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On Embedding and NP-Complete Problems of Equitable Labelings

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Abstract: *The embedding and NP-complete problems in the context of various graph labeling schemes were remained in the focus of many researchers. Here we have explored embedding and NP-Complete problems for some variants of cordial labeling.*
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I. Introduction

A *graph labeling* is an assignment of integers to the vertices or edges, or both, subject to certain condition or conditions.

Labeling of discrete structures is a potential area of research and a comprehensive survey on graph labeling can be found in Gallian [1].

Definition 1.1: For a graph G, a vertex labeling function $f: V(G) \rightarrow \{0,1\}$ induces an edge labeling function $f^*: E(G) \rightarrow \{0,1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$. Then f is called *cordial labeling* of graph G if absolute difference of number of vertices with label 1 and label 0 is at most 1 and absolute difference of number of edges with label 1 and label 0 is at most 1.

A graph G is called *cordial* if it admits a cordial labeling.

The concept of cordial labeling was introduced by Cahit [2] in 1987 and in the same paper he presented several results on this newly introduced concept.

After this some labelings like prime cordial labeling, A - cordial labeling, product cordial labeling, edge product cordial labeling etc. were also introduced with minor variations in cordial theme. Such labelings are classified as equitable labelings.

Definition 1.2: For a graph G, an edge labeling function $f: E(G) \rightarrow \{0,1\}$ induces a vertex labeling function $f^*: V(G) \rightarrow \{0,1\}$ defined as $f^*(u) = \Pi\{f(uv) / uv \in E(G)\}$. Then f is called an *edge product cordial labeling* of graph G if the absolute difference of number of vertices with label 1 and label 0 is at most 1 and the absolute difference of number of edges with label 1 and label 0 is also at most 1.

A graph G is called *edge product cordial graph* if it admits an edge product cordial labeling.

In 2012, the concept of edge product cordial labeling was introduced by Vaidya and Barasara [3]. In the same paper they have investigated edge product cordial labeling for some standard graphs.

For any graph G, we introduce following notations:

1. $v_f(1)$ = the number of vertices having label 1;
2. $v_f(0)$ = the number of vertices having label 0;
3. $e_f(1)$ = the number of edges having label 1;
4. $e_f(0)$ = the number of edges having label 0;

Throughout this discussion we consider simple, finite and undirected graph $G = (V(G), E(G))$ with $|V(G)| = p$ and $|E(G)| = q$. For all other standard terminology and notation we refer to Chartrand and Lesniak[4].

II. Embedding of Equitable graphs

Definition 2.1: For a graph G, an edge labeling function $f: E(G) \rightarrow \{0,1\}$ induces a vertex labeling function $f^*: V(G) \rightarrow \{0,1\}$ defined as $f^*(u) = \Pi\{f(uv) / uv \in E(G)\}$. Then f is called a *total edge product cordial labeling* of graph G if $\left|\left(v_f(1)+e_f(1)\right)-\left(v_f(0)+e_f(0)\right)\right| \leq 1$. A graph G is called *total edge product cordial graph* if it admits a total edge product cordial labeling.

The concept of total edge product cordial labeling was introduced by Vaidya and Barasara [5].

Theorem 2.2: Any graph G can be embedded as an induced subgraph of a total edge product cordial graph.
Proof. Label the edges of G in such a way that edges with label 1 and edges with label 0 differ by at most 1.

Let V_i be the set of vertices with label i and E_i be the set of edges with label i while $n(V_i)$ and $n(E_i)$ be the cardinality of set V_i and E_i respectively.

Case 1: When $\left(n(V_1)+n(E_1)\right)-\left(n(V_0)+n(E_0)\right) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_0. Assign label 0 to these new edges. Consequently $\left\lceil \frac{r}{2} \right\rceil$ vertices will receive label 0.

Thus, $\left|\left(v_f(1)+e_f(1)\right)-\left(v_f(0)+e_f(0)\right)\right| = \left|n(V_1)+n(E_1)-n(V_0)-n(E_0)-2\left\lceil \frac{r}{2} \right\rceil\right| \leq 1$.

Case 2: When $\left(n(V_0)+n(E_0)\right)-\left(n(V_1)+n(E_1)\right) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_0. Assign label 1 to these new edges. Consequently $\left\lceil \frac{r}{2} \right\rceil$ vertices will receive label 1.

Thus, $\left|\left(v_f(1)+e_f(1)\right)-\left(v_f(0)+e_f(0)\right)\right| = \left|n(V_1)+n(E_1)+2\left\lceil \frac{r}{2} \right\rceil-n(V_0)-n(E_0)\right| \leq 1$.

Thus in all the possibilities, the constructed supergraph H satisfies the conditions for total edge product cordial graph. That is, any graph G can be embedded as an induced subgraph of a total edge product cordial graph.

Definition 2.3: Let $f: V(G) \rightarrow \{1,2,...,p\}$ be a bijection. For each edge uv, assign the label $|f(u)-f(v)|$. Function f is called a *difference cordial labeling* if $|e_f(1)-e_f(0)| \leq 1$. where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a *difference cordial graph*.

The concept of difference cordial labeling was introduced by Ponraj *et al.* [6] in 2003.

Theorem 2.4: Any graph G can be embedded as an induced subgraph of a difference cordial graph.
Proof. Let G be the given graph with p vertices and q edges. As per definition of difference cordial labeling assign labels $\{1,2,..., p\}$ to the vertices of G.

Let E_i be the set of edges with label i while $n(E_i)$ be the cardinality of set E_i.

Case 1: When $n(E_1)-n(E_0) > 1$ (say r).

The new graph H can be obtained by adding r vertices, say $\{v_1, v_2, ..., v_r\}$ to the graph G and join them to the vertex with label 1. Assign label $p+i$ to the vertex v_i for $i=1,2,...,r$. Consequently r edges will receive label 0.

Hence, $|e_f(1)-e_f(0)| = |n(E_1)-n(E_0)-r| = 0$.

Case 2: When $n(E_0)-n(E_1) > 1$ (say r).

The new graph H can be obtained by adding a path of r vertices, say $\{v_1, v_2, ..., v_r\}$ to the graph G and join the vertex v_1 to the vertex with label p. Assign label $p+i$ to the vertex v_i for $i = 1, 2, ..., r$. Consequently r edges will receive label 1.

Hence, $|e_f(1) - e_f(0)| = |n(E_1) + r - n(E_0)| = 0$.

Thus in all the possibilities, the constructed supergraph H satisfies the conditions for difference cordial graph. That is, any graph G can be embedded as an induced subgraph of a difference cordial graph.

Definition 2.5: A graph G is called *simply sequential* if there is a bijection $f: V(G) \cup E(G) \to \{1, 2, ..., p+q\}$ such that for each edge $e = \{uv\} \in E(G)$ one has $f(e) = f(u) + f(v)$.

In 1983, Bange *et al.* [7] have introduced the concept of sequential labeling.

Definition 2.6: If a mapping $f: V(G) \cup E(G) \to \{0,1\}$ is such that for each edge $e = uv$, $f(e) = |f(u) - f(v)|$ and the condition $|(v_f(1) + e_f(1)) - (v_f(0) + e_f(0))| \le 1$ holds then G is called *total sequential cordial graph*.

In 2002, Cahit [8] has introduced the concept of total sequential cordial labeling as a weaker version of sequential labeling.

Theorem 2.7: Any graph G can be embedded as an induced subgraph of a total sequential cordial graph.
Proof. Label the vertices of graph G in such a way that vertices with label 0 and vertices with label 1 differ by at most 1.

Let V_i be the set of vertices with label i and E_i be the set of edges with label i while $n(V_i)$ and $n(E_i)$ be the cardinality of set V_i and E_i respectively.

Case 1: When $(n(V_1) + n(E_1)) - (n(V_0) + n(E_0)) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_0. Assign label 0 to these new vertices. Consequently $\left\lceil \frac{r}{2} \right\rceil$ edges will receive label 0.

Thus, $|(v_f(1) + e_f(1)) - (v_f(0) + e_f(0))| = \left| n(V_1) + n(E_1) - n(V_0) - n(E_0) - 2\left\lceil \frac{r}{2} \right\rceil \right| \le 1$.

Case 2: When $(n(V_0) + n(E_0)) - (n(V_1) + n(E_1)) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_0. Assign label 1 to these new vertices. Consequently $\left\lceil \frac{r}{2} \right\rceil$ edges will receive label 1.

Thus, $|(v_f(1) + e_f(1)) - (v_f(0) + e_f(0))| = \left| n(V_1) + n(E_1) + 2\left\lceil \frac{r}{2} \right\rceil - n(V_0) - n(E_0) \right| \le 1$.

Thus in all the possibilities, the constructed supergraph H satisfies the conditions for total sequential cordial graph. That is, any graph G can be embedded as an induced subgraph of a total sequential cordial graph.

Definition 2.8: Let $f: V(G) \to \{1, 2, ..., p\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 otherwise. f is called a *divisor cordial labeling* if $|e_f(1) - e_f(0)| \le 1$. A graph with a divisor cordial labeling is called a *divisor cordial graph*.

In 2011, Varatharajan *et al.* [9] have introduced the concept of divisor cordial labeling.

Theorem 2.9: Any graph G can be embedded as an induced subgraph of a divisor cordial graph.
Proof. Let G be the given graph with p vertices and q edges. As per definition of divisor cordial labeling assign labels $\{1, 2, ..., p\}$ to the vertices of G.

Let E_i be the set of edges with label i while $n(E_i)$ be the cardinality of set E_i.

Case 1: When $n(E_0) - n(E_1) > 1$ (say r).

The new graph H can be obtained by adding r vertices to the graph G, say $v_1, v_2, ..., v_r$ and join them to vertex with label 1. Assign label $p+i$ to the vertex v_i for $i=1,2,...,r$. Consequently r edges will receive label 1.

Hence, $|e_f(1)-e_f(0)| = |n(E_0)-n(E_1)-r| = 0$.

Case 2: When $n(E_1)-n(E_0) > 1$ (say r).

The new graph H can be obtained by adding a path of r vertices, say $v_1, v_2, ..., v_r$ to the graph G and join the vertex v_1 to the vertex with label p. Assign label $p+i$ to the vertex v_i for $i=1,2,...,r$. Consequently r edges will receive label 0.

Hence, $|e_f(1)-e_f(0)| = |n(E_0)+r-n(E_1)| = 0$.

Thus in all the possibilities, the constructed supergraph H satisfies the conditions for divisor cordial graph. That is, any graph G can be embedded as an induced subgraph of a divisor cordial graph.

Definition 2.10: A graph G is said to have a *magic labeling* with constant C if there exists a bijection $f: V(G) \cup E(G) \to \{1,2,..., p+q\}$ such that $f(u)+f(v)+f(uv)=C$ for all $uv \in E(G)$.

Kotzig and Rosa [10] have introduced the concept of magic labeling in 1970 and investigated magic labeling for some standard graphs.

Definition 2.11: A graph G is said to have a *totally magic cordial labeling* with constant C if there exists a mapping $f: V(G) \cup E(G) \to \{0,1\}$ such that $f(u)+f(v)+f(uv)=C$ (mod 2) for all $uv \in E(G)$ provided the condition $|(v_f(1)+e_f(1))-(v_f(0)+e_f(0))| \le 1$ holds.

In 2002, Cahit [8] has introduced the concept of total magic cordial labeling as a weaker version of magic labeling.

Theorem 2.12: Any graph G can be embedded as an induced subgraph of a total magic cordial graph.

Proof. Label the vertices of graph G in such a way that $|v_f(1)-v_f(0)| \le 1$ and as per definition of total magic cordial labeling assign edge labels such that $f(u)+f(v)+f(uv)=C$ (mod 2).

Let V_i be the set of vertices with label i and E_i be the set of edges with label i while $n(V_i)$ and $n(E_i)$ be the cardinality of set V_i and E_i respectively.

Case 1: If $C=0$.

Subcase 1: When $(n(V_1)+n(E_1))-(n(V_0)+n(E_0)) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_0. Assign label 0 to these new vertices. Consequently $\left\lceil \frac{r}{2} \right\rceil$ edges will receive label 0.

Thus, $|(v_f(1)+e_f(1))-(v_f(0)+e_f(0))| = \left| n(V_1)+n(E_1)-n(V_0)-n(E_0)-2\left\lceil \frac{r}{2} \right\rceil \right| \le 1$.

Subcase 2: When $(n(V_0)+n(E_0))-(n(V_1)+n(E_1)) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_0. Assign label 1 to these new vertices. Consequently $\left\lceil \frac{r}{2} \right\rceil$ edges will receive label 1.

Thus, $|(v_f(1)+e_f(1))-(v_f(0)+e_f(0))| = \left| n(V_1)+n(E_1)+2\left\lceil \frac{r}{2} \right\rceil-n(V_0)-n(E_0) \right| \le 1$.

Case 2: If $C=1$.

Subcase 1: When $(n(V_1)+n(E_1))-(n(V_0)+n(E_0)) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_1. Assign label 0 to these new vertices. Consequently $\left\lceil \frac{r}{2} \right\rceil$ edges will receive label 0.

Thus, $\left|\left(v_f(1)+e_f(1)\right)-\left(v_f(0)+e_f(0)\right)\right| = \left|n(V_1)+n(E_1)-n(V_0)-n(E_0)-2\left\lceil \frac{r}{2} \right\rceil\right| \le 1$.

Subcase 2: When $\left(n(V_0)+n(E_0)\right)-\left(n(V_1)+n(E_1)\right) > 1$ (say r).

The new graph H can be obtained by adding $\left\lceil \frac{r}{2} \right\rceil$ vertices to the graph G and join them to arbitrary member of V_1. Assign label 1 to these new vertices. Consequently $\left\lceil \frac{r}{2} \right\rceil$ edges will receive label 1.

Thus, $\left|\left(v_f(1)+e_f(1)\right)-\left(v_f(0)+e_f(0)\right)\right| = \left|n(V_1)+n(E_1)+2\left\lceil \frac{r}{2} \right\rceil-n(V_0)-n(E_0)\right| \le 1$.

Thus in all the possibilities, the constructed supergraph H satisfies the conditions for total magic cordial graph. That is, any graph G can be embedded as an induced subgraph of a total magic cordial graph.

III. NP-Complete Problems

Theorem 3.1: Any planar graph G can be embedded as an induced subgraph of a planar total edge product cordial (difference cordial, total sequential cordial, divisor cordial, total magic cordial) graph.

Proof. If G is planar graph. Then the supergraph H constructed in Theorem 2.2 (corresponding theorems 2.4, 2.7, 2.9 and 2.12) is a planar graph. Hence the result.

Theorem 3.2: Any connected graph G can be embedded as an induced subgraph of a connected total edge product cordial (difference cordial, total sequential cordial, divisor cordial, total magic cordial) graph.

Proof. If G is connected graph. Then the supergraph H constructed in Theorem 2.2 (corresponding theorems 2.4, 2.7, 2.9 and 2.12) is a connected graph. Hence the result.

Theorem 3.3: The problem of deciding whether the chromatic number $\chi(G) \le k$, where $k \ge 3$, is NP-complete even for total edge product cordial (difference cordial, total sequential cordial, divisor cordial, total magic cordial) graph.

Proof. Let G be a graph with chromatic number $\chi(G) \ge 3$. Let supergraph H constructed in Theorem 2.2 is total edge product cordial (corresponding theorems 2.4, 2.7, 2.9 and 2.12), which contains G as an induced subgraph. Then obviously we have $\chi(H) \ge \chi(H)$. Since the problem of deciding whether the chromatic number $\chi(G) \le k$, where $k \ge 3$, is NP-complete by [11]. It follows that deciding whether the chromatic number $\chi(G) \le k$, where $k \ge 3$, is NP-complete even for prime cordial graphs.

Theorem 3.4: The problem of deciding whether the clique number $\omega(G) \ge k$ is NP-complete even when restricted to total edge product cordial (difference cordial, total sequential cordial, divisor cordial, total magic cordial) graph.

Proof. Since the problem of deciding whether the clique number of a graph $\omega(G) \ge k$ is NP-complete by [11] and $\omega(H) \ge \omega(G)$ for the supergraph H constructed in Theorem 2.2 is total edge product cordial (corresponding theorems 2.4, 2.7, 2.9 and 2.12). Hence the result.

Theorem 3.5: The problem of deciding whether the domination number (total domination number) is less than or equal to k is NP-complete even when restricted to total edge product cordial (difference cordial, total sequential cordial, divisor cordial, total magic cordial) graph.

Proof. Since the problem of deciding whether the domination number (total domination number) of a graph G is less than or equal to k is NP-complete as reported in [11] and the supergraph H constructed in Theorem 2.2 is total edge product cordial (corresponding theorems 2.4, 2.7, 2.9 and 2.12) has domination number greater than or equal to domination number of G. Hence the result.

IV. Concluding Remarks

Acharya *et al.* [12] have proved that any graph G can be embedded as an induced subgraph of a graceful graph. Thus showing the impossibility of obtaining any forbidden subgraph characterization for graceful graphs on the same line Acharya *et al.* [13, 14], Anandavally *et al.* [15], Germina and Ajitha [16], Vaidya and Vihol [17] and Vaidya and Barasara [18] have discussed embedding and NP-Complete problems in the context of various graph labeling schemes. However here we have discussed embedding and NP-Complete problems for total edge product cordial labeling, difference cordial labeling, total sequential cordial labeling, divisor cordial labeling and total magic cordial labeling.

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