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Existence of unique solution for Fractional Differential Equation by Picard approximation method

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Abstract: *Our work is finding continuous function y on (a, ∞) which is the unique solution for the* $y^{(\alpha)}(x)=\lambda f(y(x)), x\in(a, \infty)$, $0<\alpha\le 1$ *with* $y^{(\alpha-1)}(a) = \mu$, *where* μ *is constant and λ is a real number using the picard approximation method theorem.*
Keywords: *Fractional differential equation, picard approximation method.*

I. Introduction

Fractional calculus as well as fractional Differential equations have received increasing attention and has been a significant development in ordinary and partial fractional differential equations in recent years; seethe papers by Abbas and Benchohra [1,2,3], Agaiwal et al. [4], monographs of Kilps, Lakshmikatham et al. [6]. This article studies the existence of the unique solution of fractional differential equation $y^{(\alpha)}(x)=\lambda f(y(x))$ and $y^{(\alpha-1)}(a)=\mu$, μ is some constant, $0 <\alpha \le 1$, λ is real number using picard approximation method.

II. Preliminaries

In this section we introduce notations, definitions and preliminary facts which are used throughout this paper.

Definition (2-1)([7]): Let x={F:F is a real valued function and continuous on [a,∞)}, for some a∈(-∞, ∞) . Let the ||·|| on xbe defined by

$$\|F\| = \sup_{x\in[a,\infty)}\left\{e^{-\gamma|\alpha|}|F(x)|\right\}$$

provided that this norm exist for some constant γ>0.

Lemma (2-1)([8]): Let $0<\alpha\le 1$ and f,g be continuous functions on (a, ∞), where a∈R such that $\sup\{| f(g(x)) | : x \in (a, \infty) \} = M <\infty$. Define $f_\alpha(x) = \frac{\mu(x-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\cdot f(g(t))dt,$

for all x>a andμ is some constant. Then $f_\alpha \in c(a,\infty)$.

Lemma (2-2)([9]): Let us define $F_\alpha(x) = (x-a)^{\alpha-1}f_\alpha(x)$ on (a,∞), where f_α define in lemma (2-1) and $0<\alpha\le 1$.Then $F_\alpha\in c[a,\infty)$.

Lemma (2-3) ([7]): Let α, γ∈ R, γ> -1. If x>a then

$$\mathop{\mathrm{I}}_a^x{}^{\alpha}\frac{(t-a)^\gamma}{\Gamma(\gamma+1)} = \begin{cases} \frac{(\alpha-a)^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}, & \alpha+\gamma \ne \text{negative integer} \\ 0 \quad, & \alpha+\gamma = \text{negative integer}\end{cases}$$

Lemma (2-4) ([9]):suppose G is a banach space and let T∈L(G) such that $\|T^n\|^{\frac{1}{n}} < 1$. Then I-T is regular and $(I-T)^{-1} = I + \sum_{n=1}^{\infty} T^n$, where the series $\sum_n T^n$ converge in L(G).

Definition (2-2) ([9]): Let f be Lebesque- measurable function defined a.e on [a, b], if α>0 then we define

$$\mathop{\mathrm{I}}_a^b{}^{\alpha} f = \frac{1}{\Gamma(\alpha)}\int_a^b f(t)\,(b-t)^{\alpha-1}dt$$

provided the integral (Lebesque) exists.

Definition (2-3) ([10]): If α∈R, f is define a.e on the interval [a, b], we define

$$\frac{d^{\alpha} f}{dx^{\alpha}} = f^{\alpha}(x) = \underset{a}{\overset{x}{\mathrm{I}}}{}^{-\alpha} f \quad \text{for all} \quad x \in [a, b]$$

provided that $\underset{a}{\overset{x}{\mathrm{I}}}{}^{-\alpha} f$ exists.

Lemma (2-5) ([8]): If $0 < \alpha \leq 1$ and f(x) is continuous on [a,b], $|f(x)| \leq M$ for all $x \in [a,b]$ (where $M \in R^{+}$, M>0). Then

$$\underset{a}{\overset{x}{\mathrm{I}}}{}^{\alpha} \underset{a}{\overset{x}{\mathrm{I}}}{}^{-\alpha} \mathrm{f}(x) = \mathrm{f}(x) \quad \text{for all} \quad x \in (a, b).$$

Theorem (2-1)([9]): Let $0<\alpha\leq 1$ and γ be positive constant. Let $g(x)=(g_1(x), g_2(x), \ldots, g_n(x))^T$, $x\in[a,\infty)$, whereg_i are continuous on $[a,\infty)$, i= 1, 2, …, n and $|g(x)| = \left(\sum_{i=1}^{n} g_i^2\right)^{\frac{1}{2}}$ and $|g(x)|\leq x+c$, where c is positive constant. $f_i=(f_1, f_2, \ldots, f_n)^T$ such that $f_i\in[a,\infty)$ and$\sup\{|f(x)|: x\in[a, \infty)\}=M<\infty$. Choose λ such that $\lambda < \left(e^{\alpha}\left(\frac{c}{\alpha}\right)^{\alpha}\right)^{-1}$. Then there exists continuous vector function $y(x)=(y_1(x), y_2(x), \ldots, y_n(x))^T$, $x\in(a,\infty)$ such that $y^{(\alpha)}(x)=\lambda f(y(x))$, $x\in(a,\infty)$ with $y^{(\alpha-1)}(a)=\mu$, where $\mu=(\mu_1, \mu_2, \ldots, \mu_n)^T$ is some constant vector and satisfied $|y(x)|<\exp(\alpha c^{-1}|x|)$.constant.

III. Main Results

In this section we prove the existence of a continuous function y on (a, ∞) which is the unique solution for $y^{(\alpha)}(x)=\lambda f(y(x))$, and $y^{(\alpha-1)}(a)=\mu$, where μ is some constant,$0 <\alpha \leq 1$, λ is real number using picard approximation method.

Theorem (3): Let $0<\alpha\leq 1$, g(x) is continuous function on $[a, \infty)$ and $|g(x)|\leq|x|$ ...(3.1) where $x\in[a,\infty)$. Let f(y(x)) be a continuous function on $[a, \infty)$ such that $\sup\{|f(y(x))|:x\in[a,\infty)\}=M<\infty$. Then there exist a continuous function y on (a, ∞) which is the unique solution for

$y^{(\alpha)}(x)=\lambda f(y(x))$ $\quad x\in(a,\infty)$ with

$y^{(\alpha-1)}(a)=\mu$, where μ is some constantand λ is real number.

<u>Proof:</u>Let[a, a+h]be any compact subinterval of $[a,\infty)$ and let $(X, \|\cdot\|)$ be the space defined in definition (2-1). Consider

$$y(x) = y_0(x) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(y(t))dt, x \in (a,\infty) \quad ...(3.2)$$

where $y_0 = \dfrac{\mu(x-a)^{\alpha-1}}{\Gamma(\alpha)}$, it follows from lemma (2-1) that $y\in(a,\infty)$.Then

$$(x-a)^{1-\alpha} y(x) = b + \frac{(x-a)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(y(t))dt, \quad x\in(a, \infty). \text{ Where } b = \frac{\mu}{\Gamma(\alpha)}$$

Let $F(x)=(x-a)^{1-\alpha}y(x)$, $x\in(a,\infty)$ ...(3.3)
Where y is given in (3.2) and define
$F(x, y(t))=(x-a)^{1-\alpha}f(y(x))$, $a\leq t<x<\infty$ ...(3.4)
Thus from Lemma (2-2) we have $F\in c[a, \infty)$.
Now define a linear operator K on [a, a+h] as:

$$(KF)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} F(x, y(t))dt, \quad x\in[a, a+h] \quad ...(3.5),$$

and consider the equation

$$F(x) = b + \frac{\lambda}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}F(x,y(t))d \quad , x\in[a, a+h] \qquad ...(3.6)$$

where $b = \frac{\mu}{\Gamma(\alpha)}$ and μ is some constant.

Now we prove

$\lim_{x\to\infty}\|K^n\|^{\frac{1}{n}} = 0$, from (3.3) we have

$$|(KF)(x)| = \left|\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}F(x,y(t))\,dt\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left|F(x,y(t))\right|\,dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,e^{\gamma|y(t)|}dt$$

Then from (3.1) we have

$$|(KF)(x)| \le \frac{\|F\|}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,e^{\gamma|t|}dt$$

$$\le \frac{\|F\|\,e^{\gamma|x|}}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,dt$$

$$= \frac{\|F\|\,e^{\gamma|x|}}{\Gamma(\alpha)}\left[\left\{\frac{-(x-t)^{\alpha}}{\alpha}\right\}_a^x\right]$$

$$= \frac{\|F\|\,e^{\gamma|x|}}{\Gamma(\alpha)}\frac{(x-a)^{\alpha}}{\alpha}$$

$$= \frac{\|F\|e^{\gamma|x|}(x-a)^{\alpha}}{\Gamma(\alpha+1)} \ldots (3.7)$$

Now by induction we prove the following inequality

$$|(K^nF)(x)| \le \frac{\|F\|e^{\gamma|x|}(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$\le \frac{\|F\|e^{\gamma|x|}h^{n\alpha}}{\Gamma(n\alpha+1)}\ , x\in[a, a+h] \text{ and } n = 1, 2, \ldots \qquad \ldots (3,8)$$

by using (3.7), it is obvious that (3.8) holds for n=1.

Next suppose that (3.8) is true for positive integer n, then we have

$$|(K^{n+1}F)(x)| = |K(K^nF)(x)|$$

$$= \left|\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,K^nF(x,y(t))dt\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,\left|K^nF(x,y(t))\right|dt$$

It follows from (3.8) that

$$|(K^{n+1}F)(x)| \le \frac{1}{\Gamma(\alpha)}\int_a^x \frac{(x-t)^{\alpha-1}\|F\|(y(t)-a)^{n\alpha}e^{\gamma|y(t)|}}{\Gamma(n\alpha+1)}\ dt$$

$$\le \frac{\|F\|}{\Gamma(\alpha)}\int_a^x \frac{(x-t)^{\alpha-1}(t-a)^{n\alpha}e^{\gamma|t|}}{\Gamma(n\alpha+1)}\ dt$$

$$\le \frac{\|F\|\,e^{\gamma|x|}}{\Gamma(\alpha)}\int_a^x \frac{(x-t)^{\alpha-1}(t-a)^{n\alpha}}{\Gamma(n\alpha+1)}\ dt$$

$= \|F\|\, e^{\gamma|x|} \overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha} \dfrac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)}$

Then by lemma (2-3) we have

$$|(K^{n+1}F)(x)| \le \|F\|\, e^{\gamma|x|} \frac{(x-a)^{n\alpha+\alpha}}{\Gamma(n\alpha+\alpha+1)}, \qquad x \in [a, a+h]$$

$= \dfrac{\|F\|\, e^{\gamma|x|}(x-a)^{\alpha(n+1)}}{\Gamma(\alpha(n+1)+1)}$

$\le \dfrac{\|F\|\, e^{\gamma|x|} h^{\alpha(n+1)}}{\Gamma(\alpha(n+1)+1)}$

Thus (3.8) hold for all n=1, 2, 3, ...

Hence

$$e^{-\gamma|x|}|(K^n F)(x)| \le \frac{\|F\|\, h^{n\alpha}}{\Gamma(n\alpha+1)}, x \in [a, a+h]$$

And so by definition (2-1) we get

$\|K^n F\| \le \dfrac{\|F\|\, h^{n\alpha}}{\Gamma(n\alpha+1)}$

and it follows from definition (2-2) that

$\|K^n\| \le \dfrac{h^{n\alpha}}{\Gamma(n\alpha+1)}$

Now

$\lim\limits_{n\to\infty} \|K^n\|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left(\dfrac{h^{n\alpha}}{\Gamma(n\alpha+1)}\right)^{\frac{1}{n}}$

$= h^{\alpha} \lim\limits_{n\to\infty} \left(\dfrac{1}{n\alpha\ \Gamma(n\alpha)}\right)^{\frac{1}{n}}$

$= h^{\alpha} \lim\limits_{n\to\infty} \left(\dfrac{1}{(n\alpha)^{\frac{1}{n}}[\Gamma(n\alpha)]^{\frac{1}{n}}}\right)$

Since n= 1, 2, 3, ... then

$\lim\limits_{n\to\infty} (n\alpha)^{\frac{1}{n}} = \lim\limits_{n\to\infty} (n)^{\frac{1}{n}}\ (\alpha)^{\frac{1}{n}} \ge 1$

And also we have

$$\Gamma(n\alpha) = \sqrt{2\pi}(n\alpha)^{n\alpha-\frac{1}{2}} e^{(-n\alpha+\theta)/12n\alpha}, 0 < \theta < 1, n\epsilon\, Z^{+}$$

seeArtine (1964) and so

$[\Gamma(n\alpha)]^{\frac{1}{n}} = (2\pi)^{\frac{1}{n}}(n\alpha)^{\alpha} \quad \dfrac{1}{(n\alpha)^{\frac{1}{2n}}} e^{-\alpha} e^{\theta/12n^2\alpha}$

And hence

$\lim\limits_{n\to\infty} [\Gamma(n\alpha)]^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left[(2\pi)^{\frac{1}{n}}(n\alpha)^{\alpha} \quad \dfrac{1}{(n\alpha)^{\frac{1}{2n}}} e^{-\alpha} e^{\theta/12n^2\alpha}\right]$

$= 1 . \infty . 1 . e^{-\infty} . 1 = \infty$

Consequently we have

$\lim_{n\to\infty} \|K^n\|^{\frac{1}{n}} = 0$ and this implies that

$\lim\limits_{n\to\infty} \|(\lambda K)^n\|^{\frac{1}{n}} = |\lambda| \lim\limits_{n\to\infty} \|K^n\|^{\frac{1}{n}} = 0$

Then by lemma (2-4), $(I - \lambda K)^{-1} = I + \sum_n \lambda^n K^n$

and then series is convergent.

From (3.5) and (3.6) we have

$F(x) = (I - \lambda K)^{-1}(b)$, therefore F is exists and is the unique solution of

$$F(x) = b + \frac{\lambda}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} F\big(x, y(t)\big)dt.$$

Then from (3.3) we get

$$F(x) = \frac{\mu}{\Gamma(\alpha)} + \frac{\lambda(x-a)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f\big(\,y(t)\big)dt$$

for allx∈(a, a+h] and by using (3.3) it follows that

$$(x-a)^{1-\alpha}y(x)=\frac{\mu}{\Gamma(\alpha)}+\frac{\lambda(x-a)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(y(t))dt$$

for all $x\in(a, a+h]$

$$y(x)=\frac{\mu(x-a)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\lambda}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(y(t))dt$$

Therefore by definition (2-2) we get

$$y(x)=\frac{\mu(x-a)^{\alpha-1}}{\Gamma(\alpha)}+\lambda \mathop{I}_a^x{}^{\alpha} f \quad , \quad x\in(a,a+h] \;...(3.9)$$

andso

$$\mathop{I}_a^x{}^{-\alpha}\, y=\mathop{I}_a^x{}^{-\alpha}\frac{\mu(t-a)^{\alpha-1}}{\Gamma(\alpha)}+\lambda \mathop{I}_a^x{}^{-\alpha}\mathop{I}_a^t{}^{\alpha} f$$

But from lemma (2-3) we have $\mathop{I}_a^x{}^{-\alpha}\dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}=0$ and by lemma(2-5) we get

$\mathop{I}_a^x{}^{-\alpha}\mathop{I}_a^t{}^{\alpha} f = f(y(x))$ for all $x\in(a, a+h]$

Thus $\mathop{I}_a^x{}^{-\alpha} y=\lambda\, f(y(x))$, $x\in(a, a+h]$

Then by using definition (2-5) we get

$y^{(\alpha)}(x)=\mathop{I}_a^x{}^{-\alpha} y=\lambda\, f(y(x))$ $x\in(a, a+h]$

Furthermore from (3.9) we have

$$\mathop{I}_a^x{}^{1-\alpha} y=\mathop{I}_a^x{}^{1-\alpha}\frac{\mu(t-a)^{\alpha-1}}{\Gamma(\alpha)}+\lambda \mathop{I}_a^x{}^{1-\alpha}\mathop{I}_a^t{}^{\alpha} f$$

It follows from lemma (2-3) that

$$\mathop{I}_a^x{}^{1-\alpha} y=\mu+\lambda \mathop{I}_a^x{}^{1-\alpha}\mathop{I}_a^t{}^{\alpha} f=\mu+\lambda \mathop{I}_a^x{}^{1} f$$

$$=\mu+\lambda\int_a^x f(y(t))dt$$

and so $\mathop{I}_a^x{}^{1-\alpha} y$ exists for all $x\in[a, a+h]$

since by definition (2-3)

$y^{(\alpha-1)}(x)=\mathop{I}_a^x{}^{1-\alpha} y$ therefore

$y^{(\alpha-1)}(a)=\mu$.

Now from theorem (2-1) equation (3.11) we have

$|F(x)| \le b+|\lambda||kF(x)|$

then from theorem (2-1) equation (3.8) we have

$|F(x)| \le b+|\lambda|\|F\|\dfrac{e^{\gamma c}}{\gamma^c}e^{\gamma|x|}$

$< b+\|F\|e^{\gamma|x|}$ since $\gamma c=\alpha$

$= e^{\gamma|x|}(be^{-\gamma|x|}+\|F\|)$

$< e^{\gamma|x|}(b+\|F\|)$

thus by using theorem (2-1) equation (3.3) we obtain

$|(x-a)^{1-\alpha}y(x)| < e^{\gamma|x|}[b+\|F\|]$
$h^{1-\alpha}|y(x)| < e^{\gamma|x|}[b+\|F\|]$
$|y(x)| < e^{\gamma|x|}h^{\alpha-1}[b+\|F\|]$
and so the solution function satisfied
$|y(x)|<\exp(\alpha c^{-1}|x|).\text{constant}$

References:

[1]. S. Abbas and M. Benchohra; Partial Hyperbolic Differential Equations with FinitedelyInvolving the CaputoFractional Derivative, commun.Math.Anal.7 (2009).
[2]. S. Abbas and M. Benchohra; DarbouxProblem for Perturbed Partial Differential Equations of Fractional Order with Finite Delay, Nonlinear Anal.Hybrid Syst. 3 (2009).
[3]. S. Abbas and M. Benchohra; Upper and Lower Solutions Method for Impulsive Hyperbolic Differential Equations with Fractional Order,Non Linear Anal. Hybrid Syst. 4 (2010).
[4]. R. P Agrawal, M. Benchohra and S. Hamani; A Survey Existence Result for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions, Acta. Appl. Math. 109 (3) (2010).
[5]. A. A. Kilbas,Hari. M. Srivastava and Trujillo;Theory and Applications of Fractional Differential Equations, Elsevier Science B. V., Amsterdam, (2006).
[6]. V. Laskmikantham, S. Leela and J. Vasundhara; Theory of Fractional Dynamic Systems, Combridge Academic Publishers, Cambridge (2009), 14/ page.
[7]. Al-Tabakchaly. M. A., On the Existence and Uniqueness for Fractional Differential Equations, Msc. Thesis, University of Musol.
[8]. Dr. Ahmed Zain al-Abdin, MahaAbd Al-Wahab, Theorem on Certain Fractional Functions and Derivatives, Eng and Tech. Journal.Vol. 30, No. 2, (2012).
[9]. MahaAb Al-Wahab, On the Existence and Uniqueness for Solution of System Fractional Differential Equations, IOSR Journal of Math.(IOSR-JM), (2012).
[10]. MahaAb Al-Wahab, On Banach Fixed Point Theorem Solving a Special Kind of Fractional Differential Equation, IOSR Journal of Math. (IOSR-JM), (2014).
[11]. Artine.G; The Gamma Function, HOLT_ Rinchart and wiston (1964).