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Hardy-Steklov operator on two exponent Lorentz spaces for non-decreasing functions

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Abstract: *In this paper, we obtain the characterization on pair of weights* v *and* w *so that the Hardy-Steklov operator* $\int_{a(x)}^{b(x)} f(t)\,dt$ *is bounded from* $L_v^{p,q}(0,\infty)$ *to* $L_w^{r,s}(0,\infty)$ *for* $0 < p,q,r,s < \infty$.

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I. Introduction

By a weight function u defined on $(0,\infty)$ we mean a non-negative locally integrable measurable function. We take $\mathrm{M}_0^+ \equiv \mathrm{M}_0^+((0,\infty), u(x)dx)$ to be the set of functions which are measurable, non-negative and finite a.e. on $(0,\infty)$ with respect to the measure $u(x)dx$. Then the distribution function λ_f^u of $f \in \mathrm{M}_0^+$ is given by

$$\lambda_f^u(t) := \int_{\{x\in(0,\infty)\,:\, f(x)>t\}} u(x)dx, \quad t \ge 0.$$

The non-increasing rearrangement f_u^* of f with respect to $du(x)$ is defined as

$$f_u^*(y) := \inf\{t : \lambda_f^u(t) \le y\}, \quad y \ge 0.$$

For $0 < p < \infty,\ 0 < q \le \infty$, the two exponent Lorentz spaces $L_v^{p,q}(0,\infty)$ consist of $f \in \mathrm{M}_0^+$ for which

$$\|f\|_{L_v^{p,q}} := \begin{cases} \left(\int_0^\infty \frac{q}{p}[t^{1/p} f_v^*(t)]^q \frac{dt}{t}\right)^{1/q}, & 0<q<\infty \\ \sup_{t>0} t^{1/p} f_v^*(t), & q=\infty \end{cases} \tag{1}$$

is finite.

In this paper, we characterize the weights v and w for which a constant $C > 0$ exists such that

$$\|Tf\|_{L_w^{r,s}} \le C\|f\|_{L_v^{p,q}}, \quad f \ge 0 \tag{2}$$

where T is the Hardy-Steklov operator defined as

$$(Tf)(x) = \int_{a(x)}^{b(x)} f(t)dt. \tag{3}$$

The functions $a = a(x)$ and $b = b(x)$ in (3) are strictly increasing and differentiable on $(0,\infty)$. Also, they satisfy

$$a(0) = b(0) = 0; \quad a(\infty) = b(\infty) = \infty \quad \text{and} \quad a(x) < b(x) \text{ for } 0 < x < \infty.$$

Clearly, a^{-1} and b^{-1} exist, and are strictly increasing and differentiable. The constant C attains different bounds for different appearances.

II. Lemmas

Lemma 1. *We have*

$$\|f\|_{L_v^{r,s}} = \begin{cases} \left(\int_0^\infty st^{s-1}[\lambda_f^v(t)]^{s/r}\, dt\right)^{1/s}, & 0 < s < \infty \\ \sup_{t>0} t\,[\lambda_f^v(t)]^{1/r}, & s = \infty. \end{cases} \tag{4}$$

Proof. Applying the change of variable $y = \lambda_f^v(t)$ to the R.H.S. of (1) and integrating by parts we get the lemma. □

Lemma 2. *If f is nonnegative and non-decreasing, then*

$$\|f\|_{L_v^{r,s}}^s = \frac{s}{r}\int_0^\infty f^s(x)\left(\int_x^\infty v(t)dt\right)^{\frac{s}{r}-1} v(x)dx. \tag{5}$$

Proof. We obtain the above equality by evaluating the two iterated integrals of $st^{s-1}\left(\frac{s}{r}\right)h^{\frac{s}{r}-1}(x)v(x)$ over the set $\{(x,t)\,;0 < t < f(x), 0 < x\}$, so that we have

$$\int_0^\infty\int_0^{f(x)} st^{s-1}\left(\frac{s}{r}\right)h^{\frac{s}{r}-1}(x)v(x)\, dt\, dx = \int_0^\infty\int_{x(t)}^\infty st^{s-1}\left(\frac{s}{r}\right)h^{\frac{s}{r}-1}(x)v(x)\, dx\, dt, \tag{6}$$

where $x(t) = \sup\{x : f(x) \le t\}$ for a fixed t, and $h(x) = \int_x^\infty v(t)dt.$

Integrating with respect to 't' first, the L.H.S. of (6) gives us the R.H.S. of (5). Further

$$\frac{s}{r}\int_{x(t)}^\infty h^{\frac{s}{r}-1}(x)v(x)\, dx = h(x(t))^{\frac{s}{r}} = \left(\int_{x(t)}^\infty v(s)\, ds\right)^{s/r} = [v\{x : f(x) > t\}]^{\frac{s}{r}} = [\lambda_f^v(t)]^{\frac{s}{r}}.$$

Hence the lemma now follows in view of Lemma 1. □

III. Main Results

Theorem 1. *Let $0 < p, q, r, s < \infty$ be such that $1 < q \le s < \infty$. Let T be the Hardy-Steklov operator given in (3) with functions a and b satisfying the conditions given thereat. Also, we assume that $a'(x) < b'(x)$ for $x \in (0,\infty)$. Then the inequality*

$$\left(\int_0^\infty \frac{s}{r}[Tf(x)]_w^{*s}\, x^{s/r}\, \frac{dx}{x}\right)^{1/s} \le C\left(\int_0^\infty \frac{q}{p}[f_v^*(x)]^q\, x^{q/p}\, \frac{dx}{x}\right)^{1/q} \tag{7}$$

holds for all nonnegative non-decreasing functions f *if and only if*

$$\sup_{\substack{0<t<x<\infty \\ a(x)<b(t)}} \left(\frac{s}{r} \int_t^x \left(\int_y^\infty w(z)dz \right)^{\frac{s}{r}-1} w(y)dy \right)^{1/s} \times \left(\int_{a(x)}^{b(t)} \left[\left(\int_y^\infty v(z)dz \right)^{\frac{q}{p}-1} v(y) \right]^{1-q'} dy \right)^{1/q'} < \infty. \quad (8)$$

Proof. Using differentiation under the integral sign, the condition $a'(x) < b'(x)$ for $x \in (0,\infty)$ ensures that Tf is nonnegative and non-decreasing. Consequently, by Lemma 2, the inequality (7) is equivalent to

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f(t)dt \right)^s W(x)dx \right)^{1/s} \le C \left(\int_0^\infty f^q(x)V(x)dx \right)^{1/q} \quad (9)$$

where $W(x) = \frac{s}{r} \left(\int_x^\infty w(z)dz \right)^{\frac{s}{r}-1} w(x)$ and $V(x) = \frac{q}{p} \left(\int_x^\infty v(z)dz \right)^{\frac{q}{p}-1} v(x).$

Thus it suffices to show that (9) holds if and only if (8) holds. The result now follows in view of Theorem 3.11 [2]. □

Similarly, in view of Theorem 2.5 [1], by making simple calculations, we may obtain the following:

Theorem 2. *Let* $0 < p,q,r,s < \infty$ *be such that* $0 < s < q, 1 < q < \infty$. *Let* T *be the Hardy-Steklov operator given in (3) with functions a and b satisfying the conditions given thereat. Also, we assume that* $a'(x) < b'(x)$ *for* $x \in (0,\infty)$. *Then the inequality (7) holds for all nonnegative non-decreasing functions* f *if and only if*

$$\left(\int_0^\infty \int_{b^{-1}(a(t))}^t [a^{s/r}(t) - b^{s/r}(x)]^{l/p'} [x^{q/p} - t^{q/p}]^{l/p} \times \frac{q}{p} \left(\int_x^\infty v(y)dy \right)^{\frac{q}{p}-1} v(x)dx \, \sigma(t)dt \right)^{1/l} < \infty$$

and

$$\left(\int_0^\infty \int_t^{a^{-1}(b(t))} [a^{s/r}(x) - b^{s/r}(t)]^{l/p'} [t^{q/p} - x^{q/p}]^{l/p} \times \frac{q}{p} \left(\int_x^\infty v(y)dy \right)^{\frac{q}{p}-1} v(x)dx \, \sigma(t)dt \right)^{1/l} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}, \ \frac{1}{l} = \frac{1}{s} - \frac{1}{q},$ *and* σ *is the normalizing function as defined in* [3].

Remark. The condition $a'(x) < b'(x)$ for $x \in (0,\infty)$ cannot be relaxed since otherwise the monotonicity of Tf would be on stake. For example, consider the functions

$$a(x) = \begin{cases} \sqrt{\frac{x}{10}}, & 0 \le x < 10 \\ \sqrt{10x} - 9, & 10 \le x < 20 \\ \sqrt{\frac{x}{10}} + 9(\sqrt{2}-1), & x \ge 20 \end{cases}$$

and

$$b(x) = \begin{cases} 10\sqrt{10x} \ , & 0 \le x < 10 \\ \sqrt{\dfrac{x}{10}} + 99 \ , & 10 \le x < 20 \\ 10\sqrt{10x} + 99(\sqrt{2} - 1) \ , & x \ge 20. \end{cases}$$

Note that a and b satisfy all the aforementioned conditions, except that, we have $a'(x) > b'(x)$ for $10 \le x < 20$.

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