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CR-Submanifolds of a Nearly Hyperbolic Cosymplectic Manifold with Semi-Symmetric Semi-Metric Connection

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Abstract: *We consider a nearly hyperbolic cosymplectic manifold and we study some properties of CR-submanifolds of a nearly cosymplectic manifold with a semi-symmetric semi-metric connection. We also obtain some results on $\xi-$horizontal and ξ $-$vertical CR- submanifolds of a nearly cosymplectic manifold with a semi-symmetric semi-metric connection and study parallel distributions on nearly hyperbolic cosymplectic manifold with a semi-symmetric semi-metric connection.*

Keywords: *CR-submanifolds, Nearly hyperbolic cosymplectic manifold, totally geodesic, Parallel distribution and Semi-symmetric semi-metric connection.*
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I. Introduction

A. Bejancu [1] initiated a new class of submanifolds of a complex manifold which he called CR-submanifolds and obtained some interesting results. A. Bejancu also introduced the notion of CR-submanifolds of Kaehler manifold in [2]. Since then, many papers have been concerned with Kaehler manifolds. The notion of CR-submanifolds of Sasakian manifold was studied by C. J. Hsu in [3] and M. Kobayashi in [10]. Later, several geometers (see [4], [5], [7]) studied CR-submanifolds of almost contact manifolds. In [9], Upadhyay and Dube studied almost hyperbolic $(f,\ g,\ \eta,\ \xi\ )$-structure. Moreover in [6], M. Ahmad and Kashif Ali, studied some properties of CR-submanifolds of a nearly hyperbolic cosymplectic manifold.

In the present paper, we study some properties of CR-submanifolds of a nearly hyperbolic cosymplectic manifold with a semi-symmetric semi-metric connection.

The paper is organized as follows: In 2, we give a brief description of nearly hyperbolic cosymplectic manifold with a semi-symmetric semi-metric connection. In 3, some properties of CR-submanifolds of nearly hyperbolic cosymplectic manifold are investigated. In 4, some results on parallel distribution on ξ- horizontal and ξ-vertical CR-submanifolds of a nearly cosymplectic manifold are obtained.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure $(\emptyset,\xi,\eta,g)$, where a tensor $\emptyset$ of type $(1,1)$ a vector field ξ, called structure vector field and η, the dual 1-form of ξ and the associated Riemannian metric g satisfying the following

$$\emptyset^2X=X+\eta(X)\xi \quad (2.1)$$

$$\eta(\xi)=-1,\quad g(X,\xi)=\eta(X) \quad (2.2)$$

$$\emptyset(\xi)=0,\quad \eta o\emptyset=0 \quad (2.3)$$

$$g(\emptyset X,\emptyset Y)=-g(X,Y)-\eta(X)\eta(Y) \quad (2.4)$$

for any X,Y tangent to $\bar{M}$. In this case

$$g(\emptyset X,Y)=-g(\emptyset Y,X). \quad (2.5)$$

An almost hyperbolic contact metric structure $(\emptyset,\xi,\eta,g)$ on $\bar{M}$ is called nearly hyperbolic cosymplectic manifold if and only if

$$(\nabla_X\emptyset)Y+(\nabla_Y\emptyset)X=0 \quad (2.6)$$

$$\nabla_X\xi=0 \quad (2.7)$$

for all X,Y tangent to $\bar{M}$, where ∇ is Riemannian connection $\bar{M}$.

Now, we define a semi-symmetric semi-metric connection

$$\bar{\nabla}_XY=\nabla_XY-\eta(X)Y+g(X,Y)\xi \quad (2.8)$$

such that $(\bar{\nabla}_Xg)(Y,Z)=2\eta(X)g(Y,Z)-\eta(Y)g(X,Z)-\ \eta(Z)g(X,Y)$

from (2.8), replacing $Y\ by\ \emptyset Y$, we have

$$\bar{\nabla}_X\emptyset Y=\nabla_X\emptyset Y-\eta(X)\emptyset Y+g(X,\emptyset Y)\xi$$

$$(\bar{\nabla}_X\emptyset)Y+\emptyset(\bar{\nabla}_XY)=(\nabla_X\emptyset)Y+\emptyset(\nabla_XY)-\eta(X)\emptyset Y+g(X,\emptyset Y)\xi$$

Interchanging X & Y, we have

$$(\bar{\nabla}_Y\emptyset)X+\emptyset(\bar{\nabla}_YX)=(\nabla_Y\emptyset)X+\emptyset(\nabla_YX)-\eta(Y)\emptyset X+\ g(Y,\emptyset X)\xi$$

Adding above two equations, we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset(\bar{\nabla}_X Y) + \emptyset(\bar{\nabla}_Y X) = (\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) - \eta(X)\emptyset Y$$
$$-\eta(Y)\emptyset X + g(X, \emptyset Y)\xi + g(Y, \emptyset X)\xi$$
$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset(\bar{\nabla}_X Y - \nabla_X Y) + \emptyset(\bar{\nabla}_Y X - \nabla_Y X) = (\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X$$
Using equation (2.6) in above, we have
$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset(\bar{\nabla}_X Y - \nabla_X Y) + \emptyset(\bar{\nabla}_Y X - \nabla_Y X) = -\eta(X)\emptyset Y - \eta(Y)\emptyset X$$
Using equation (2.8) in above, we have
$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset[\nabla_X Y - \eta(X)Y + g(X,Y)\xi - \nabla_X Y] + \emptyset[\nabla_Y X - \eta(Y)X + g(Y,X)\xi - \nabla_Y X]$$
$$= -\eta(X)\emptyset Y - \eta(Y)\emptyset X$$
$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -\eta(X)\emptyset Y - \eta(Y)\emptyset X + \eta(Y)\emptyset X + \eta(X)\emptyset Y - \emptyset[g(X,Y)\xi] - \emptyset[g(Y,X)\xi]$$

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = 0 \tag{2.9}$$
Now replacing Y by ξ in (2.8) we get $\bar{\nabla}_X \xi = \nabla_X \xi - \eta(X)\xi + g(X,\xi)\xi$
$$\bar{\nabla}_X \xi = 0 \tag{2.10}$$
An almost hyperbolic contact metric manifold with almost hyperbolic contact structure $(\emptyset, \xi, \eta, g)$ is called nearly hyperbolic Cosymplectic manifold with semi-symmetric semi-metric connection if it is satisfied (2.9) and (2.10).

III. CR-Submanifolds

Let M be submanifold immersed in $\bar{M}$, we assume that the vector ξ is tangent to M, denoted by $\{\xi\}$ the 1-dimentional distribution spanned by ξ on M, then M is called a CR-submanifold [14] of $\bar{M}$ if there exist two differentiable distribution D & $D^{\perp}$ on M satisfying

(i) $TM = D \oplus D^{\perp}$,

(ii) The distribution D is invariant under $\emptyset$ that is $\emptyset D_X = D_X$ for each $X \in M$,

(iii) The distribution $D^{\perp}$ is anti-invariant under $\emptyset$, that is $\emptyset D^{\perp}{}_X \subset T^{\perp}M$ for each $X \in M$,

Where TM & $T^{\perp}M$ be the Lie algebra of vector fields tangential & normal to M respectively. If $dim\, D^{\perp} = 0$ (resp. $dim\, D = 0$) then CR-submanifold is called an invariant (resp. anti-invariant) submanifold. The distribution D $(resp. D^{\perp})$ is called the horizontal (resp. vertical) distribution. Also the pair $D, D^{\perp}$ is called ξ–horizontal
(resp. ξ–vertical) if $\xi_x \in D_x$ $(resp\ \xi_x \in D^{\perp}{}_x)$.
Let Riemannian metric g induced on M and ∇^* is induced Levi-Civita connection on M then the Guass formula is given by
$$\nabla_X Y = \nabla_X^* Y + h(X,Y) \tag{3.1}$$
$$\nabla_X N = -A_N X + \nabla_X^{\perp} N \tag{3.2}$$
for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form & A_N is the Weingarten map associated with N as
$$g(A_N X, Y) = g(h(X,Y), N) \tag{3.3}$$
Any vector X tangent to M is given as
$$X = PX + QX \tag{3.4}$$
where $PX \in D\ and\ QX \in D^{\perp}$.
Similarly, for N normal to M, we have
$$\emptyset N = BN + CN \tag{3.5}$$
where BN $(resp. CN)$ is tangential component $(resp. normal\ component)$ of $\emptyset N$.
The Guass formula for a nearly hyperbolic cosymplectic manifold with semi-symmetric semi-metric connection is
$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3.6}$$
For Weingarten formula putting $Y = N$ in (2.8), we have
$$\bar{\nabla}_X N = \nabla_X N - \eta(X)N + g(X,N)\xi$$
$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N - \eta(X)N \tag{3.7}$$
for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N.

Some Basic lemmas

Lemma 3.1. Let M be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection, then
$$2(\bar{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X, \emptyset Y) - h(Y, \emptyset X) - \emptyset[X,Y]$$
$$2(\bar{\nabla}_Y \emptyset)X = \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y, \emptyset X) - h(X, \emptyset Y) + \emptyset[X,Y]$$
$for\ each\ X, Y \in D.$

Proof. By Gauss formulas (3.6), we have

$$\overline{\nabla}_X \emptyset Y = \nabla_X \emptyset Y + h(X, \emptyset Y)$$

Interchanging X *and* Y in above, we have

$$\overline{\nabla}_Y \emptyset X = \nabla_Y \emptyset X + h(Y, \emptyset X)$$

From above two equations, we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X, \emptyset Y) - h(Y, \emptyset X) \qquad (3.8)$$

Also, by covariant differentiation, we have

$$\overline{\nabla}_X \emptyset Y = (\overline{\nabla}_X \emptyset)Y + \emptyset(\overline{\nabla}_X Y)$$

Interchanging X *and* Y in above, we have

$$\overline{\nabla}_Y \emptyset X = (\overline{\nabla}_Y \emptyset)X + \emptyset(\overline{\nabla}_Y X)$$

from above two equations, we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = (\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X + \emptyset[X, Y] \qquad (3.9)$$

From (3.8) and (3.9), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X + \emptyset[X, Y] = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X, \emptyset Y) - h(Y, \emptyset X)$$

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X, \emptyset Y) - h(Y, \emptyset X) - \emptyset[X, Y] \qquad (3.10)$$

Adding (2.9) and (3.10), we obtain

$$2(\overline{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X, \emptyset Y) - h(Y, \emptyset X) - \emptyset[X, Y]$$

Subtracting equation (3.10) from (2.9), we have

$$2(\overline{\nabla}_Y \emptyset)X = \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y, \emptyset X) - h(X, \emptyset Y) + \emptyset[X, Y]$$

for each $X, Y \in D$.

Hence lemma is proved

Lemma 3.2. Let M be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X, Y] - \eta(X)\emptyset Y + \eta(Y)\emptyset X$$

$$2(\overline{\nabla}_Y \emptyset)X = A_{\emptyset Y} X - A_{\emptyset X} Y + \nabla_Y^{\perp} \emptyset X - \nabla_X^{\perp} \emptyset Y + \emptyset[X, Y] + \eta(X)\emptyset Y - \eta(Y)\emptyset X$$

for all $X, Y \in D^{\perp}$.

Proof. Using Weingarten formula (3.7), we have

$$\overline{\nabla}_X \emptyset Y = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset X - \eta(X)\emptyset Y$$

Interchanging *and* Y , we have

$$\overline{\nabla}_Y \emptyset X = -A_{\emptyset X} Y + \nabla_Y^{\perp} \emptyset X - \eta(Y)\emptyset X$$

From above two equations, we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X + \eta(Y)\emptyset X - \eta(X)\emptyset Y \qquad (3.11)$$

Comparing equation (3.9) and (3.11), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X + \emptyset[X, Y] = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X + \eta(Y)\emptyset X - \eta(X)\emptyset Y$$

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X, Y] + \eta(Y)\emptyset X - \eta(X)\emptyset Y \qquad (3.12)$$

Adding (2.9) and (3.12), we have

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X, Y] - \eta(X)\emptyset Y + \eta(Y)\emptyset X$$

Subtracting (3.12) from (2.9), we have

$$2(\overline{\nabla}_Y \emptyset)X = A_{\emptyset Y} X - A_{\emptyset X} Y + \nabla_Y^{\perp} \emptyset X - \nabla_X^{\perp} \emptyset Y + \emptyset[X, Y] + \eta(X)\emptyset Y - \eta(Y)\emptyset X$$

for all $X, Y \in D^{\perp}$.

Corollary 3.3. Let M be a $\xi-$horizontal CR-submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X} Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y^{\perp} \emptyset X - \emptyset[X, Y]$$

$$2(\overline{\nabla}_Y \emptyset)X = A_{\emptyset Y} X - A_{\emptyset X} Y + \nabla_Y^{\perp} \emptyset X - \nabla_X^{\perp} \emptyset Y + \emptyset[X, Y]$$

for all $X, Y \in D^{\perp}$.

Lemma 3.4. Let M be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y] - \eta(X)\emptyset Y - \eta(Y)\emptyset X$$

$$2(\overline{\nabla}_Y \emptyset)X = A_{\emptyset Y} X - \nabla_X^{\perp} \emptyset Y + \nabla_Y \emptyset X + h(Y, \emptyset X) + \emptyset[X, Y] - \eta(X)\emptyset Y - \eta(Y)\emptyset X$$

for all $X \in D$ *and* $Y \in D^{\perp}$.

Proof. By Gauss formulas (3.6), we have

$$\overline{\nabla}_Y \emptyset X = \nabla_Y \emptyset X + h(Y, \emptyset X)$$

Also, by Weingarten formula (3.7), we have

$$\overline{\nabla}_X \emptyset Y = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \eta(X)\emptyset Y$$

From above two equations, we hav

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \eta(X)\emptyset Y \qquad (3.13)$$

Comparing equation (3.9) and (3.13), we have

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset[X,Y] = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \eta(X)\emptyset Y$$

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \emptyset[X,Y] - \eta(X)\emptyset Y \quad (3.14)$$

Adding equation (2.9) & (3.14), we have

$$2(\bar{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \emptyset[X,Y] - \eta(X)\emptyset Y$$

Subtracting (3.14) from (2.9), we have

$$2(\bar{\nabla}_Y \emptyset)X = A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y \emptyset X + h(Y,\emptyset X) + \emptyset[X,Y] + \eta(X)\emptyset Y$$

$for\ all\ X \in D\ and\ Y \in D^{\perp}.$

Lemma 3.6. Let M be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$\emptyset P(\nabla_X Y) + \emptyset P(\nabla_Y X) = P(\nabla_X \emptyset PY) + P(\nabla_Y \emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y \quad (3.15)$$

$$2Bh(X,Y) = Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y \quad (3.16)$$

$$\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X,\emptyset PY) + h(Y,\emptyset PX) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX$$
$$-\eta(X)\emptyset QY - \eta(Y)\emptyset QX \quad (3.17)$$

$for\ all\ X,Y \in TM.$

Proof. From equation (3.4), we have

$$\emptyset Y = \emptyset PY + \emptyset QY$$

Differentiating covariantly with respect to vector, we have

$$\bar{\nabla}_X \emptyset Y = \bar{\nabla}_X(\emptyset PY + \emptyset QY)$$
$$\bar{\nabla}_X \emptyset Y = \bar{\nabla}_X \emptyset PY + \bar{\nabla}_X \emptyset QY$$
$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\bar{\nabla}_X Y) = \bar{\nabla}_X \emptyset PY + \bar{\nabla}_X \emptyset QY$$

Using equations (3.6) and (3.7) in above, we have

$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X,Y) = \nabla_X \emptyset PY + h(X,\emptyset PY) - A_{\emptyset QY}X + \nabla_X^{\perp}\emptyset QY - \eta(X)\emptyset QY \quad (3.18)$$

Interchanging X & Y, we have

$$(\bar{\nabla}_Y \emptyset)X + \emptyset(\nabla_Y X) + \emptyset h(Y,X) = \nabla_Y \emptyset PX + h(Y,\emptyset PX) - A_{\emptyset QX}Y + \nabla_Y^{\perp}\emptyset QX$$
$$-\eta(Y)\emptyset QX \quad (3.19)$$

Adding equations (3.18) & (3.19), we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX + h(X,\emptyset PY) + h(Y,\emptyset PX)$$
$$-A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX - \eta(X)\emptyset QY - \eta(Y)\emptyset QX \quad (3.20)$$

By Virtue of (2.9) & (3.19), we have

$$\emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX + h(X,\emptyset PY) + h(Y,\emptyset PX) - A_{\emptyset QX}Y - A_{\emptyset QY}X$$
$$+\nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX - \eta(X)\emptyset QY - \eta(Y)\emptyset QX$$

Using equations (3.4) and (3.5) in above, we have

$$\emptyset P(\nabla_X Y) + \emptyset Q(\nabla_X Y) + \emptyset P(\nabla_Y X) + \emptyset Q(\nabla_Y X)) + 2Bh(X,Y) + 2Ch(X,Y) = P(\nabla_X \emptyset PY) + Q(\nabla_X \emptyset PY) + P(\nabla_Y \emptyset PX) + Q(\nabla_Y \emptyset PX) + h(X,\emptyset PY) + h(Y,\emptyset PX) - PA_{\emptyset QY}X - QA_{\emptyset QY}X - PA_{\emptyset QX}Y - QA_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX - \eta(X)\emptyset QY - \eta(Y)\emptyset QX$$

Comparing horizontal, vertical and normal components, we get desired result.

IV. Parallel Distribution

Definition 4.1. The horizontal (resp. vertical) distribution D (resp. $D^{\perp}$) is said to be parallel [13] with respect to the connection on M if $\nabla_X Y \in D$ (resp. $\nabla_Z W \in D^{\perp}$) for any vector field $X,Y \in D$ (resp. $W,Z \in D^{\perp}$).

Theorem 4.2. Let M be a $\xi-$vertical CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection. If horizontal distribution D is parallel, then

$$h(X,\emptyset Y) = h(Y,\emptyset X) \quad (4.1)$$

$for\ all\ X,Y \in D.$

Proof. Let $X,Y \in D$, as D is parallel distribution, then

$$\nabla_X \emptyset Y \in D \quad \& \quad \nabla_Y \emptyset X \in D.$$

Then, from (3.16), we have

$$2Bh(X,Y) = Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y$$

As Q being a projection operator on $D^{\perp}$ then we have

$$2Bh(X,Y) = 0 \quad (4.2)$$

From equations (3.5) and (4.2), we have

$$\emptyset h(X,Y) = Ch(X,Y) \qquad for\ all\ X,Y \in D \quad (4.3)$$

Now, from equation (3.17), we have

$$\emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X,\emptyset PY) + h(Y,\emptyset PX) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX - \eta(X)\emptyset QY - \eta(Y)\emptyset QX$$

As Q being a projection operator on $D^{\perp}$ then we have

$$2Ch(X,Y) = h(X,\emptyset PY) + h(Y,\emptyset PX)$$

Using equation (4.3) in above, we have

$$h(X,\emptyset PY) + h(Y,\emptyset PX) = 2\emptyset h(X,Y) \quad (4.4)$$

Replacing X by $\emptyset X$ in (4.4), we have

$$h(\emptyset X,\emptyset Y) + h(Y,\emptyset^2 X) = 2\emptyset h(\emptyset X,Y)$$

Using equation (2.1) in above, we have

$$h(\emptyset X,\emptyset Y) + h(Y,X) = 2\emptyset h(\emptyset X,Y) \quad (4.5)$$

Replacing Y by $\emptyset Y$ & using (2.1) in (4.4), we have

$$h(X,Y) + h(\emptyset Y,\emptyset X) = 2\emptyset h(X,\emptyset Y) \quad (4.6)$$

By Virtue of (4.5) and (4.6), we have

$$h(X,\emptyset Y) = h(Y,\emptyset X)$$

$for\ all\ X,Y \in D.$

Theorem 4.3. Let M be a $\xi-$vertical CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection. If horizontal distribution D is parallel with respect to connection on M, then

$$A_{\emptyset X}Y + A_{\emptyset Y}X \in D^{\perp} \quad (4.7)$$

$for\ all\ X,Y \in D^{\perp}.$

Proof: Let $X,Y \in D^{\perp}$, from Weingarten formula (3.7), we have

$$\bar{\nabla}_X \emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \eta(X)\emptyset Y$$

$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\bar{\nabla}_X Y) = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \eta(X)\emptyset Y$$

Using Gauss equation (3.6) in above, we have

$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X,Y) = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \eta(X)\emptyset Y$$

$$(\bar{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \emptyset(\nabla_X Y) - \emptyset h(X,Y) - \eta(X)\emptyset Y \quad (4.8)$$

Interchanging X & Y in (4.8), we have

$$(\bar{\nabla}_Y \emptyset)X = -A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \eta(Y)\emptyset X - \emptyset(\nabla_Y X) - \emptyset h(Y,X) \quad (4.9)$$

Adding equations (4.8) and (4.9), we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X - \emptyset(\nabla_X Y) - \emptyset(\nabla_Y X) - \eta(X)\emptyset Y - \eta(Y)\emptyset X - 2\emptyset h(X,Y) \quad (4.10)$$

Using equation (2.9) in (4.10), we have

$$0 = -A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X - \emptyset(\nabla_X Y) - \emptyset(\nabla_Y X) - \eta(X)\emptyset Y - \eta(Y)\emptyset X - 2\emptyset h(X,Y) \quad (4.11)$$

Taking inner product with $Z \in D$ in (4.11), we have

$$0 = -g(A_{\emptyset Y}X,Z) - g(A_{\emptyset X}Y,Z) + g(\nabla_X^{\perp}\emptyset Y,Z) + g(\nabla_Y^{\perp}\emptyset X,Z) - g(\emptyset(\nabla_X Y),Z) - g(\emptyset(\nabla_Y X),Z) - 2g(\emptyset h(X,Y),Z) - \eta(X)g(\emptyset Y,Z) - \eta(Y)g(\emptyset X,Z)$$

$$g(A_{\emptyset Y}X + A_{\emptyset X}Y,Z) = -g(\emptyset(\nabla_X Y),Z) - g(\emptyset(\nabla_Y X),Z) \quad (4.12)$$

If $D^{\perp}$ is parallel then $\nabla_X Y \in D^{\perp}$ and $\nabla_Y X \in D^{\perp}$, so from equation (4.12), we have

$$g(A_{\emptyset Y}X + A_{\emptyset X}Y,Z) = 0$$

Consequently, we have

$$A_{\emptyset Y}X + A_{\emptyset X}Y \in D^{\perp} \qquad for\ all\ X,Y \in D^{\perp}$$

Hence lemma is proved.

Definition 4.4. A CR-submanifold is said to be mixed totally geodesic if $h(X,Y) = 0,\ for\ all\ X \in D\ and\ Y \in D^{\perp}$.

Definition 4.5. A Normal vector field $N \neq 0$ is called $D-parallel$ normal section if $\nabla_X^{\perp}N = 0$,for all $X \in D$.

Lemma 4.6. Let M be a CR-submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection. Then M is a mixed totally geodesic if and only if $A_N X \in D\ for\ all\ X \in D$.

Proof. Let $A_N X \in D\ for\ all\ X \in D$.

Now, from equation (3.3), we have

$g(h(X,Y),N) = g(A_N X,Y) = 0, \qquad for\ Y \in D^{\perp}.$

Which is equivalent to $\quad h(X,Y) = 0$

Hence M is totally mixed geodesic.

Conversely, Let M is totally mixed geodesic.

That is $\quad h(X,Y) = 0 \qquad for\ X \in D\ and\ Y \in D^{\perp}.$

Now, from equation (3.3), we have
Now, $g(h(X,Y),N) = g(A_N X, Y)$.
This implies that $g(A_N X, Y) = 0$
Consequently, we have

$$A_N X \in D, \ \ for\ all\ Y \in D^{\perp}$$

Hence lemma is proved.

Theorem 4.7. Let M be a mixed totally geodesic CR-submanifold of a nearly hyperbolic cosymplectic $\bar{M}$ with semi-symmetric semi-metric connection. Then the normal section $N \in \emptyset D^{\perp}$ is $D\ parallel$ if and only if

$$\nabla_X \emptyset N \in D\ for\ all\ X \in D\ \&\ Y \in D^{\perp}.$$

Proof. Let $\in \emptyset D^{\perp},\ X \in D\ and\ Y \in D^{\perp}$, then from (3.16), we have

$$2Bh(X,Y) = Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y$$

For mixed totally geodesic, we have from above equation

$$Q\nabla_Y \emptyset X = 0, \qquad for\ X \in D.$$

In particular, we have

$$Q\nabla_Y X = 0. \tag{4.13}$$

Using it in (3.17), we have $\emptyset Q \nabla_X Y = \nabla_X^{\perp} \emptyset Y.$

$$\emptyset Q \nabla_X \emptyset N = \nabla_X^{\perp} N \tag{4.14}$$

Thus, if the normal section $N \neq 0$ is D-parallel, then using "**definition 4.5**" and (4.14), we have

$$\emptyset \nabla_X (\emptyset N) = 0.$$

Which is equivalent to

$$\nabla_X \emptyset N \in D, \qquad for\ all\ X \in D.$$

The converse part easily follows from (4.14). This completes the proof of the theorem.
Hence the theorem is proved.

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