IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 1 Ver. V (Jan - Feb. 2015), PP 112-121
www.iosrjournals.org

Connected Total Dominating Sets and Connected Total Domination Polynomials of Stars and Wheels

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Abstract: *Let G = (V, E) be a simple graph. A set S of vertices in a graph G is said to be a total dominating set if every vertex $v \in V$ is adjacent to an element of S. A total dominating set S of G is called a connected total dominating set if the induced subgraph <S> is connected. In this paper, we study the concept of connected total domination polynomials of the star graph S_n and wheel graph W_n. The connected total domination polynomial of a graph G of order n is the polynomial $D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)}^{n} d_{ct}(G,i)x^i$, where $d_{ct}(G, i)$ is the number of connected total dominating set of G of size i and $\gamma_{ct}(G)$ is the connected total domination number of G. We obtain some properties of $D_{ct}(S_n, x)$ and $D_{ct}(W_n, x)$ and their coefficients. Also, we obtain the recursive formula to derive the connected total dominating sets of the star graph S_n and the Wheel graph W_n.*

Keywords: *Connected total dominating set, connected total domination number, connected total domination polynomial, star graph and wheel graph.*

I. Introduction

Let G = (V, E) be a simple graph of order |V| = n. A set S of vertices in a graph G is said to be a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

A set S of vertices in a Graph G is said to be a total dominating set if every vertex $v \in V$ is adjacent to an element of S. A total dominating set S of G is called a connected total dominating set if the induced subgraph <S> is connected. The minimum cardinality of a connected total dominating set S of G is called the connected total domination number and is denoted by $\gamma_{ct}(G)$.

Let S_n be the star graph with n vertices and W_n be the wheel graph with n vertices. In the next section, we construct the families of connected total dominating sets of S_n by recursive method. In section III, we use the results obtained in section II to study the connected total domination polynomials of the star graph. In section IV, we construct the families of connected total dominating sets of W_n by recursive method. We also investigate the connected total domination polynomials of the wheel graph W_n in section V. As usual we use $\binom{n}{i}$ for the combination n to i and we denote the set {1, 2, . . . , n} by [n] throughout this paper.

II. Connected Total Dominating Sets Of The Star Graph S_n.

Let S_n, $n \geq 3$ be the star graph with n vertices $V(S_n)$ = [n] and $E(S_n)$ = (1,3) and (1,4))
$E(S_n)$ = {(1, 2), (1, 3), (1, 4), . . . , (1, n)}. . Let $d_{ct}(S_n, i)$ be the number of connected total dominating sets of S_n with cardinality i.

Lemma 2.1

The following properties hold for all Graph G with |V(G)| = n vertices.

(i) $d_{ct}(G,n) = 1$.
(ii) $d_{ct}(G,n-1) = n$.
(iii) $d_{ct}(G,i) = 0$ if $i > n$.
(iv) $d_{ct}(G,0) = 0$.
(v) $d_{ct}(G,1) = 0$.

Proof

Let G = (V, E) be a simple graph of order n.

(i) We have $D_{ct}(G, n) = [n]$. Therefore, $d_{ct}(G, n) = 1$.
(ii) Also, $D_{ct}(G, n-1) = \{[n] - \{x\} / x \in [n]\}$.
Therefore, $d_{ct}(G, n-1) = n$.
(iii) There does not exist a subgraph H of G such that $|V(H)| > |V(G)|$. Therefore, $d_{ct}(G,i) = 0$ if $i > n$.
(iv) There does not exist a subgraph H of G such that $|V(H)| = 0$, Φ is not a connected total dominating set of G.
(v) By the definition of total domination, a single vertex cannot dominate totally. Therefore, $d_{ct}(G,1) = 0$.

Theorem 2.2

Let S_n be the star graph with order n, then $d_{ct}(S_n,i) = \binom{n}{i} - \binom{n-1}{i}$, for all $n \geq 3$.

Proof:

Let S_n be the star graph with n vertices and $n \geq 3$. Let $v_1 \in V(S_n)$ and v_1 is the centre of S_n and let the other vertices be $v_2, v_3, \ldots, v_n$. Since the subgraph with vertex set as $\{v_2, v_3, \ldots, v_n\}$ is not connected, every connected total dominating set of S_n must contain the vertex v_1. Since $|V(S_n)| = n$, S_n contains $\binom{n}{i}$ number of subsets of cardinality i. Since, the subgraph with vertex set as $\{v_2, v_3, \ldots, v_n\}$ is not connected, each time $\binom{n-1}{i}$ number of subsets of S_n with cardinality i are not connected total dominating sets. Hence, S_n contains $\binom{n}{i} - \binom{n-1}{i}$ number of subsets of connected total dominating sets with cardinality i. Therefore, $d_{ct}(S_n,i) = \binom{n}{i} - \binom{n-1}{i}$, for all $n \geq 3$.

Theorem 2.3

Let S_n be the star graph with order $n \geq 3$, then

i) $d_{ct}(S_n, i) = \binom{n-1}{i-1}$ for all $2 \leq i \leq n$.

ii) $d_{ct}(S_n, i) = \begin{cases} d_{ct}(S_{n-1}, i) + d_{ct}(S_{n-1}, i-1) & \text{if } 2 < i \leq n. \\ d_{ct}(S_{n-1}, i) + 1 & \text{if } i = 2. \end{cases}$

Proof:

(i) By theorem 2.2, we have $d_{ct}(S_n, i) = \binom{n}{i} - \binom{n-1}{i}$.

We know that, $\binom{n}{i} - \binom{n-1}{i} = \binom{n-1}{i-1}$.

Therefore, $d_{ct}(S_n, i) = \binom{n-1}{i-1}$

(ii) We have, $d_{ct}(S_n, i) = \binom{n-1}{i-1}$, $d_{ct}(S_{n-1}, i) = \binom{n-2}{i-1}$ and $d_{ct}(S_{n-1}, i-1) = \binom{n-2}{i-2}$.

We know that,

$$\binom{n-2}{i-1} + \binom{n-2}{i-2} = \binom{n-1}{i-1}$$

Therefore, $d_{ct}(S_n, i) = d_{ct}(S_{n-1}, i) + d_{ct}(S_{n-1}, i-1)$.
When $i = 2$,

$$d_{ct}(S_n, 2) = \binom{n-1}{1} = n - 1$$

Consider, $d_{ct}(S_{n-1}, 2) + 1 = \binom{n-2}{1} + 1$
$= n - 2 + 1$
$= n - 1$

$d_{ct}(S_{n-1}, 2) + 1 = d_{ct}(S_n, 2)$

Therefore, $d_{ct}(S_n, i) = d_{ct}(S_{n-1}, i) + 1$ if $i = 2$.

III. Connected Total Domination Polynomials Of The Star Graph S_n.

Definition: 3.1

Let $d_{ct}(S_n, i)$ be the number of connected total dominating sets of a star graph S_n with cardinality i. Then, the connected total domination polynomial of S_n is defined as $D_{ct}(S_n, x) = \sum_{i=\gamma_{ct}(S_n)}^{n} d_{ct}(S_n, i)\, x^i$.

Remark 3.2

$\gamma_{ct}(S_n) = 2$.

Proof:

Let S_n be a star graph with n vertices and $n \geq 3$. Let $v_1 \in V(S_n)$ and v_1 is the centre of S_n and let the other vertices be $v_2, v_3, \ldots, v_n$. The centre vertex v_1 and one more vertex from $v_2, v_3, \ldots, v_n$ is enough to cover all the other vertices.Therefore, the minimum cardinality is 2. Therefore, $\gamma_{ct}(S_n) = 2$.

Theorem 3.3

Let S_n be a star graph with order n, then $D_{ct}(S_n, x) = x[(1 + x)^{n-1} - 1]$.

Proof:

By the definition of connected total domination polynomial, we have,

$$D_{ct}(S_n, x) = \sum_{i=2}^{n} d_{ct}(S_n, i)x^i.$$

$$= \sum_{i=2}^{n} \binom{n-1}{i-1} x^i \text{, by Theorem 2. 3(i).}$$

$$= \binom{n-1}{1}x^2 + \binom{n-1}{2}x^3 + \binom{n-1}{3} x^4 + \ldots + \binom{n-1}{n-1} x^n.$$

$$= x\left[\binom{n-1}{1}x + \binom{n-1}{2}x^2 + \binom{n-1}{3} x^3 + \ldots + \binom{n-1}{n-1} x^{n-1} \right]$$

$$= x\left[\sum_{i=0}^{n} \binom{n-1}{i}x^i - 1 \right]$$

Hence,

$$D_{ct}(S_n, x) = x[(1 + x)^{n-1} - 1].$$

Theorem 3.4

Let S_n be a star graph with order n, then

$D_{ct}(S_n, x) = (1 + x) D_{ct}(S_{n-1}, x) + x^2$ with $D_{ct}(S_2, x) = x^2$.

Proof:

We have, $D_{ct}(S_n, x) = \sum_{i=2}^{n} d_{ct}(S_n, i)x^i$.

$$= d_{ct}(S_n, 2)\, x^2 + \sum_{i=3}^{n} (S_n, i)\, x^i.$$

$$= \binom{n-1}{1}x^2 + \sum_{i=3}^{n} [\, d_{ct}(S_{n-1}, i) + d_{ct}(S_{n-1}, i-1)]\, x^i, \text{ by Theorem 2. 3.}$$

$$= (n-1)\, x^2 + \sum_{i=3}^{n} d_{ct}(S_{n-1}, i)\, x^i + \sum_{i=3}^{n} d_{ct}(S_{n-1}, i-1)x^i.$$

Consider, $\sum_{i=3}^{n} d_{ct}(S_{n-1}, i)\, x^i = \sum_{i=2}^{n} d_{ct}(S_{n-1}, i)\, x^i - d_{ct}(S_{n-1}, 2)\, x^2$.

$$= D_{ct}(S_{n-1}, x) - \binom{n-2}{1} x^2.$$
$$= D_{ct}(S_{n-1}, x) - (n-2)\, x^2.$$

Consider, $\sum_{i=3}^{n} d_{ct}(S_{n-1}, i-1)\, x^i = x[\sum_{i=3}^{n} d_{ct}(S_{n-1}, i-1)\, x^{i-1}]$

$$= x \sum_{i=2}^{n-1} d_{ct}(S_{n-1}, i)\, x^i.$$
$$= x D_{ct}(S_{n-1}, x).$$

Now, $D_{ct}(S_n, x) = (n-1)x^2 + D_{ct}(S_{n-1}, x) - (n-2)\, x^2 + xD_{ct}(S_{n-1}, x).$

$= nx^2 - x^2 + D_{ct}(S_{n-1}, x) - nx^2 + 2x^2 + x\, D_{ct}(S_{n-1}, x).$

$D_{ct}(S_n, x) = D_{ct}(S_{n-1}, x) + xD_{ct}(S_{n-1}, x) + x^2$

Therefore, $D_{ct}(S_n, x) = (1 + x)\, D_{ct}(S_{n-1}, x) + x^2$ with $D_{ct}(S_2, x) = x^2$.

Example 3.5

Let S_9 be the star graph with order 9 as given in Figure 1.

Figure 1

By Theorem 3.4, we have $D_{ct}(S_9, x) = (1 + x)\, D_{ct}(S_8, x) + x^2$

$= (1 + x)\,(7x^2 + 21x^3 + 35x^4 + 35x^5 + 21x^6 + 7x^7 + x^8) + x^2.$

$= 8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9.$

Theorem 3.6

Let S_n be a star graph with order $n \geq 3$. Then

(i) $D_{ct}(S_n, x) = \sum_{i=2}^{n} \binom{n}{i} x^i - \sum_{i=2}^{n} \binom{n-1}{i} x^i$

(ii) (ii) $D_{ct}(S_n, x) = \sum_{i=2}^{n} \binom{n-1}{i-1} x^i$

Proof:

i) follows from the definition of connected total domination polynomial and Theorem 2.2.

ii) follows from the definition of connected total domination polynomial and Theorem 2.3(i).

We obtain $d_{ct}(S_n, i)$, for $2 \leq i \leq 15$ as shown in Table 1.

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1													
3	2	1												
4	3	3	1											
5	4	6	4	1										
6	5	10	10	5	1									
7	6	15	20	15	6	1								
8	7	21	35	35	21	7	1							
9	8	28	56	70	56	28	8	1						
10	9	36	84	126	126	84	36	9	1					
11	10	45	120	210	252	210	120	45	10	1				

12	11	55	165	330	462	462	330	165	55	11	1			
13	12	66	220	495	792	924	792	495	220	66	12	1		
14	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
15	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

Table 1

In the following Theorem we obtain some properties of $d_{ct}(S_n, i)$.

Theorem 3.7

The following properties hold for the coefficients of $D_{ct}(S_n, x)$ for all $n \geq 3$.

(i) $d_{ct}(S_n, 2) = n - 1$.

(ii) $d_{ct}(S_n, n) = 1$.

(iii) $d_{ct}(S_n, n - 1) = n - 1$.

(iv) $d_{ct}(S_n, i) =$ o if $i < 2$ or $i > n$.

(v) $d_{ct}(S_n, i) = d_{ct}(S_n, n - i + 1)$ for all $n \geq 3$.

IV. Connected Total Dominating Sets Of The Wheel Graph W_n.

Let W_n, $n \geq 5$ be the wheel graph with $V(W_n) = [n]$ and $E(W_n) = \{(1, 2), (1, 3), \ldots, (1, n), (2, 3), (3, 4), \ldots, (n - 1, n), (n, 2)\}$. Let $d_{ct}(W_n, i)$ be the number of connected total dominating sets of W_n with cardinality i.

Lemma 4.1

For any cycle graph C_n with n vertices,

(i) $d_{ct}(C_n, n) = 1$.

(ii) $d_{ct}(C_n, n - 1) = n$.

(iii) $d_{ct}(C_n, n - 2) = n$.

(iv) $d_{ct}(C_n, i) = 0$ if $i < n - 2$ or $i > n$.

Theorem 4.2

For any cycle graph C_n with n vertices, $D_{ct}(C_n, x) = nx^{n-2} + nx^{n-1} + x^n$.

Proof:

The proof is given in [6].

Theorem 4.3

Let W_n, $n \geq 5$ be the wheel graph with n vertices, then $d_{ct}(W_n, i) = d_{ct}(S_n, i) + d_{ct}(C_{n-1}, i)$, for all i.

Proof:

Let S_n be the star graph and $v_1 \in V(S_n)$ be such that v_1 is the centre of S_n. Let S_n be a spanning subgraph of W_n and since $W_n - v_1 = C_{n-1}$, $S_n \cup C_{n-1} = W_n$. Therefore, the number of connected total dominating sets of the wheel graph W_n with cardinality i is the sum of the number of connected total dominating sets of the star graph S_n with cardinality i and the number of connected total dominating sets of the cycle C_{n-1} with cardinality i. Hence, $d_{ct}(W_n, i) = d_{ct}(S_n, i) + d_{ct}(C_{n-1}, i)$.

Theorem 4.4

Let W_n be the wheel graph with order $n \geq 5$, then

(i) $d_{ct}(W_n, i) = \binom{n}{i} - \binom{n-1}{i}$, for all $i < n - 3$.

(ii) $d_{ct}(W_n, i) = \binom{n}{i} - \binom{n-1}{i} + n - 1$ for $i = n - 3, n - 2$.

(iii) $d_{ct}(W_n, i) = \binom{n}{i} - \binom{n-1}{i} + 1$ for $i = n - 1$.

Proof:

(i) By theorem 4.3, we have $d_{ct}(W_n, i) = d_{ct}(S_n, i) + d_{ct}(C_{n-1}, i)$. Since, $d_{ct}(C_{n-1}, i) = 0$ for all $i < n - 3$, we have,

$$d_{ct}(W_n, i) = d_{ct}(S_n, i) \text{ for all } i < n - 3.$$
$$= \binom{n}{i} - \binom{n-1}{i} \text{ for all } i < n - 3, \text{ by Theorem 2.2.}$$

(ii) Since, $d_{ct}(C_{n-1}, i) = n - 1$ for $i = n - 2, n - 3$ we have, $d_{ct}(W_n, i) = \binom{n}{i} - \binom{n-1}{i} + n - 1$ for $i = n - 2, n - 3$.

(iii) Since, $d_{ct}(C_{n-1}, i) = 1$ for $i = n-1$, we have, $d_{ct}(W_n, i) = \binom{n}{i} - \binom{n-1}{i} + 1$ for $i = n-1$.

Theorem 4.5

Let W_n be a wheel graph with order $n \geq 5$, then,

(i) $d_{ct}(W_n, i) = \binom{n-1}{i-1}$ for all $i < n-3$.

(ii) $d_{ct}(W_n, i) = \binom{n-1}{i-1} + n - 1$ for $i = n-2, n-3$.

(iii) $d_{ct}(W_n, i) = \binom{n-1}{i-1} + 1$ for $i = n-1$.

Proof:

(i) By theorem 4.4 (i) and since, $\binom{n}{i} - \binom{n-1}{i} = \binom{n-1}{i-1}$, we have, $d_{ct}(W_n, i) = \binom{n-1}{i-1}$ for all $i < n-3$.

(ii) By theorem 4.4 (ii) and since, $\binom{n}{i} - \binom{n-1}{i} = \binom{n-1}{i-1}$, we have,

$$d_{ct}(W_n, i) = \binom{n-1}{i-1} + (n-1) \text{ for all } i = n-2, n-3.$$

(iii) By theorem 4.4 (iii) and since, $\binom{n}{i} - \binom{n-1}{i} = \binom{n-1}{i-1}$, we have,

$$d_{ct}(W_n, i) = \binom{n-1}{i-1} + 1 \text{ for } i = n-1.$$

Theorem 4.6

Let W_n be a wheel graph with order $n \geq 5$, then

(i) $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + 1$ if $i = 2$.
(ii) $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1)$ for all $2 < i \leq n$ and $i \neq n-3, n-4$.
(iii) $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1) - (n-3)$ if $i = n-3$.
(iv) $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1) - (n-2)$ if $i = n-4$.

Proof:

(i) When $i = 2$, $d_{ct}(W_n, 2) = \binom{n-1}{1}$, by Theorem 4.5

$= n-1$

Consider, $d_{ct}(W_{n-1}, 2) + 1 = \binom{n-2}{1} + 1$.

$= n-2+1.$

$= n-1.$

$d_{ct}(W_{n-1}, 2) + 1 = d_{ct}(W_n, 2)$

Therefore, $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + 1$ if $i = 2$.

(ii) By Theorem 4.5 (i), we have, $d_{ct}(W_n, i) = \binom{n-1}{i-1}$ for all $i < n-3$.

Also, $d_{ct}(W_{n-1}, i) = \binom{n-2}{i-1}$ and $d_{ct}(W_{n-1}, i-1) = \binom{n-2}{i-2}$.

We know that, $\binom{n-2}{i-1} + \binom{n-2}{i-2} = \binom{n-1}{i-1}$.

Therefore, $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1)$ for all $2 < i \leq n$ and $i \neq n-3, n-4$.

(iii) When $i = n-3$, we have,

$$d_{ct}(W_n, n-3) = \binom{n-1}{n-4} + (n-1) \text{ by theorem 4.5 (ii)}$$

$$= \binom{n-1}{3} + (n-1)$$

$$d_{ct}(W_{n-1}, n-3) = \binom{n-2}{n-4} + (n-2)$$

$$= \binom{n-2}{2} + (n-2)$$

$$d_{ct}(W_{n-1}, n-4) = \binom{n-2}{n-5} + (n-2)$$

$$= \binom{n-2}{3} + (n-2)$$

Consider, $\binom{n-2}{2} + (n-2) + \binom{n-2}{3} + (n-2)$

$$= \binom{n-2}{2} + \binom{n-2}{3} + (n-1) + (n-3)$$

$$= \binom{n-1}{3} + (n-1) + (n-3)$$

Therefore, $d_{ct}(W_{n-1}, n-3) + d_{ct}(W_{n-1}, n-4) = d_{ct}(W_n, n-3) + (n-3)$.

Hence, $d_{ct}(W_n, i) = d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1) - (n-3)$ if $i = n-3$.

(iv) when $i = n-4$, we have,

$$d_{ct}(W_n, n-4) = \binom{n-1}{n-5}$$

$$= \binom{n-1}{4}.$$

$$d_{ct}(W_{n-1}, n-4) = \binom{n-2}{n-5} + (n-2).$$

$$= \binom{n-2}{3} + (n-2).$$

$$d_{ct}(W_{n-1}, n-5) = \binom{n-2}{n-6}$$

$$= \binom{n-2}{4}.$$

Consider, $\binom{n-2}{3} + (n-2) + \binom{n-2}{4}$.

$$= \binom{n-2}{3} + \binom{n-2}{4} + (n-2).$$

$$= \binom{n-1}{4} + n-2.$$

Therefore, $d_{ct}(W_{n-1}, n-4) + d_{ct}(W_{n-1}, n-5) = d_{ct}(W_n, n-4) + (n-2)$.

Hence, $d_{ct}(W_n, i) + d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1) - (n-2)$ if $i = n-4$.

V. Connected Total Domination Polynomials Of The Wheel Graph W_n.

Definition: 5.1

Let $dct(W_n, i)$ be the number of connected total dominating sets of W_n with cardinality i. Then, the connected total domination polynomial of W_n is defined as

$$D_{ct}(W_n, x) = \sum_{i=\gamma_{ct}(W_n)}^{n} d_{ct}(W_n, i)\, x^i$$

Remark 5.2

$\gamma_{ct}(W_n) = 2$.

Proof:

Let W_n be a wheel graph with n vertices. Let $v_1 \in V(W_n)$ and v_1 is the centre of W_n and let the other vertices be $v_2, v_3, \ldots, v_n$. The centre vertex v_1 and one more vertex from $v_2, v_3, \ldots, v_n$ is enough to cover all the other vertices. Therefore the minimum cardinality is 2. Therefore, $\gamma_{ct}(W_n) = 2$.

Theorem 5.3

Let W_n be a wheel graph with order n, then $D_{ct}(W_n, x) = D_{ct}(S_n, x) + D_{ct}(C_{n-1}, x)$.

Proof:

By the definition of connected total domination polynomial,

we have, $D_{ct}(W_n, x) = \sum_{i=2}^{n} d_{ct}(W_n, i)\, x^i$.

$$= \sum_{i=2}^{n} [d_{ct}(S_n, i) + d_{ct}(C_{n-1}, i)]\, x^i, \text{ by Theorem 4.3.}$$

$$= \sum_{i=2}^{n} [d_{ct}(S_n, i)\, x^i + \sum_{i=2}^{n} d_{ct}(C_{n-1}, i)]\, x^i.$$

Therefore,

$D_{ct}(W_n, x) = D_{ct}(S_n, x) + D_{ct}(C_{n-1}, x)$.

Theorem 5.4

Let $D_{ct}(W_n, x)$ be the connected total domination polynomial of a wheel graph W_n with order $n \geq 5$. Then, $D_{ct}(W_n, x) = x[(1 + x)^{n-1} - 1] + (n-1)\, x^{n-3} + (n-1)\, x^{n-2} + x^{n-1}$.

Proof:

By Theorem 5.3, we have,

$D_{ct}(W_n, x) = D_{ct}(S_n, x) + D_{ct}(C_{n-1}, x)$

Therefore,

$D_{ct}(W_n, x) = x[(1 + x)^{n-1} - 1] + (n-1)\, x^{n-3} + (n-1)\, x^{n-2} + x^{n-1}$, by Theorem 3.2 and Theorem 4.2.

Theorem 5.5

Let $D_{ct}(W_n, x)$ be the connected total domination polynomial of a wheel graph W_n with order $n \geq 5$. Then,

(i) $D_{ct}(W_n, x) = \sum_{i=2}^{n} \binom{n}{i} x^i - \sum_{i=2}^{n} \binom{n-1}{i} x^i + (n-1)\, x^{n-3} + (n-1)\, x^{n-2} + x^{n-1}$.

(ii) $D_{ct}(W_n, x) = \sum_{i=2}^{n} \binom{n-1}{i-1} x^i + (n-1)\, x^{n-3} + (n-1)\, x^{n-2} + x^{n-1}$.

Proof:

(i) follows from Theorem 5.3 Theorem 3.6 (i) and Theorem 4.2.

(ii) follows from Theorem 5.3, Theorem 3.6(ii) and Theorem 4.2.

Theorem 5.6

Let $D_{ct}(W_n, x)$ be the connected total domination polynomial of a wheel graph W_n with order $n \geq 5$. Then, $D_{ct}(W_n, x) = (1 + x)\, D_{ct}(W_{n-1}, x) - (n-2)\, x^{n-4} - (n-3)\, x^{n-3} + x^2$.

Proof:

From the definition of connected total domination polynomial, we have, $D_{ct}(W_n, x) = \sum_{i=2}^{n} d_{ct}(W_n, i)\, x^i$.

$$= \sum_{i=2}^{n} [d_{ct}(W_{n-1}, i) + d_{ct}(W_{n-1}, i-1)]\, x^i, \text{ by Theorem 4.6.}$$

$$= \sum_{i=2}^{n} d_{ct}(W_{n-1}, i)\, x^i + \sum_{i=2}^{n} d_{ct}(W_{n-1}, i-1)\, x^i.$$

$$= \sum_{i=2}^{n-1} d_{ct}(W_{n-1}, i)\, x^i + x \sum_{i=3}^{n} d_{ct}(W_{n-1}, i-1)\, x^{i-1}.$$

$$= D_{ct}(W_{n-1}, x) + x\, D_{ct}(W_{n-1}, x)$$

Hence,

$$D_{ct}(W_n, x) = (1 + x)\, D_{ct}(W_{n-1}, x) \qquad \text{----------------(1)}$$

If i = 2 , then

$d_{ct}(W_n, 2)x^2 = [d_{ct}(W_{n-1}, 2) + 1]\, x^2$, by Theorem 4.6 (i).

Hence,

$$d_{ct}(W_n, 2)x^2 = d_{ct}(W_{n-1}, 2)\ x^2 + x^2 \qquad \text{-------------(2)}$$

If i = n – 3, then,

$d_{ct}(W_n, n-3)\, x^{n-3} = [d_{ct}(W_{n-1}, n-3) + d_{ct}(W_{n-1}, n-4) - (n-3)]\, x^{n-3}$ by Theorem 4.6 (iii).

Hence,

$$d_{ct}(W_n, n-3)\, x^{n-3} = d_{ct}(W_{n-1}, n-3)\, x^{n-3} + d_{ct}(W_{n-1}, n-4)\, x^{n-3} - (n-3)x^{n-3} \qquad \text{--------------(3)}$$

If i = n – 4, then

$d_{ct}(W_n, n-4)\, x^{n-4} = [d_{ct}(W_{n-1}, n-4) + d_{ct}(W_{n-1}, n-5) - (n-2)]\, x^{n-4}$ by Theorem 4.6 (iv).

Hence,

$$d_{ct}(W_n, n-4)x^{n-4} = d_{ct}(W_{n-1}, n-4)\, x^{n-4} + d_{ct}(W_{n-1}, n-5)\, x^{n-4} - (n-2)x^{n-4} \qquad \text{--------------(4)}$$

Combining (1), (2), (3) and (4) we get ,

$$D_{ct}(W_n, x) = (1 + x)\, D_{ct}(W_{n-1}, x) - (n-2)\, x^{n-4} - (n-3)\, x^{n-3} + x^2.$$

Example 5.7

$D_{ct}(W_5, x) = 8x^2 + 10x^3 + 5x^4 + x^5.$

By Theorem 5.6, we have

$D_{ct}(W_6, x) = (1 + x)\, (8x^2 + 10x^3 + 5x^4 + x^5) - 4x^2 - 3x^3 + x^2$

$= 5x^2 + 15x^3 + 15x^4 + 6x^5 + x6$

We obtain $d_{ct}(W_n, i)$ for $5 \le n \le 15$ as shown in Table 2.

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	8	10	5	1										
6	5	15	15	6	1									
7	6	15	26	21	7	1								
8	7	21	35	42	28	8	1							
9	8	28	56	70	64	36	9	1						
10	9	36	84	126	126	93	45	10	1					
11	10	45	120	210	252	210	130	55	11	1				
12	11	55	165	330	462	462	330	176	66	12	1			
13	12	66	220	495	792	924	792	495	232	78	13	1		
14	13	78	286	715	1287	1716	1716	1287	715	299	91	14	1	
15	14	91	364	1001	2002	3003	3432	3003	2002	1001	378	105	15	1

Table 2

In the following Theorem we obtain some properties of $d_{ct}(W_n, i)$

Theorem 5.8

The following properties hold for the coefficients of $D_{ct}(W_n, x)$ for all $n \ge 5$.

(i) $d_{ct}(W_n, 2) = n - 1$ for all $n \ge 6$.

(ii) $d_{ct}(W_n, n) = 1$

(iii) $d_{ct}(W_n, n-1) = n$

(iv) $d_{ct}(W_n, i) = 0$ if $i < 2$ or $i > n$.

(v) $d_{ct}(W_n, n-2) = \binom{n}{2}$

(vi) $d_{ct}(W_n, n-3) = \binom{n-1}{3} + (n-1)$

(vii) $d_{ct}(W_n, n-4) = \binom{n-1}{4}$

(viii) $d_{ct}(W_n, n-5) = \binom{n-1}{5}$

(ix) $d_{ct}(W_n, n-6) = \binom{n-1}{6}$

(x) $d_{ct}(W_n, n-i) = \binom{n-1}{i}$.

Proof:

Proof of (i), (ii) and (iii) follows from Theorem 4.5.

(iv) from Table 2 , we have $d_{ct}(W_n, i) = 0$ if $i < 2$ or $i > n$.

Proof of (v), (vi), (vii), (viii), (ix) and (x) follows from Theorem 4.5

VI. Conclusion

In this paper, the connected total domination polynomials of stars S_n and wheels W_n has been derived by identifying its connected total dominating sets. Similarly, we can generalize this study to any power of the star S_n and power of the wheel W_n.

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