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A Tau Approach for Solving Fractional Diffusion Equations using Legendre-Chebyshev Polynomial Method

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Abstract: *In this paper, a modified numerical algorithm for solving the fractional diffusion equation is proposed. Based on Tau idea where the shifted Legendre polynomials in time and the shifted Chebyshev polynomials in space are utilized respectively.*
The problem is reduced to the solution of a system of linear algebraic equations. From the computational point of view, the solution obtained by this approach is tested and the efficiency of the proposed method is confirmed.
Keywords: *Tau method, Shifted Chebyshev polynomial, Shifted Legendre polynomial, Fractional diffusion equation.*

I. Introduction

Fractional calculus is a generalization of classical calculus which provides an excellent tool to describe memory and hereditary properties of various materials and processes. The field of the fractional differential equations draws special interest of researchers in several areas including chemistry, physics, engineering, finance and social sciences [1].

There are many advantages of fractional derivatives. One of them is that it can be seen as a set of ordinary derivatives that give the fractional derivatives the ability to describe what integer-order derivatives cannot [2].

Fractional diffusion equations [3] are the generalization of classical diffusion equations. These equations play important roles in molding anomalous diffusion systems and sub-diffusion systems. In recent years, they have been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering etc, [4].

Most fractional diffusion equations do not have closed form solutions, so approximation and numerical techniques such as He's variational iteration method [5,6], Adomian's decomposition technique [7], the homotopy analysis method [8], finite difference scheme [9], and other methods [10], [11], [12] and [13]. [14] Described sinc-Legendre collocation method for the fractional diffusion equations and [15] investigating the numerical solution for a class of fractional diffusion-wave equations with a variable coefficients using sinc-Chebyshev collocation method.

In this paper we develop a numerical algorithm to solve fractional diffusion equation based on Tau method using shifted Legendre polynomials-shifted Chebyshev polynomials. The required approximate solution is expanded as a series with the elements of shifted Legendre polynomials in time and shifted Chebyshev polynomials in space with unknown coefficients. The proposed approach will reduce the problem to the solution to a system of linear algebraic equations.

This paper is organized as follows:

In the next section, we introduce the definitions of the fractional derivatives and integration which are necessary for the late of this paper. In sections 3 and 4 we present the shifted Legendre and shifted Chebyshev polynomials respectively. Section 5 is devoted to constructing and analyzing the numerical algorithm. As a result, a system of linear algebraic equations is formed and the solution of the considered problem is obtained. In section 6, numerical examples are given to demonstrate the effectiveness and convergence of the proposed method. Finally a brief conclusion is given in section 7.

II. Fractional derivatives and integration [1]

In this section we shall give some definitions which are needed in this paper later on.

Definition (1): The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as:

$$I_t^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (x-t)^{\alpha-1} f(t)\, dt,\ \alpha > 0,\ x > 0 . \qquad \dots (1)$$

$$I_t^0 f(x) = f(t)$$

Definition (2): The Riemann-Liouville fractional derivative operator of order $\alpha > 0$ is defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt \qquad \dots (2)$$

Where n is an integer and n-1< $\alpha \le$ n.

Definition (3): The fractional derivative of f(x) in the Caputo sense is defined as

$${}^{c}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\, dt, \quad n-1 < \alpha \le n,\ n \in \mathbb{N}. \qquad \dots (3)$$

Where α> 0 is the order of the derivative and n is the smallest integer greater than or equal to α. For the Caputo derivative, we have

$$D^{\alpha} C = 0,\ \text{(C is a constant)} \qquad \dots (4)$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil. \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \alpha \rfloor. \end{cases} \qquad \dots (5)$$

Note: We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and the function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α.

III. The shifted Legendre polynomials [16]

The well-known Legendre polynomials are defined on the interval [-1,1] and can be determined with the aid of the following recurrence formula

$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z), \quad i=1, 2, \dots .$$

Where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $x \in [0, h]$ we define the so-called shifted Legendre polynomials by introducing the change of variable $z = (2x-h)/h$. The shifted Legendre polynomials in x are then obtained as:

$$L_{i+1}^h(x) = \frac{(2i+1)(2x-h)}{(i+1)h} L_i^h(x) - \frac{i}{i+1} L_{i-1}^h(x), \quad i = 1, 2, \dots . \qquad \dots (6)$$

Where $L_0^h(x) = 1$ and $L_1^h(x) = (2x-h)/h$.

The analytic form of the shifted Legendre polynomial $L_i^h(x)$ of degree i is given by

$$L_i^h(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{x^k}{(k!)^2 h^k} \qquad \dots (7)$$

The orthogonality condition is

$$\int_0^h L_i^h(x) L_j^h(x)\, dx = \begin{cases} \dfrac{h}{2i+1}, & \text{for } i = j. \\ 0, & \text{for } i \ne j. \end{cases}$$

A function $f(x)$ defined over [0,1] may be approximated as:

$$f(x) \approx \sum_{s=1}^{n} \sum_{r=0}^{m} c_{sr} \Phi_{sr}(x) = C^T \Phi(x) = \hat{f}(x)$$

Where C and $\Phi(x)$ are n(m+1)×1 matrices given by:

$$C = [c_{10}, c_{11}, \dots, c_{1m}, c_{20}, c_{21}, \dots c_{2m}, \dots, c_{nm}]$$

And

$$\Phi_{m,h}(x) = [L_0^h(x), L_1^h(x), \dots, L_m^h(x)]^T \qquad \dots (8)$$

IV. The shifted Chebyshev polynomials [17]

The well-known Chebyshev polynomials are defined on the interval [-1, 1] and can be determined with the aid of the following recurrence formulae

$$T_{i+1}(t) = 2t\, T_i(t) - T_{i-1}(t), \quad i = 1, 2, \dots . \qquad \dots (9)$$

Where $T_0^h(t)=1$ and $T_1^h(t)=t$. In a shifted Chebyshev polynomial, the domain is transformed to values between 0 and h by introducing the change of variables $t=(2x/h)-1$. Let the shifted Chebyshev polynomials $T_i\left(\frac{2x}{h}-1\right)$ be denoted by $T_i^h(x)$, then the $T_i^h(x)$ can be generated by the following recurrence relation

$$T_{i+1}^h(x)=2\left(\frac{2x}{h}-1\right)T_i^h(x)-T_{i-1}^h(x) \qquad \dots (10)$$

Where $T_0^h(x)=1$ and $T_1^h(x)=(2x/h)-1$. The analytic form of the shifted Chebyshev polynomials $T_i^h(x)$ of degree i is given by

$$T_i^h(x)=i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!h^k}x^k \qquad \dots (11)$$

The orthogonality condition is

$$\int_0^h T_i^h(x)T_j^h(x)W_h(x)dx=\delta_{ij}h_j \qquad \dots (12)$$

Where $W_h(x)=\frac{1}{\sqrt{hx-x^2}}$, $h_j=\frac{\varepsilon_j}{2}\pi$, with $\varepsilon_0=2$ and $\varepsilon_j=1$, $j\geq 1$.

A function $f(x)$ defined over [0,1] may be approximated as:

$$f(x)\approx\sum_{s=1}^{n}\sum_{r=0}^{m}c_{sr}\Psi_{sr}(x)=C^T\Psi(x)=\hat{f}(x)$$

Where C and $\Psi(x)$ are n(m+1)×1 matrices given by:

$$C\,\square\,[c_{10},c_{11},\dots,c_{1m},c_{20},c_{21},\dots c_{2m},\dots,c_{nm}]$$

And

$$\Psi_{m,h}(x)=[T_0^h(x),T_1^h(x),\dots,T_m^h(x)]^T \qquad \dots (13)$$

Lemma (1), [17]: Let $T_i^h(x)$ be a shifted Chebyshev polynomial, then:

$$D^\alpha T_i^h(x)=0, \quad i=0,1,\dots,\lceil\alpha\rceil-1.$$

Proof: The lemma can be easily proved by making use of the relations (5) and (11).

Theorem (1), [17]: Fractional derivative of order α>0 for the Chebyshev polynomials is given by:

$$D^\alpha\Psi_{m,h}(x)\,\square\,D^{(\alpha)}\Psi_{m,h}(x) \qquad \dots (14)$$

Where $D^{(\alpha)}$ is the (m+1) × (m+1) operational matrix of fractional derivative of order α in Caputo sense and is defined as follows: (Note that in $D^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows are all zero).

$$D^{(\alpha)}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,m,k} \\ \vdots & \vdots & & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,m,k} \end{pmatrix} \qquad \dots (15)$$

Where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k}=\frac{(-1)^{i-k}2i(i+k-1)!\Gamma(k-1+\frac{1}{2})}{\varepsilon_j h^\alpha\Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-j-\alpha+1)\Gamma(k+j-\alpha+1)}$$

V. The Approach

In this section we will find an approximate solution for the one-dimensional space fractional diffusion equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = c(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t),\ 0 < x < \ell,\ \ 0 \le t \le \tau,\ 1 < \alpha \le 2, \qquad \ldots (16)$$

with initial condition

$$u(x,0) = f(x), \quad 0 < x < \ell, \qquad \ldots (17)$$

And boundary conditions

$$u(0,t) = g_0(t), \quad 0 < t \le \tau, \qquad \ldots (18)$$

$$u(\ell,t) = g_1(t), \quad 0 < t \le \tau, \qquad \ldots (19)$$

Based on Tau idea where the shifted Legendre polynomial in time and the shifted Chebyshev polynomial in space are utilized respectively.

A function u(x,t) of two independent variables defined for 0<x<1, 0<t< τ may be expanded in terms of the shifted Legendre and shifted Chebyshev polynomials as

$$u_{n,m}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij} L_i^{\tau}(t) T_j^{\ell}(x) = \Phi_{n,\tau}^{T}(t) A \Psi_{m,\ell}(x) \qquad \ldots (20)$$

Where the shifted Chebyshev vector $\Psi_{m,\ell}(x)$ is defined as eq. (13) and the shifted Legendre vector $\Phi_{n,\tau}(t)$ is defined as eq. (8).

The shifted Chebyshev and the shifted Legendre coefficients matrix A is given by

$$A = \begin{pmatrix} a_{00} & \cdots & a_{0m} \\ \vdots & & \vdots \\ a_{n0} & \cdots & a_{nm} \end{pmatrix}$$

Where

$$a_{ij} = \left(\frac{2i+1}{\tau}\right)\left(\frac{2}{\pi \varepsilon_j}\right)\int_0^{\tau}\int_0^{\ell} u(x,t) L_i^{\tau}(t) T_j^{\ell}(x) W_{\ell}(x)\,dt\,dx,\ i = 0,1,\ldots,n,\ j = 0,1,\ldots,m \qquad \ldots (21)$$

The integration of $\Phi_{n,\tau}(t)$ from 0 to t can be expressed as:

$$\int_0^{t} \Phi_{n,\tau}(t')dt' \simeq P\Phi_{n,\tau}(t) \qquad \ldots (22)$$

Where P is an (n+1)×(n+1) operational matrix of integration given by

$$P = \tau \begin{pmatrix} \delta_0 & \delta_0 & & & \\ -\delta_1 & 0 & \delta_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -\delta_{n-1} & 0 & \delta_{n-1} \\ & & & -\delta_n & 0 \end{pmatrix} \qquad \ldots (23)$$

With $\quad \delta_k = \dfrac{1}{2(2k+1)}$

To solve problem (16) – (19) we approximate f(x) by (m+1) terms of the shifted Legendre-shifted Chebyshev polynomial.

$$f(x) \simeq \sum_{j=0}^{m} f_i T_j^{\ell}(x) = T_0^{h}(t).\sum_{j=0}^{m} f_i T_j^{\ell}(x) = \Phi_{n,\tau}^{T}(t) F \Psi_{m,\ell}(x) \simeq f(x;t), \qquad \ldots (24)$$

Where F is known (n+1)×(m+1) matrix and can be shown by

$$F = \begin{pmatrix} f_0 & f_1 & \cdots & f_{m-1} & f_m \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \ldots (25)$$

In addition, we approximate c(x), u(x,t) and q(x,t) by the shifted Legendre-shifted Chebyshev polynomials as

$$c_m(x) = C^{T}\Psi_{m,\ell}(x), \qquad \ldots (26)$$

$$u_{n,m}(x,t) = \Phi_{n,\tau}^{T}(t) A \Psi_{m,\ell}(x), \qquad \ldots (27)$$

$$q_{n,m}(x,t) = \Phi_{n,\tau}^{T}(t) Q \Psi_{m,\ell}(x), \qquad \ldots (28)$$

Where the vector $C=[c_0,c_1,...,c_m]^T$ and the matrix Q are known, but A is (n+1) × (m+1) unknown matrix. Now integrating eq. (16) from 0 to t and using eq. (17), we can get

$$u(x,t)-f(x)=\int_0^t c(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}dt+\int_0^t q(x,t)dt. \quad \dots (29)$$

Using eqs. (14), (22), (26) and (27), then we obtain

$$\int_0^t c(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}dt \simeq (C^T\Psi_{m,\ell}(x))\left(\int_0^t \Phi_{n,\tau}^T(t)dt\right)A(D^{(\alpha)}\Psi_{m,\ell}(x))=(C^T\Psi_{m,\ell}(x))(\Phi_{n,\tau}^T(t)P^TAD^{(\alpha)}\Psi_{m,\ell}(x))$$
$$=\Phi_{n,\tau}^T(t)P^TAD^{(\alpha)}\Psi_{m,\ell}(x)\Psi_{m,\ell}^T(x)C. \quad \dots (30)$$

Let

$$\Psi_{m,\ell}(x)\Psi_{m,\ell}^T(x)C \simeq H^T\Psi_{m,\ell}(x), \quad \dots (31)$$

Where H is a (m+1) × (m+1) matrix. In order to illustrate H, eq. (31) can be written as

$$\sum_{k=0}^m c_k T_k^\ell(x)T_j^\ell(x)=\sum_{k=0}^m H_{kj}T_k^\ell(x), \qquad j=0,1,...,m.$$

Multiplying both sides of the above equation by $T_i^\ell(x)W_\ell(x),\ i=0,1,...,m$ and integrating the result from 0 to ℓ, we obtain

$$\sum_{k=0}^m c_k\int_0^\ell T_i^\ell(x)T_j^\ell(x)T_k^\ell(x)W_\ell(x)dx=H_{ij}\int_0^\ell T_i^\ell(x)T_i^\ell(x)W_\ell(x)dx, \quad i,j=0,1,...,m. \quad \dots (32)$$

Now, we suppose

$$w_{i,j,k}=\int_0^\ell T_i^\ell(x)T_j^\ell(x)T_k^\ell(x)W_\ell(x)dx, \quad \dots (33)$$

By eqs. (12), (32) and (33), we have

$$H_{ij}=\frac{2}{\varepsilon_i\pi}\sum_{k=0}^m c_k w_{i,j,k}, \quad i,j=0,1,...,m. \quad \dots (34)$$

So eq. (30) can be written as

$$\int_0^t c(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}dt \simeq \Phi_{n,\tau}^T(t)P^TAD^{(\alpha)}H^T\Psi_{m,\ell}(x). \quad \dots (35)$$

In addition, by eqs. (22) and (28), we get

$$\int_0^t q_{n,m}(x,t)dt=\Phi_{n,\tau}^T(t)P^TQ\Psi_{m,\ell}(x) \quad \dots (36)$$

Finally, applying eqs. (24), (27), (35) and (36), the residual $R_{n,m}(x,t)$ for eq. (29) can be written as

$$R_{n,m}(x,t)=\Phi_{n,\tau}^T(t)(A-F-P^TAD^{(\alpha)}H^T-P^TQ)\Psi_{m,\ell}(x)=\Phi_{n,\tau}^T(t)E\Psi_{m,\ell}(x),$$

Where

$$E=A-F-P^TAD^{(\alpha)}H^T-P^TQ$$

According to the typical Tau method (see [18]), we generate (n+1)×(m+1) linear algebraic equations as follows

$$E_{ij}=0, \quad i=0,1,...,n,\ \ j=0,1,...,m-2. \quad \dots (37)$$

Also, substituting eq. (27) in eqs. (18) and (19), we obtain

$$\Phi_{n,\tau}^T(t)A\Psi_{m,\ell}(0)=g_0(t), \quad \dots (38)$$

$$\Phi_{n,\tau}^T(t)A\Psi_{m,\ell}(\ell)=g_1(t), \quad \dots (39)$$

Respectively, eqs. (38) and (39) are collocated at (n+1) points. In addition, by comparing the coefficients eqs. (38) and (39) then we can obtain (n+1)×(m+1) coefficients of A. Consequently, the $u_{n,m}(x,t)$ given in eq. (27) can be calculated.

VI. Illustrative examples

To validate the effectiveness of the proposed method for problem (16) – (19) we consider the following two examples. Our results are compared with the exact solution or with the previous works on the same problem.

Example (1): Consider the following space fractional differential equation in

$$\frac{\partial u(x,t)}{\partial t}=c(x)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}}+q(x,t), \quad 0<x<1,\ t>0,$$

With the coefficient function

$$c(x) = \Gamma(1.2)x^{1.8}$$

The source function

$$q(x,t) = (6x^3 - 3x^2)e^{-t}$$

The initial condition

$$u(x,0) = x^2 - x^3, \quad 0 < x < 1,$$

And the boundary conditions

$$u(0,t) = u(1,t) = 0.$$

The exact solution of this problem is given in [17] as:

$$u(x,t) = (x^2 - x^3)e^{-t}.$$

Following table (1) represent a comparison of the absolute errors between the proposed method and methods [16] and [17] for u(x,2) of example (1).

Table (1) comparison of the absolute errors with methods [16] and [17] for u(x,2) of example (1).

x	n=m=3			n=m=5		
	Method[16]	Method[17]	Proposed method	Method[16]	Method[17]	Proposed method
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	2.89e-05	5.845e-06	6.379e-06	4.47e-06	1.261e-05	1.252e-05
0.2	1.09e-04	2.653e-05	7.168e-05	2.78e-07	1.634e-06	9.609e-07
0.3	2.20e-04	8.329e-05	1.732e-04	5.81e-06	1.425e-05	1.565e-05
0.4	3.40e-04	1.505e-04	2.884e-04	1.02e-05	2.513e-05	2.719e-05
0.5	4.45e-04	2.145e-04	3.946e-04	1.17e-05	2.774e-05	3.024e-05
0.6	5.15e-04	2.613e-04	4.693e-04	1.08e-05	2.332e-05	2.599e-05
0.7	5.27e-04	2.771e-04	4.898e-04	8.54e-06	1.553e-05	1.809e-05
0.8	4.60e-04	2.480e-04	4.335e-04	6.06e-06	8.317e-06	1.043e-05
0.9	2.91e-04	1.603e-04	2.777e-04	3.67e-06	3.758e-06	5.062e-06
1.0	0.0	0.0	0.0	0.0	0.0	0.0

<u>Example (2):</u> Consider the space fractional differential equation

$$\frac{\partial u(x,t)}{\partial t} = c(x)\frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} + q(x,t), \quad 0 < x < 1,\ t > 0,$$

With the coefficient function

$$c(x) = \Gamma(1.5)x^{0.5},$$

The source function

$$q(x,t) = (x^2 + 1)\cos(t+1) - 2x\sin(t+1),$$

The initial condition

$$u(x,0) = (x^2 + 1)\sin(1),$$

And the boundary conditions

$$u(0,t) = \sin(t+1), \quad u(1,t) = 2\sin(t+1).$$

The exact solution of this problem is given in [17] as:

$$u(x,t) = (x^2 + 1)\sin(t+1).$$

Following table (2) represent a comparison between the solutions obtained by the proposed method for u(x,1) and the exact solution of example (2).

Table (2) comparison between the solution obtained by the proposed method and the exact solution for u(x,1) of example (2).

x	n=m=2 The Approach	n=m=3 The Approach	n=m=6 The Approach	n=m=7 The Approach	exact
0.1	0.1140	0.1341	0.1426	0.1426	0.1425
0.2	0.1217	0.1377	0.1468	0.1468	0.1468
0.3	0.1302	0.1441	0.1539	0.1538	0.1538
0.4	0.1397	0.1534	0.1637	0.1637	0.1637
0.5	0.1500	0.1654	0.1764	0.1764	0.1764
0.6	0.1611	0.1803	0.1919	0.1919	0.1919
0.7	0.1731	0.1978	0.2103	0.2103	0.2103
0.8	0.1859	0.2181	0.2314	0.2314	0.2314
0.9	0.1997	0.2411	0.2554	0.2554	0.2554

VII. Conclusion

In this paper we develop and analyze the numerical algorithm for the fractional diffusion equation. Based on Tau method the shifted Legendre polynomial and the shifted Chebyshev polynomial are used to reduce the problem to the solution of a system of linear algebraic equations. The solution obtained by this method is in good agreement with the exact solution and other previous works that are used to solve fractional diffusion equation. The effectiveness of the presented method is tested and confirmed through out the numerical results.

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