 0 \tag{73}$$

for almost all $s \in [\sigma, t_1]$. We use the kernel to interpret this as

$$u(s) + q \in [\alpha(q, s) + \ker T(t_1, s) Y_0] \tag{74}$$

Because α has values in P , $u(s) + q \in [P + \ker T(t_1, s) Y_0]$. Since the right side of (74) is closed, we use Hajek's lemma [8, p.59] to obtain

$$u(s) + Q \subseteq [P + \ker T(t_1, s) Y_0] \text{ a.e.} \tag{75}$$

or

$$u(s) \in [P + T(t_1, s)Y_0] * Q \equiv U(s) \text{ a. e.} \tag{76}$$

So that $u \in L_2([\sigma, t_1], U)$. Hence u is an admissible control for (67) and (70) to yield

$$x(t_1, \sigma, \varphi) = x_1 + \int_{\sigma}^{t_1} T(t_1, s)Y_0 u(s) ds \tag{77}$$

or

$$x_1 = x(t_1, \sigma, \varphi) - \int_{\sigma}^{t_1} T(t_1, s)Y_0 u(s) ds. \tag{78}$$

Note that $x_{\sigma} = \varphi$ and (74) proves (67).

Conversely, assume that (67) is Euclidean space controllable at time t_1 , let $\varphi \in W_2^{(1)}$ and $x_1 \in E^n$ be given. Let u be the appropriate control in L_2 such that $x_{\sigma} = \varphi, x(t_1) = x_1$. Then

$$x_1 = x(t_1, \sigma, \varphi) - \int_{\sigma}^{t_1} T(t_1, s)Y_0 u(s) ds \tag{79}$$

where $u(s) \in U$ yields $u(s) + q \in [P + \ker T(t_1, s)Y_0]$. We now apply Fillipov's lemma [8, p.110] to construct pursuer control: there exist measurable mappings

$$\alpha : Q \times [\sigma, t_1] \rightarrow P, \quad l : Q \times [\sigma, t_1] \rightarrow \ker T(t_1, s)Y_0 \tag{80}$$

such that

$$u(s) + q \equiv \alpha(q, s) + l(q, s). \tag{81}$$

Because α takes values in a compact set P , $\alpha \in L_2([\sigma, t_1], P)$. We now show that for any quarry control q , α is indeed appropriate.

For any $q \in L_2([\sigma, t_1], Q)$, $\alpha - q = u - l$, so that the solution at t_1 of (63) with this pair of α and q with initial function φ is:

$$x(t_1, \sigma, \varphi, \alpha, q) = x(t_1, \sigma, \varphi) - \int_{\sigma}^{t_1} T(t_1, s)Y_0 [\alpha(q, s) - q] ds \tag{82}$$

$$= x(t_1, \sigma, \varphi) - \int_{\sigma}^{t_1} T(t_1, s)Y_0 [u(s) - l(q, s)] ds \tag{83}$$

$$= x(t_1, \sigma, \varphi) - \int_{\sigma}^{t_1} T(t_1, s)Y_0 \alpha(q, s) ds - \int_{\sigma}^{t_1} T(t_1, s)Y_0 u(s) ds \tag{84}$$

$$= x_1 + 0. \tag{85}$$

This completes the proof.

Theorem 4.2

In game (63) assume $0 \in Q, P$ compact. Then there is complete function space capture everywhere at time t_1 for (67) if, and only if, the associated nonlinear control equation:

$$\dot{x}(t) = g(t, x_t) - u(t), \quad t \geq \sigma \tag{86}$$

$$x_{\sigma} = \varphi \in W_2^{(1)}([-r, 0], E^n), \quad u(t) \in U \tag{87}$$

$$U = [P + \ker T(t_1, s)Y_0] * Q \tag{88}$$

is controllable at time t_1 . Furthermore, $\alpha(q, s) = u(s) + q$ modulo $\ker T(t_1, s)Y_0$, for all $q \in Q, s \in [\sigma, t_1]$ can be used to determine a suitable control strategy from admissible control $u \in L_2([\sigma, t_1], U)$ for (67) and vice versa.

The proof is quite similar to that of theorem 4.1.

V. Conclusion

We notice that we have employed nonlinear variation of constants formula as developed by Shanholt to arrive at similar results such as that developed by Chukwu, et al for Euclidean and function space controllability. We also establish similar results for nonlinear differential games of pursuit.

References

- [1]. Hermes,H. and Lasalle,J.P., Functional Analysis and Time Optimal Control Academic Press, 1969.
- [2]. Chukwu, E. N., On null-controllability of nonlinear delay systems with restrained controls, J. Math. Anal. Appl. Vol.76, no. 1, July 1980
- [3]. -----, Capture in linear functional differential games of pursuit, J. Math. Anal. Appl., Vol. 70, No. 2, 1979.
- [4]. Shanholt, G. A., A nonlinear variation of constants formula for functional differential equations, Math. Systems theory, Vol. 6, No. 4, 1972-1973.
- [5]. Chukwu, E. N., Null-controllability in function space of nonlinear retarded systems with limited control. Preprint.
- [6]. Hale, J., Functional Differential Equations Springer- Verlag, New York, 1971.
- [7]. Luenberger, D. G., Optimization of vector space method John Wiley, New York, 1969.
- [8]. Hajek,O., Pursuit Games Academic Press, New York, 1975.