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Semi-Invariant Submanifolds of a Nearly Hyperbolic Cosymplectic Manifold With Semi-Symmetric Semi-Metric Connection

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Abstract: *We consider a nearly hyperbolic cosymplectic manifold and study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold admitting semi-symmetric semi-metric connection. We also find the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold with semi-symmetric semi-metric connection and study parallel distributions on nearly hyperbolic cosymplectic manifold with semi-symmetric semi-metric connection.*
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I. Introduction

A semi invariant submanifold is the extension of the concept of a CR-submanifold of a Kaehler manifold to submanifolds of almost contact manifolds. CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was initiated by A. Bejancu in [9]. A. Bejancu[9] also initiated a new class of submanifold of a complex manifold which he called CR-submanifold and obtained some interesting results. The notion of semi invariant submanifolds of Sasakian manifolds was initiated by Bejancu-Papaghuic in [10]. The study of CR-submanifolds of Sasakian manifold was studies by C.J.Hsu in [12]. Semi-invariant submanifolds in almost contact manifold was enriched by several geometers (see, [1], [4], [5], [6], [8], [15]). On the otherhand, Ahmad M. and Ali K., studied semi-invariant submanifolds of a nearly hyperbolic cosymplectic in [2]. In this paper, we study semi-invariant submanifolds of a nearly hyperbolic cosymplectic manifold with semi-symmetric semi-metric connection.

The paper is organized as follows. In section II, we give a brief introduction of nearly hyperbolic cosymplectic manifold. In section III, Some properties of semi invariant submanifolds of a nearly hyperbolic cosymplectic manifold with semi-symmetric semi-metric connection are investigated. We also study parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold with semi-symmetric semi-metric connection.

In section IV, we discuss the integrability conditions of some distributions on nearly hyperbolic cosymplectic manifold with semi-symmetric semi-metric connection.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic Contact metric manifold with the almost hyperbolic contact metric structure$(\emptyset, \xi, \eta, g)$, where a tensor $\emptyset$ of type $(1,1)$ a vector field ξ, called structure vector field and η, the dual 1-form of ξ and the associated Riemannian metric g satisfying the following

$$\emptyset^2 X = X + \eta(X)\xi \quad (2.1)$$

$$\eta(\xi) = -1, \qquad g(X,\xi) = \eta(X) \quad (2.2)$$

$$\emptyset(\xi) = 0, \qquad \eta o\emptyset = 0 \quad (2.3)$$

$$g(\emptyset X, \emptyset Y) = -g(X,Y) - \eta(X)\eta(Y) \quad (2.4)$$

For any X, Y tangent to $\bar{M}$ [16]. In this case

$$g(\emptyset X, Y) = -g(\emptyset Y, X). \quad (2.5)$$

An almost hyperbolic contact metric structure $(\emptyset, \xi, \eta, g)$ on $\bar{M}$ is called nearly hyperbolic cosymplectic manifold [10] if and only if

$$(\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X = 0 \quad (2.6)$$

$$\nabla_X \xi = 0 \quad (2.7)$$

for all X, Y tangent to $\bar{M}$,where ∇ is Riemannian connection $\bar{M}$.

Now, we define a semi-symmetric semi-metric connection

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi \quad (2.8)$$

Such that

$$(\bar{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y) - \eta(Z)g(X,Y)$$

Replacing Y by ∅Y,in equation (2.8) we have

$$\overline{\nabla}_X \emptyset Y = \nabla_X \emptyset Y - \eta(X)\emptyset Y + g(X, \emptyset Y)\xi$$

$$(\overline{\nabla}_X \emptyset)Y + \emptyset(\overline{\nabla}_X Y) = (\nabla_X \emptyset)Y + \emptyset(\nabla_X Y) - \eta(X)\emptyset Y + g(X, \emptyset Y)\xi$$

Interchanging X & Y, we have

$$(\overline{\nabla}_Y \emptyset)X + \emptyset(\overline{\nabla}_Y X) = (\nabla_Y \emptyset)X + \emptyset(\nabla_Y X) - \eta(Y)\emptyset X + g(Y, \emptyset X)\xi$$

Adding above two equations, we have

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X + \emptyset(\overline{\nabla}_X Y) + \emptyset(\overline{\nabla}_Y X) = (\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) - \eta(X)\emptyset Y - \eta(Y)\emptyset X + g(Y, \emptyset X)\xi + g(X, \emptyset Y)\xi$$

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X + \emptyset(\overline{\nabla}_X Y - \nabla_X Y) + \emptyset(\overline{\nabla}_Y X - \nabla_Y X) = (\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + g(Y, \emptyset X)\xi + g(X, \emptyset Y)\xi$$

Using equation (2.6) & (2.8) in above, we have

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X = 0 \qquad (2.9)$$

Now replacing Y by ξ in (2.8) we get

$$\overline{\nabla}_X \xi = \nabla_X \xi - \eta(X)\xi + g(X, \xi)\xi$$

$$\overline{\nabla}_X \xi = 0 \qquad (2.10)$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure $(\emptyset, \xi, \eta, g)$ is called nearly hyperbolic Cosymplectic manifold with semi-symmetric semi-metric connection if it is satisfied (2.9) & (2.10).

III. Semi-invariant Sub manifold

Let M be submanifold immersed in $\overline{M}$, we assume that the vector ξ is tangent to M, denoted by {ξ} the 1-dimentional distribution spanned by ξ on M, then M is called a semi-invariant sub manifold [8] of $\overline{M}$ if there exist two differentiable distribution D & $D^{\perp}$ on M satisfying

(i) $TM = D \oplus D^{\perp} \oplus \xi$, where D , $D^{\perp}$ & ξ are mutually orthogonal to each other.

(ii) The distribution D is invariant under ∅ that is $\emptyset D_X = D_X$ for each $X \in M$,

(iii) The distribution $D^{\perp}$ is anti-invariant under ∅, that is $\emptyset {D^{\perp}}_X \subset T^{\perp}M$ for each $X \in M$,

Where TM & $T^{\perp}M$ be the Lie algebra of vector fields tangential & normal to M respectively.

Let Riemannian metric g and ∇ be induced Levi-Civita connection on M then the Guass formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad (3.1)$$

For Weingarten formula putting Y = N in (2.8), we have

$$\overline{\nabla}_X N = \nabla_X N - \eta(X)N + g(X, N)$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \qquad (3.2)$$

For any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form & A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X, Y), N) \qquad (3.3)$$

Any vector X tangent to M is given as

$$X = PX + QX + \eta(X)\xi \qquad (3.4)$$

Where $PX \in D$ and $QX \in D^{\perp}$.

Similarly, for N normal to M, we have

$$\emptyset N = BN + CN \qquad (3.5)$$

Where BN (resp. CN) is tangential component (resp. normal component) of ∅N.

Using the semi-symmetric non-metric connection the Nijenhuis tensor is expressed as

$$N(X, Y) = (\overline{\nabla}_{\emptyset X} \emptyset)Y - (\overline{\nabla}_{\emptyset Y} \emptyset)X - \emptyset(\overline{\nabla}_X \emptyset)Y + \emptyset(\overline{\nabla}_Y \emptyset)X \qquad (3.6)$$

Now from (2.9) replacingX by ∅X, we have

$$(\overline{\nabla}_{\emptyset X} \emptyset)Y + (\overline{\nabla}_Y \emptyset)\emptyset X = 0 \qquad (3.7)$$

From (2.1) again,

$$\emptyset^2 X = X + \eta(X)\xi$$

$$\emptyset(\emptyset X) = X + \eta(X)\xi$$

Differentiating conveniently along the vector, we have

$$\overline{\nabla}_Y\{\emptyset(\emptyset X)\} = \overline{\nabla}_Y\{X + \eta(X)\xi\}$$

$$(\overline{\nabla}_Y \emptyset)\emptyset X + \emptyset(\overline{\nabla}_Y \emptyset X) = \overline{\nabla}_Y X + (\overline{\nabla}_Y \eta)(X)\xi + \eta(\overline{\nabla}_Y X)\xi + \eta(X)\overline{\nabla}_Y \xi$$

Using equation (2.10) in above, we have

$$(\overline{\nabla}_Y \emptyset)\emptyset X + \emptyset\{(\overline{\nabla}_Y \emptyset)X + \emptyset(\overline{\nabla}_Y X)\} = \overline{\nabla}_Y X + (\overline{\nabla}_Y \eta)(X)\xi + \eta(\overline{\nabla}_Y X)\xi$$

$$(\overline{\nabla}_Y \emptyset)\emptyset X + \emptyset(\overline{\nabla}_Y \emptyset)X + \emptyset^2(\overline{\nabla}_Y X) = \overline{\nabla}_Y X + (\overline{\nabla}_Y \eta)(X)\xi + \eta(\overline{\nabla}_Y X)\xi$$

$$(\overline{\nabla}_Y \emptyset)\emptyset X + \emptyset(\overline{\nabla}_Y \emptyset)X + \overline{\nabla}_Y X + \eta(\overline{\nabla}_Y X)\xi = \overline{\nabla}_Y X + (\overline{\nabla}_Y \eta)(X)\xi + \eta(\overline{\nabla}_Y X)\xi$$

$$(\overline{\nabla}_Y \emptyset)\emptyset X = (\overline{\nabla}_Y \eta)(X)\xi - \emptyset(\overline{\nabla}_Y \emptyset)X \qquad (3.8)$$

From (3.7) & (3.8), we have

$$(\overline{\nabla}_{\emptyset X}\emptyset)Y = -(\overline{\nabla}_Y\eta)(X)\xi + \emptyset(\overline{\nabla}_Y\emptyset)X \quad (3.9)$$

Interchanging X & Y, we have

$$(\overline{\nabla}_{\emptyset Y}\emptyset)X = -(\overline{\nabla}_X\eta)(Y)\xi + \emptyset(\overline{\nabla}_X\emptyset)Y \quad (3.10)$$

Using equation (3.9), (3.10) in (3.6), we have

$$N(X,Y) = (\overline{\nabla}_X\eta)(Y)\xi - (\overline{\nabla}_Y\eta)(X)\xi + \emptyset(\overline{\nabla}_Y\emptyset)X - \emptyset(\overline{\nabla}_X\emptyset)Y - \emptyset(\overline{\nabla}_X\emptyset)Y + \emptyset(\overline{\nabla}_Y\emptyset)X$$

$$N(X,Y) = (\overline{\nabla}_X\eta)(Y)\xi - (\overline{\nabla}_Y\eta)(X)\xi - 2\emptyset(\overline{\nabla}_X\emptyset)Y + 2\emptyset(\overline{\nabla}_Y\emptyset)X$$

$$N(X,Y) = 2d\eta(X,Y)\xi + 4\emptyset(\overline{\nabla}_Y\emptyset)X - 2\emptyset(\overline{\nabla}_Y\emptyset)X - 2\emptyset(\overline{\nabla}_X\emptyset)Y$$

$$N(X,Y) = 2g(\emptyset X,Y)\xi + 4\emptyset(\overline{\nabla}_Y\emptyset)X - 2\emptyset\{(\overline{\nabla}_Y\emptyset)X + (\overline{\nabla}_X\emptyset)Y\}$$

Using equation (2.9), we have

$$N(X,Y) = 2g(\emptyset X,Y)\xi + 4\emptyset(\overline{\nabla}_Y\emptyset)X \quad (3.11)$$

As we know,

$$(\overline{\nabla}_Y\emptyset)X = \overline{\nabla}_Y\emptyset X - \emptyset(\overline{\nabla}_Y X)$$

Using Guass formula, we have

$$(\overline{\nabla}_Y\emptyset)X = \nabla_Y\emptyset X + h(Y,\emptyset X) - \emptyset(\nabla_Y X + h(Y,X))$$

$$(\overline{\nabla}_Y\emptyset)X = \nabla_Y\emptyset X + h(Y,\emptyset X) - \emptyset(\nabla_Y X) - \emptyset h(Y,X)$$

$$\emptyset(\overline{\nabla}_Y\emptyset)X = \emptyset(\nabla_Y\emptyset X) + \emptyset h(Y,\emptyset X) - \emptyset^2(\nabla_Y X) - \emptyset^2\, h(Y,X)$$

$$\emptyset(\overline{\nabla}_Y\emptyset)X = \emptyset(\nabla_Y\emptyset X) + \emptyset h(Y,\emptyset X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(Y,X) - \eta\big(h(Y,X)\big)\xi$$

$$\emptyset(\overline{\nabla}_Y\emptyset)X = \emptyset(\nabla_Y\emptyset X) + \emptyset h(Y,\emptyset X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(Y,X) \quad (3.12)$$

Using equation (3.12) in (3.11)), we have

$$N(X,Y) = 4\emptyset(\nabla_Y\emptyset X) + 4\emptyset h(Y,\emptyset X) - 4(\nabla_Y X) - 4\eta(\nabla_Y X)\xi - 4h(Y,X) + 2g(\emptyset X,Y)\xi \quad (3.13)$$

Lemma 3.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X\emptyset)Y = \nabla_X\emptyset Y - \nabla_Y\emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y].$$

for each $X, Y \in D$.

Proof. By Gauss formulas (3.1), we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

Replacing by $\emptyset Y$, we have

$$\overline{\nabla}_X\emptyset Y = \nabla_X\emptyset Y + h(X,\emptyset Y)$$

Similarly,

$$\overline{\nabla}_Y\emptyset X = \nabla_Y\emptyset X + h(Y,\emptyset X)$$

From above two equations, we have

$$\overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = \nabla_X\emptyset Y - \nabla_Y\emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) \quad (3.14)$$

Also, by covariant differentiation, we have

$$\overline{\nabla}_X\emptyset Y = (\overline{\nabla}_X\emptyset)Y + \emptyset(\overline{\nabla}_X Y)$$

Similarly,

$$\overline{\nabla}_Y\emptyset X = (\overline{\nabla}_Y\emptyset)X + \emptyset(\overline{\nabla}_X X)$$

From above two equations, we have

$$\overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = (\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X + \emptyset[X,Y] \quad (3.15)$$

From (3.14) and (3.15), we have

$$(\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X + \emptyset[X,Y] = \nabla_X\emptyset Y - \nabla_Y\emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X)$$

$$(\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X = \nabla_X\emptyset Y - \nabla_Y\emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y] \quad (3.16)$$

Adding (2.9) and (3.16), we obtain

$2(\overline{\nabla}_X\emptyset)Y = \nabla_X\emptyset Y - \nabla_Y\emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y].$

for each $X, Y \in D$.

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric non-metric connection, then

$$2(\overline{\nabla}_Y\emptyset)X = \nabla_Y\emptyset X - \nabla_X\emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + \emptyset[X,Y]$$

for each $X, Y \in D$.

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric non-metric connection, then

$$2(\overline{\nabla}_X\emptyset)Y = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.

Proof. Using Weingarten formula (3.2), we have

$$\overline{\nabla}_X\emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y$$

Interchanging X & Y , we have

$$\overline{\nabla}_Y\emptyset X = -A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X$$

From above two equations, we have

$$\overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X \quad (3.17)$$

Comparing equation (3.15) & (3.17), we have

$$(\overline{\nabla}_X\emptyset)Y-(\overline{\nabla}_Y\emptyset)X+\emptyset[X,Y]=A_{\emptyset X}Y-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y^{\perp}\emptyset X$$

$$(\overline{\nabla}_X\emptyset)Y-(\overline{\nabla}_Y\emptyset)X=A_{\emptyset X}Y-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y^{\perp}\emptyset X-\emptyset[X,Y] \qquad (3.18)$$

Adding (2.9) & (3.18), we have

$$2(\overline{\nabla}_X\emptyset)Y=A_{\emptyset X}Y-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y^{\perp}\emptyset X-\emptyset[X,Y]$$

for all $X,Y\in D^{\perp}$.

Lemma 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_Y\emptyset)X=A_{\emptyset Y}X-A_{\emptyset X}Y+\nabla_Y^{\perp}\emptyset X-\nabla_X^{\perp}\emptyset Y+\emptyset[X,Y]$$

for all $X,Y\in D^{\perp}$.

Lemma 3.5. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X\emptyset)Y=-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y\emptyset X-h(Y,\emptyset X)-\emptyset[X,Y]$$

for all $X\in D$ and $Y\in D^{\perp}$.

Proof. By Gauss formulas (3.1), we have

$$\overline{\nabla}_Y\emptyset X=\nabla_Y\emptyset X+h(Y,\emptyset X)$$

Also, by Weingarten formula (3.2), we have

$$\overline{\nabla}_X\emptyset Y=-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y$$

From above two equations, we have

$$\overline{\nabla}_X\emptyset Y-\overline{\nabla}_Y\emptyset X=-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y\emptyset X-h(Y,\emptyset X) \qquad (3.19)$$

Comparing equation (3.15) and (3.19), we have

$$(\overline{\nabla}_X\emptyset)Y-(\overline{\nabla}_Y\emptyset)X+\emptyset[X,Y]=-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y\emptyset X-h(Y,\emptyset X)$$

$$(\overline{\nabla}_X\emptyset)Y-(\overline{\nabla}_Y\emptyset)X=-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y\emptyset X-h(Y,\emptyset X)-\emptyset[X,Y] \qquad (3.20)$$

Adding equation (2.9) & (3.20), we get

$$2(\overline{\nabla}_X\emptyset)Y=-A_{\emptyset Y}X+\nabla_X^{\perp}\emptyset Y-\nabla_Y\emptyset X-h(Y,\emptyset X)-\emptyset[X,Y]$$

for all $X\in D$ and $Y\in D^{\perp}$.

Lemma 3.6. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_Y\emptyset)X=A_{\emptyset Y}X-\nabla_X^{\perp}\emptyset Y+\nabla_Y\emptyset X+h(Y,\emptyset X)+\emptyset[X,Y]$$

for all $X\in D$ and $Y\in D^{\perp}$.

Lemma 3.7. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$P(\nabla_X\emptyset PY)+P(\nabla_Y\emptyset PX)-PA_{\emptyset QY}X-PA_{\emptyset QX}Y=\emptyset P(\nabla_XY)+\emptyset P(\nabla_YX) \qquad (3.21)$$

$$Q(\nabla_X\emptyset PY)+Q(\nabla_Y\emptyset PX)-QA_{\emptyset QY}X-QA_{\emptyset QX}Y=2Bh(X,Y) \qquad (3.22)$$

$$h(X,\emptyset PY)+h(Y,\emptyset PX)+\nabla_X^{\perp}\emptyset QY+\nabla_Y^{\perp}\emptyset QX=\emptyset Q(\nabla_XY)+\emptyset Q(\nabla_YX)+2Ch(X,Y) \qquad (3.23)$$

$$\eta(\nabla_X\emptyset PY)+\eta(\nabla_Y\emptyset PX)-\eta\left(A_{\emptyset QY}X\right)-\eta\left(A_{\emptyset QX}Y\right)=0 \qquad (3.24)$$

for all $X,Y\in TM$.

Proof. From equation (3.4), we have

$$\emptyset Y=\emptyset PY+\emptyset QY+\eta(Y)\emptyset\xi$$

Using equation (2.3), we have

$$\emptyset Y=\emptyset PY+\emptyset QY$$

Differentiating covariantly with respect to vector, we have

$$\overline{\nabla}_X\emptyset Y=\overline{\nabla}_X(\emptyset PY+\emptyset QY)$$

$$\overline{\nabla}_X\emptyset Y=\overline{\nabla}_X\emptyset PY+\overline{\nabla}_X\emptyset QY$$

$$(\overline{\nabla}_X\emptyset)Y+\emptyset(\overline{\nabla}_XY)=\overline{\nabla}_X\emptyset PY+\overline{\nabla}_X\emptyset QY$$

Using equations (3.1) and (3.2), we have

$$(\overline{\nabla}_X\emptyset)Y+\emptyset(\nabla_XY)+\emptyset h(X,Y)=\nabla_X\emptyset PY+h(X,\emptyset PY)-A_{\emptyset QY}X+\nabla_X^{\perp}\emptyset QY \qquad (3.25)$$

Interchanging X & Y, we have

$$(\overline{\nabla}_Y\emptyset)X+\emptyset(\nabla_YX)+\emptyset h(Y,X)=\nabla_Y\emptyset PX+h(Y,\emptyset PX)-A_{\emptyset QX}Y+\nabla_Y^{\perp}\emptyset QX \qquad (3.26)$$

Adding equations (3.25) & (3.26), we have

$$(\overline{\nabla}_X\emptyset)Y+(\overline{\nabla}_Y\emptyset)X+\emptyset(\nabla_XY)+\emptyset(\nabla_YX)+2\emptyset h(X,Y)=\nabla_X\emptyset PY+\nabla_Y\emptyset PX+h(X,\emptyset PY)+$$
$$h(Y,\emptyset PX)--A_{\emptyset QY}X-A_{\emptyset QX}Y+\nabla_X^{\perp}\emptyset QY+\nabla_Y^{\perp}\emptyset QX \qquad (3.27)$$

By Virtue of (2.9) & (3.27), we have

$$\emptyset(\nabla_XY)+\emptyset(\nabla_YX)+2\emptyset h(X,Y)=\nabla_X\emptyset PY+\nabla_Y\emptyset PX+h(X,\emptyset PY)+h(Y,\emptyset PX)$$
$$-A_{\emptyset QY}X-A_{\emptyset QX}Y+\nabla_X^{\perp}\emptyset QY+\nabla_Y^{\perp}\emptyset QX$$

Using equations (3.4) & (3.5), we have

$$\emptyset P(\nabla_XY)+\emptyset Q(\nabla_XY)+\eta(\nabla_XY)\emptyset\xi+\emptyset P(\nabla_YX)+\emptyset Q(\nabla_YX)+\eta(\nabla_YX)\emptyset\xi$$

$$+2Bh(X,Y) + 2Ch(X,Y) = P(\nabla_X \emptyset PY) + Q(\nabla_X \emptyset PY) + \eta(\nabla_X \emptyset PY)\xi + P(\nabla_Y \emptyset PX)$$
$$+Q(\nabla_Y \emptyset PX) + \eta(\nabla_Y \emptyset PX)\xi + h(X,\emptyset PY) + h(Y,\emptyset PX) - PA_{\emptyset QY}X - QA_{\emptyset QY}X$$
$$-\eta\left(A_{\emptyset QY}X\right)\xi - PA_{\emptyset QX}Y - QA_{\emptyset QX}Y - \eta\left(A_{\emptyset QX}Y\right)\xi + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX$$

Using equation (2.3), we have

$$\emptyset P(\nabla_X Y) + \emptyset Q(\nabla_X Y) + \emptyset P(\nabla_Y X)$$
$$+\emptyset Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(X,Y) = P(\nabla_X \emptyset PY) + Q(\nabla_X \emptyset PY) + \eta(\nabla_X \emptyset PY)\xi$$
$$+P(\nabla_Y \emptyset PX) + Q(\nabla_Y \emptyset PX) + \eta(\nabla_Y \emptyset PX)\xi + h(X,\emptyset PY) + h(Y,\emptyset PX) - PA_{\emptyset QY}X$$
$$-QA_{\emptyset QY}X - \eta\left(A_{\emptyset QY}X\right)\xi - PA_{\emptyset QX}Y - QA_{\emptyset QX}Y - \eta\left(A_{\emptyset QX}Y\right)\xi + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX$$

Comparing horizontal, vertical and normal components we get

$$P(\nabla_X \emptyset PY) + P(\nabla_Y \emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y = \emptyset P(\nabla_X Y) + \emptyset P(\nabla_Y X)$$
$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y = 2Bh(X,Y)$$
$$h(X,\emptyset PY) + h(Y,\emptyset PX) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX = \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y)$$
$$\eta(\nabla_X \emptyset PY) + \eta(\nabla_Y \emptyset PX) - \eta\left(A_{\emptyset QY}X\right) - \eta\left(A_{\emptyset QX}Y\right) = 0$$

for all $X, Y \in TM$.

Definition 3.8. The horizontal distribution D is said to be parallel [10] on M if $\nabla_X Y \in D$, for all $X, Y \in D$.

Theorem 3.9. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection. If horizontal distribution D is parallel, then

$$h(X,\emptyset Y) = h(Y,\emptyset X)$$

for all $X, Y \in D$.

Proof. Let $X, Y \in D$, as D is parallel distribution, then

$$\nabla_X \emptyset Y \in D \;\&\; \nabla_Y \emptyset X \in D.$$

Then, from (3.22) and (3.23), we have

$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y + h(X,\emptyset PY) + h(Y,\emptyset PX) + \nabla_X^{\perp}\emptyset QY$$
$$+\nabla_Y^{\perp}\emptyset QX = \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(X,Y)$$

As Q being a projection operator on $D^{\perp}$ then we have

$$h(X,\emptyset Y) + h(Y,\emptyset X) = 2Bh(X,Y) + 2Ch(X,Y)$$
$$h(X,\emptyset Y) + h(Y,\emptyset X) = 2\emptyset h(X,Y) \quad (3.28)$$

Replacing X by $\emptyset X$ in (3.28), we have

$$h(\emptyset X,\emptyset Y) + h(Y,\emptyset^2 X) = 2\emptyset h(\emptyset X,Y)$$

Using equation (2.1) in above, we have

$$h(\emptyset X,\emptyset Y) + h(Y,X) + \eta(X)h(Y,\xi) = 2\emptyset h(\emptyset X,Y)$$
$$h(\emptyset X,\emptyset Y) + h(Y,X) = 2\emptyset h(\emptyset X,Y) \quad (3.29)$$

Replacing Y by $\emptyset Y$ & using (2.1) in (3.28), we have

$$h(X,Y) + h(\emptyset Y,\emptyset X) = 2\emptyset h(X,\emptyset Y) \quad (3.30)$$

By Virtue of (3.29) and (3.30), we have

$$h(X,\emptyset Y) = h(Y,\emptyset X)$$

for all $X, Y \in D$.

Definition 3.10. A semi-invariant submanifold is said to be mixed totally geodesic [8] if $h(X,Y) = 0$, for all $X \in D$ and $Y \in D^{\perp}$.

Theorem 3.11. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection. Then M is a mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.

Proof. Let $A_N X \in D$ for all $X \in D$.
Now, $g(h(X,Y),N) = g(A_N X,Y) = 0$, for $Y \in D^{\perp}$.
Which is equivalent to $h(X,Y) = 0$.
Hence M is totally mixed geodesic.
Conversely, Let M is totally mixed geodesic.
That is $h(X,Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$.
Now, $g(h(X,Y),N) = g(A_N X,Y)$.
This implies that $g(A_N X,Y) = 0$
Consequently, we have

$$A_N X \in D, \text{ for all } Y \in D^{\perp}$$

IV. Integrability Condition of Distribution

Theorem 4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric non-metric connection, then the distribution $D\oplus\langle\xi\rangle$ is integrable if

$$h(X, \emptyset Z) = h(\emptyset X, Z) \tag{4.1}$$

for each $X, Y, Z \in (D\oplus\langle\xi\rangle)$.

Proof. The torsion tensor $S(X, Y)$ of an almost hyperbolic contact manifold is given by

$$S(X, Y) = N(X, Y) + 2d\eta(X, Y)\xi$$

Where $N(X, Y)$ is Neijenhuis tensor

If $(D\oplus\langle\xi\rangle)$ is integrable, then $N(X, Y) = 0$, for any $X, Y \in (D\oplus\langle\xi\rangle)$

Hence from (3.13), we have

$$4\emptyset(\nabla_Y \emptyset X) + 4\emptyset h(Y, \emptyset X) - 4(\nabla_Y X) - 4\eta(\nabla_Y X)\xi - 4h(Y, X) + 2g(\emptyset X, Y)\xi = 0 \tag{4.2}$$

Comparing normal part both side of (4.2), we have

$$4\emptyset Q(\nabla_Y \emptyset X) - 4h(Y, X) + 4Ch(Y, \emptyset X) = 0$$

$$\emptyset Q(\nabla_Y \emptyset X) - h(Y, X) + Ch(Y, \emptyset X) = 0, \tag{4.3}$$

For $X, Y \in (D\oplus\langle\xi\rangle)$

Replacing Y by $\emptyset Z$, where $Z \in D$ in (4.3), we have

$$\emptyset Q(\nabla_{\emptyset Z} \emptyset X) - h(\emptyset Z, X) + Ch(\emptyset Z, \emptyset X) = 0 \tag{4.4}$$

Interchanging X and Z, we have

$$\emptyset Q(\nabla_{\emptyset X} \emptyset Z) - h(\emptyset X, Z) + Ch(\emptyset X, \emptyset Z) = 0 \tag{4.5}$$

Subtracting (4.4) from (4.5), we obtain

$$\emptyset Q(\nabla_{\emptyset X} \emptyset Z - \nabla_{\emptyset Z} \emptyset X) - h(\emptyset X, Z) + h(\emptyset Z, X) = 0$$

$$\emptyset Q[\emptyset X, \emptyset Z] - h(\emptyset X, Z) + h(\emptyset Z, X) = 0 \tag{4.6}$$

Since $(D\oplus\langle\xi\rangle)$ is integrable, so that $[\emptyset X, \emptyset Z] \in (D\oplus\langle\xi\rangle)$, for $X, Z \in D$

Consequently, (4.6) gives

$$h(\emptyset X, Z) = h(\emptyset Z, X)$$

for each $X, Y, Z \in (D\oplus\langle\xi\rangle)$.

Theorem 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$ with semi-symmetric semi-metric connection, then

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = \frac{1}{3}\emptyset P[Y, Z]$$

for each $Y, Z \in D^\perp$.

Proof. Let $Y, Z \in D^\perp$ and $X \in TM$, from (3.3), we have

$$g(A_N X, Y) = g(h(X, Y), N)$$

As $N \in T^\perp M$ & $\in D^\perp \Longrightarrow \emptyset Z \in T^\perp M$, then from above

$$2g(A_{\emptyset Z} Y, X) = g(h(Y, X), \emptyset Z) + g(h(X, Y), \emptyset Z) \tag{4.7}$$

Using (3.1) in (4.7), we have

$$2g(A_{\emptyset Z} Y, X) = g(\bar{\nabla}_Y X - \nabla_Y X, \emptyset Z) + g(\bar{\nabla}_X Y - \nabla_X Y, \emptyset Z)$$

$$2g(A_{\emptyset Z} Y, X) = g(\bar{\nabla}_Y X, \emptyset Z) + g(\bar{\nabla}_X Y, \emptyset Z) - g(\nabla_Y X, \emptyset Z) - g(\nabla_X Y, \emptyset Z)$$

As $\nabla_X Y$ & $\nabla_Y X \in TM$, $\emptyset Z \in T^\perp M$, then

$$2g(A_{\emptyset Z} Y, X) = g(\bar{\nabla}_Y X, \emptyset Z) + g(\bar{\nabla}_X Y, \emptyset Z)$$

$$2g(A_{\emptyset Z} Y, X) = -g(\emptyset \bar{\nabla}_Y X, Z) - g(\emptyset \bar{\nabla}_X Y, Z)$$

$$2\, g(A_{\emptyset Z} Y, X) = -g(\emptyset(\bar{\nabla}_Y X) + \emptyset(\bar{\nabla}_X Y), Z)$$

$$2\, g(A_{\emptyset Z} Y, X) = -g(\bar{\nabla}_Y \emptyset X - (\bar{\nabla}_Y \emptyset)X + \bar{\nabla}_X \emptyset Y - (\bar{\nabla}_X \emptyset)Y, Z)$$

$$2\, g(A_{\emptyset Z} Y, X) = -g(\bar{\nabla}_Y \emptyset X + \bar{\nabla}_X \emptyset Y, Z) + g((\bar{\nabla}_Y \emptyset)X + (\bar{\nabla}_X \emptyset)Y, Z)$$

Using (3.1) and (2.9) in above, we have

$$2\, g(A_{\emptyset Z} Y, X) = -g(\nabla_Y \emptyset X + h(Y, \emptyset X) + \nabla_X \emptyset Y + h(X, \emptyset Y), Z)$$

$$2\, g(A_{\emptyset Z} Y, X) = -g(\nabla_Y \emptyset X, Z) - g(\nabla_X \emptyset Y, Z) \tag{4.8}$$

From (3.2), we have

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

Replacing N by $\emptyset Y$

$$\bar{\nabla}_X \emptyset Y = -A_{\emptyset Y} X + \nabla_X^\perp \emptyset Y$$

As ∇ is a Levi-Civita connection, using above, then from (4.8), we have

$$2g(A_{\emptyset Z} Y, X) = -g(\nabla_Y \emptyset X, Z) - g(-A_{\emptyset Y} X + \nabla_X^\perp \emptyset Y, Z)$$

$$2g(A_{\emptyset Z} Y, X) = -g(\nabla_Y \emptyset X, Z) + g(A_{\emptyset Y} X, Z) - g(\nabla_X^\perp \emptyset Y, Z)$$

$$2g(A_{\emptyset Z} Y, X) = -g(\nabla_Y \emptyset X, Z) + g(A_{\emptyset Y} X, Z)$$

$$2\, g(A_{\emptyset Z} Y, X) = -g(\emptyset \nabla_Y Z, X) + g(A_{\emptyset Y} Z, X) \tag{4.9}$$

Transvecting X from both sides from (4.9), we obtain

$$2A_{\emptyset Z}Y = -\emptyset\nabla_Y Z + A_{\emptyset Y}Z \quad (4.10)$$

Interchanging Y & Z, we have

$$2A_{\emptyset Y}Z = -\emptyset\nabla_Z Y + A_{\emptyset Z}Y \quad (4.11)$$

Subtracting (4.10) from (4.11), we have

$$2(A_{\emptyset Y}Z - A_{\emptyset Z}Y) = \emptyset(\nabla_Y Z - \nabla_Z Y) + (A_{\emptyset Z}Y - A_{\emptyset Y}Z)$$

$$3(A_{\emptyset Y}Z - A_{\emptyset Z}Y) = \emptyset[Y, Z]$$

$$(A_{\emptyset Y}Z - A_{\emptyset Z}Y) = \frac{1}{3}\emptyset[Y, Z]$$

Comparing the tangential part both side in above equation, we have

$$(A_{\emptyset Y}Z - A_{\emptyset Z}Y) = \frac{1}{3}\emptyset P[Y, Z]$$

Where $[Y, Z]$ is Lie Bracket.

Theorem 4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic cosymplectic manifold $\overline{M}$ with semi-symmetric semi-metric connection, then the distribution is integrable if and only if

$$A_{\emptyset Y}Z - A_{\emptyset Z}Y = 0 \quad (4.12)$$

for all $Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable, that is $[Y, Z] \in D^{\perp}$

For any $Y, Z \in D^{\perp}$, therefore $P[Y, Z] = 0$.

Consequently, from (4.11) we have

$$A_{\emptyset Y}Z - A_{\emptyset Z}Y = 0$$

Conversely, let (4.12) holds. Then by virtue of (4.11), we have

$$\emptyset P[Y, Z] = 0$$

For all $Y, Z \in D^{\perp}$. Since rank $\emptyset = 2n$

Therefore, either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$.

But $P[Y, Z] = k\xi$ is not possible as P being a projection operator on D.

So, $P[Y, Z] = 0$, this implies that $[Y, Z] \in D^{\perp}$, for all $Y, Z \in D^{\perp}$.

Hence $D^{\perp}$ is integrable.

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