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Properties of Strong Regularity of Fuzzy Measure on Metric Space

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Abstract: *The purpose of this paper is to discuss the properties of regularity and strong regularity of fuzzy measure on metric spaces following the previous results. Some properties are defined with the help of null-additivity such as inner\outer regularity and the regularity of fuzzy measure. We define the strong regularity of fuzzy measures and show our result that the null-additive fuzzy measures possess a strong regularity on complete separable metric spaces.*

Keywords: *Fuzzy measure space, null-additivity, regularity, strong regularity.*

I. Introduction

Sugeno [1] proposed the concept of fuzzy measure and fuzzy integral, researchers from several countries did a lot of works in this field. Wang [2] introduced some concepts of structural characteristics of fuzzy measure, such as null-additive, auto-continuity and uniform auto-continuity which played important role in fuzzy measure theory. Ji [3] studied the regularity and strong regularity of fuzzy measure on metric space.

In this paper, we shall investigate strong regularity of fuzzy measure on metric spaces. Under the null-additivity, weekly null-additivity and converse null-additivity condition, we shall discuss the relation among the inner regularity, the outer regularity and the strong regularity of fuzzy measure.

We discussed the strong regularity of a null-additive fuzzy measure and proved Egoroff's theorem and Lusin's theorem for fuzzy measures on a metric space. The Egoroff's theorem and Lusin's theorem in the classical measure theory are important and useful for discussion of convergence and continuity of measurable functions. The Egoroff's theorem for a fuzzy measure space was proposed by [4] [5], but there the finiteness of fuzzy measures was assumed. The Egoroff's theorem and Lusin's theorems hold for those fuzzy measures that are defined on metric spaces and supposed to be exhaustive and auto continuous from above. It will be proved that exhaustivity and autocontinuity are sufficient for a fuzzy measure to have regularity and tightness, which are enjoyed by classical measures. By using strong regularity we shall show a version of Egoroff's theorem and Lusin's theorem for null additive fuzzy measures on complete separable metric spaces.

II. Preliminaries

We assume that X is a metric space, and that $\mathcal{O}$ is the class of all the open sets in X. Borel σ- algebra $\mathfrak{B}$ is the smallest $\sigma-$ ring containing $\mathcal{O}$, and unless stated otherwise all the subsets are supposed to belong to $\mathfrak{B}$. We shall denote by $\mathcal{K}$ the class of all the compact subsets of X and by $\mathcal{C}$ the class of all the closed set in X [8], [9].

A set function $\mu: \mathfrak{B} \longrightarrow [0,\infty]$ is said to be

(a) exhaustive if $\mu(A_n) \to 0$ for any infinite disjoint sequence $\{A_n\}$ of $\mathfrak{B}$;

(b)order continuous (at $\emptyset$)if $A_n \searrow \emptyset \Longrightarrow \lim_{n\to\infty} \mu(A_n) = 0$;

(c)monotonously continuous if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$, whenever $A_n \nearrow A$ or $A_n \searrow A$;

(d)auto − continuous from above if $\lim_{n\to\infty} \mu(A_n) = 0 \Longrightarrow \lim_{n\to\infty} \mu(A \cup A_n) = \mu(A)$ for any A;

(e)autocontinuous from below if $\lim_{n\to\infty} \mu(A_n) = \mu(A) \Longrightarrow \lim_{n\to\infty} \mu(A - A_n) = \mu(A)$ for any A;

(f) auto-continuous if it is both auto-continuous from above and from below.

(g) absolutely ν- continuous if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that,

$\nu(A) < \delta \Longrightarrow \mu(A) < \varepsilon$ where ν and μ are two set functions defined on $\mathfrak{B}$.

(h) uniformly auto-continuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that
$\mu(A \cup B) \le \mu(A) + \varepsilon$ and $\mu(A - B) \ge \mu(A) - \varepsilon$ whenever $\mu(B) \le \delta$; and

(i) null-additive if $\mu(A \cup B) = \mu(A)$ for any A whenever $\mu(B) = 0$.

For every $A, B, A_n \in \mathfrak{B}, n = 1,2, \dots ...$

Let $(X, \mathfrak{B}, \mu)$ be a null- additive fuzzy measure space. A Borel set A with $\mu(A) > 0$ is called an atom of μ if $B \subset A$ implies (1) $\mu(B) = 0$ or (2) $\mu(A - B) = 0$. Atoms of fuzzy measures defined on a complete separable metric space. If a fuzzy measure μ is exhaustive and auto-continuous from above, then every atom of μ has an outstanding property that all the mass of the atom is concentrated on a single point in it. This fact makes calculation of the fuzzy integral over an atom and a finite union of disjoint atoms easy. Now we discuss some theorems which concerning some properties of fuzzy measures on metric space. If a fuzzy measure μ is finite, then μ is exhaustive. If a fuzzy measure μ is exhaustive, then μ is order continuous. The converse is also true.

Definition 2.1: A fuzzy measure space $(X, \mathfrak{B}, \mu)$ is said to be perfect if for any $\mathfrak{B}$-measurable real valued function f and any set A on the real line such that $f^{-1}(A) \in \mathfrak{B}$, there are Borel sets A_1 and A_2 on the real line such that $A_1 \subset A \subset A_2$ and $\mu\left(f^{-1}(A_2 - A_1)\right) = 0$.

Definition 2.2: Let a fuzzy measure μ be exhaustive and auto-continuous from above on $\mathfrak{B}$. If ν is a set function with $\nu \ll \mu$, then ν is also regular. Furthermore, if ν is a monotonously continuous and null-additive fuzzy measure, then

$$\nu(A) = \text{Sup}\,\{\nu(S): S \subset A, S \in \mathcal{C}\}$$
$$= \inf\{\nu(T): T \supset A, T \in \mathcal{O}\}$$

For each $A \in \mathfrak{B}$. Let fuzzy measures ν and μ be exhaustive and auto-continuous from above on $\mathfrak{B}$. If either $\nu(S) = \mu(S)$ for all $S \in \mathcal{C}$ or $\nu(T) = \mu(T)$ for all $T \in \mathcal{O}$, then $\nu \equiv \mu$ on $\mathfrak{B}$.

Definition 2.3: A fuzzy measure μ is called strongly regular, if for each $A \in \mathfrak{B}$ and each $\varepsilon > 0$, there exist a compact set $K \in \mathcal{K}$ and an open set $T \in \mathcal{O}$ such that $K \subset A \subset T$ and $\mu(T - K) < \varepsilon$.
Let μ be null-additive and order continuous. If for any $A \in \mathfrak{B}$

$$\mu(A) = \text{Sup}\,\{\mu(K): K \subset A, K \in \mathcal{K}\}$$

then μ is strongly regular.

Definition 2.4: A fuzzy measure μ is called outer regular, if for each $A \in \mathfrak{B}$ and each $\varepsilon > 0$, there exists a open set $T \in \mathcal{O}$ such that $A \subset T$, and $\mu(T - A) < \varepsilon$.
A fuzzy measure μ is called inner regular, if for each $A \in \mathfrak{B}$ and each $\varepsilon > 0$, there exists a closed set $S \in \mathcal{C}$ such that $S \subset A$, and $\mu(A - S) < \varepsilon$.
A fuzzy measure μ is called inner regular, if for each $A \in \mathfrak{B}$ and each $\varepsilon > 0$, there exists a closed set $S \in \mathcal{C}$ and an open set $T \in \mathcal{O}$ such that $S \subset A \subset T$ and $\mu(T - S) < \varepsilon$.

III. Null Additive Fuzzy Measure On Metric Space:

In mathematics a Borel set is any set in a metric space that can be formed from open sets or from closed sets through the operations of countable union, countable intersection and relative complement. Borel sets are named after Emile Borel [6]. For a metric space X, the collection of all Borel sets on X formed a $\sigma -$ algebra known as the Borel algebra or Borel σ- algebra. The Borel algebra on X is the smallest $\sigma -$ algebra containing all open sets or all closed sets. Borel sets are important in measure theory, since any measure defined on the open sets of a space or on the closed sets of a space, must also be defined on all Borel sets of that space.

Theorem 3.1: Let X be a separable metric space and μ be a null-additive fuzzy measure of X. Then there exists a unique closed set A_μ such that
(1) $\mu\left(A'_\mu\right) = 0$ where the symbol $'$ denote the complement of A_μ;
(2) If B is any closed set such that $\mu(B') = 0$, then $A_\mu \subset B$.
Moreover, A_μ is the set of all points $x \in X$ having the property that $\mu(G) > 0$ for each open set G containing x.

Proof: Let $H = \{G: G \text{ is open}, \mu(G) = 0$. Since X is separable there are many countably open sets $G_1, G_2, \dots \dots$ such that

$$\bigcup_{n=1}^{\infty} G_n = \cup \{G: G \in H\}$$

Let us denote this union by G_μ and $A_\mu = X - G_\mu$. By the null-additivity of μ, we have

$$\mu(G_\mu) = \mu\left(\bigcup_{n=1}^{\infty} G_n\right) = 0, i.e. \mu\left(A'_\mu\right) = 0.$$

Further if B is any closed set with $\mu(B') = 0$, $X - B \in H$ and hence $X - B \subset G_\mu$, namely $A_\mu \subset B$. The uniqueness of A_μ is obvious. Hence this proves the both assertion. Now there exists an open set containing x for any $x \in A_\mu$ such that $\mu(G_\mu) = 0$ and $\mu(G)$ must be positive if $x \in A_\mu$ and G is an open set containing x, we end the proof. The closed set A_μ is called the spectrum or support of μ.

□

Let X be any metric space and μ a null additive fuzzy measure on X such that $\mu(X - B) = 0$, for some separable Borel set $B \subset X$. Then μ has a spectrum A_μ which is separable and $A_\mu \subset B$. We shall investigate a smaller class of fuzzy measures on metric spaces, i.e. tight fuzzy measures. Tight fuzzy measures have the property that they are determined by their values taken on compact sets. A fuzzy measure μ is called to be tight if for each set $A \in \mathfrak{B}$ and each $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset A$ such that $\mu(A - K_\varepsilon) \le \varepsilon$.

Theorem 3.2: Let μ be a tight uniformly autocontinuous fuzzy measure on X. Then μ has separable support and for any Borel set A and any $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset A$ such that $\mu(A - K_\varepsilon) < \varepsilon$.

Proof: Let K_n be a compact set such that $\mu(X - K_n) < \frac{1}{n}$, $n = 1,2, \dots$ .... A compact set in a metric space is separable and, hence

$$\bigcup_{n=1}^{\infty} K_n \ \ is\ separable. If\ A_0 = \bigcup_{n=1}^{\infty} K_n\ \ then\ \mu(A_0) = 0\ by\ the\ monotonicity\ of\ \mu.$$

Firstly we prove that for every $\varepsilon > 0$, there exists a compact set $K \subset A$, such that $\mu(X - K) < \frac{\varepsilon}{2}$.

In fact, since X is separable, there exists denumerable dense sequence of points $\{x_n\}$ in X, let $C_a(x_n)$ be a closed sphere with radius $\frac{1}{a}$ and centre x_n in X, where a is arbitrary positive integer, then for every a we have,

$$\bigcup_{n=1}^{\infty} C_a(x_n) \nearrow X, i.e. \left[X - \bigcup_{n=1}^{\infty} C_a(x_n)\right] \searrow \emptyset$$

As μ is a fuzzy measure, we have

$$\mu\left(X - \bigcup_{n=1}^{\infty} C_a(x_n)\right) \longrightarrow 0$$

Hence, there exists positive integer b, such that

$$\mu\left(X - \bigcup_{n=1}^{b} C_a(x_n)\right) < \frac{\varepsilon}{2^{a+1}} \qquad (3.1)$$

$$Taking\ \ C_a = \bigcup_{n=1}^{b} C_a(x_n)$$

Let $a \to \infty$ then by the autocontinuity of μ, there exists sub-sequence $\{C_{a_i}\}$ of {C_a}, such that

$$\mu\left(\bigcup_{i=1}^{\infty}(X - C_{a_i})\right) = \mu\left(X - \bigcap_{i=1}^{\infty} C_{a_i}\right) = 0 \qquad (3.2)$$

Hence from equations (3.1) and (3.2), we have,

$$\mu\left(X - \bigcap_{i=1}^{\infty} C_{0_i}\right) < \frac{\varepsilon}{2}$$

$$Let\ K = \bigcap_{i=1}^{\infty} C_{0_i}$$

Since C_a is finite, K is also finite. Hence K is a complete bounded set. By the selection of K, it is obvious that K is a closed set. Hence K is a compact set ([7] page 29), as desired.

Secondly, we prove the inequality $\mu(A-K_\varepsilon)<\varepsilon$. By the uniform autocontinuity of μ, for every $\varepsilon>0$, there exists $\delta>0$, such that $\mu(E\cup F)\le\mu(E)+\frac{\varepsilon}{2}$ whenever $E,F\in\mathfrak{B},\mu(F)<\delta$. For arbitrary A$\in\mathfrak{B}$, there exists some closed set $C\subset A$, such that $\mu(A-C)<\delta$.

Let $K_\varepsilon=K\cap C$, then K_ε is a compact set and $K_\varepsilon\subset A$. Thus we have,

$$\mu(A-K_\varepsilon)\le\mu\big((X-K)\cup(A-K)\big)$$
$$\le\mu(X-K)+\frac{\varepsilon}{2}$$
$$\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\le\varepsilon$$

This completes the proof. □

Theorem 3.3: Let X be a separable metric space with the property that there exists a complete separable metric space Y such that X is contained in Y as a topological subset and X is a Borel subset of Y. Then every uniformly autocontinuous fuzzy measure μ on X is tight. In particular, if X itself is a complete separable metric space, every uniformly autocontinuous fuzzy measure on X is tight.

Proof: Let $X\subset Y$, where Y is a complete separable metric space and X is a Borel set in Y. Given a fuzzy measure μ on $\mathfrak{B}_X$, we define ν on the class $\mathfrak{B}_Y$ by setting $\nu(A)=\mu(A\cap X)$, $A\in\mathfrak{B}_Y$. Since $X\in\mathfrak{B}_Y$, $\nu(Y-X)=0$. We claim that it is enough to prove that ν is a tight measure on Y. Indeed, since X is a Borel set in Y, there will exist for each $\varepsilon>0$, a set $K_\varepsilon\subset X$, compact in Y, such that $\nu(X-K_\varepsilon)<\varepsilon$, by theorem 3.2. K_ε is also compact in X since X is a topological subset of Y. Further $\mu(X-K_\varepsilon)=\nu(X-K_\varepsilon)<\varepsilon$. This implies that μ is tight. Thus, we may assume that X is itself a complete separable metric space. For the rest part we prove in above theorem.

□

Let a fuzzy measure μ be exhaustive and autocontinuous from above on a complete separable metric space. If ν is a set function with $\nu\ll\mu$, then ν is tight. Moreover, if ν is a null additive fuzzy measure, then

$\nu(A)=Sup\,\{\nu(K):K\subset A,K\in\mathcal{K}\}\ \ for\ each\ A\in\mathfrak{B}.$

Let $X=(0,1)$ and $\mathfrak{B}$ be a Borel $\sigma-$ algebra on X, defined as

$$\mu(A)=tan\left(\frac{\pi\, m(A)}{2}\right)$$

Where $A\in\mathfrak{B}$ and m(A) denotes the Lebesgue measure of A. Then μ is $\sigma-$ finite, exhaustive and autocontinuous from above, but $\mu(X)=\infty$.

Theorem 3.4: Let X be any metric space and μ a uniformly autocontinuous tight fuzzy measure on X, then $(X,\mathfrak{B},\mu)$ is a perfect fuzzy measure space.

Proof: Let f be any real valued measurable function. It is sufficient to prove that for any $A\subset R$, where R is the real line, such that $f^{-1}(A)\in\mathfrak{B}$, there exists a Borel set $A_1\subset A$ with $\mu\big(f^{-1}(A-A_1)\big)=0$; A_2 be defined as a Borel set such that $A\subset A_2$ and $\mu\big(f^{-1}(A_2-A)\big)=0$ Suppose that A is a set such that $B=f^{-1}(A)\in\mathfrak{B}$. Let $\{C_n\}$ and $\{K_n\}$, n=1, 2, ... be two sequences of sets such that

(1) $K_1\subset K_2\subset\cdots\ldots\ldots$, each K_n is compact, $f\setminus K_n$ (f restricted to K_n) is continuous, and $\mu(X-K_n)\to 0$,

(2) $C_1\subset C_2\subset\cdots\ldots\ldots\subset B$, each C_n is closed and $\mu(B-C_n)\to 0$.

If we write $Q_n=K_n\cap C_n$, then $Q_1\subset Q_2\subset\cdots\ldots\ldots\subset B$, each Q_n is compact, $f\setminus Q_n$ is continuous and $\mu(B-Q_n)\to 0$ as $n>\infty$. If $B_n=f(Q_n)$ then B_n is a compact subset of the real line since $f\setminus Q_n$ is continuous and hence

$$A_1=\bigcup_{n=1}^{\infty}B_n$$

Is a Borel set since

$$f\left(\bigcup_{n=1}^{\infty}Q_n\right)=A_1$$

it follows that,

$$f^{-1}(A_1) \supset \bigcup_{n=1}^{\infty} Q_n$$

Clearly, $A_1 \subset A$ and $f^{-1}(A_1) \subset f^{-1}(A) = B$. By the continuity of μ we obtain

$$\mu\left(B - \bigcup_{n=1}^{\infty} K_n\right) = \lim_{n\to\infty}(B - K_n) = 0$$

so $\mu\big(B - f^{-1}(A_1)\big) = 0$. This completes the proof.

□

IV. Strong Regularity Of Fuzzy Measure On Metric Space:

Every probability measure P on a metric space is regular; that is, for every Borel set A and $\varepsilon > 0$, there exists a closed set S and an open set T such that $S \subset A \subset T$ and $P(T - S) \le \varepsilon$ [7].

Theorem 4.1: If a fuzzy measure μ is exhaustive and auto-continuous from above on the Borel $\sigma -$ algebra $\mathfrak{B}$, then μ is regular, and for any $A \in \mathfrak{B}$

$$\mu(A) = Sup\,\{\mu(S): S \subset A, S \in \mathcal{C}\}$$
$$= inf\,\{\mu(T): T \supset A, T \in \mathcal{O}\}$$

Proof: Let β be the class of all the sets $A \in \mathfrak{B}$ such that for any $\varepsilon > 0$ there are $S \in \mathcal{C}$ and $T \in \mathcal{O}$ satisfying $S \subset A \subset T$ and $\mu(T - S) \le \varepsilon$. To prove $\beta = \mathfrak{B}$ it is sufficient to show that $\mathfrak{B} \subset \beta$.

Denote the distance from x to G by d(x, G). If $G \in \mathcal{C}$, then the open sets $T_n = \left\{x: d(x,G) < \frac{1}{n}\right\}$ decrease to G, i.e. $T_n \searrow G$. Since μ is exhaustive, we have

$$\lim_{n\to\infty} \mu(T_n - G) = 0$$

Thus $\mathcal{C} \subset \beta$, now let $\{A_n\} \subset \beta$ and $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$\mu(A) \vee \mu(B) \le \delta \;\; \Rightarrow \;\; \mu(A \cup B) \le \varepsilon.$$

Hence there exists two sequences of closed sets $\{S_n\}$ and of open sets $\{T_n\}$ such that

$$S_n \subset A_n \subset T_n \qquad n = 1,2,\dots\dots \quad and \quad \mu\left(\bigcup_{n=1}^{\infty}(T_n - S_n)\right) \le \delta$$

On the other hand

$$\left(\bigcup_{n=1}^{\infty} S_n - \bigcup_{n=1}^{m} S_n\right) \searrow \emptyset \qquad (as\; m \to \infty)$$

implies the existence of $n_0 \ge 1$ such that

$$\mu\left(\bigcup_{n=1}^{\infty} S_n - \bigcup_{n=1}^{n_0} S_n\right) \le \delta.$$

$$Put\;\; T = \bigcup_{n=1}^{\infty} T_n \quad and \quad S = \bigcup_{n=1}^{n_0} S_n \quad then$$

$$\mu(T - S) \le \mu\left(\left(\bigcup_{n=1}^{\infty}(T_n - S_n)\right) \cup \left(\bigcup_{n=1}^{\infty} S_n - S\right)\right) \le \varepsilon$$

Thus β is closed under the formation of countable unions. It is obvious that β is closed under complementation, we get $\beta = \mathfrak{B}$. Now let $A \in \mathfrak{B}$ and we prove,

$$\mu(A) = inf\,\{\mu(T): T \supset A, T \in \mathcal{O}\}$$

for each $n \ge 1$, there exists $T_n \in \mathcal{O}$ such that $A \subset T_n$ and $\mu(T_n - A) < \frac{1}{n}$. Thus we have

$$\mu\left(\bigcap_{n=1}^{\infty} T_n - A\right) = 0 \; and\; hence \quad \mu(A) = \mu\left(\bigcap_{n=1}^{\infty} T_n\right) = \lim_{n\to\infty} \mu\left(\bigcap_{i=1}^{n} T_i\right) \; since \; \bigcap_{i=1}^{n} T_i \; is\; also$$

open and contains A, the equation above to be proved is obtained. The other equation is proved similarly.

□

Theorem 4.2: Let μ be a finite continuous fuzzy measure, then for any $\varepsilon > 0$ and any double sequence $\{B_{n,m}: n \geq 1, m \geq 1\} \subset \mathfrak{B}$ satisfying $B_{n,m} \searrow \emptyset \quad (m \to \infty), n = 1,2, \ldots\ldots$ there exists a subsequence $\{B_{n,m_n}\}$ of $\{B_{n,m}: n \geq 1, m \geq 1\}$ such that

$$\mu\left(\bigcup_{n=1}^{\infty} B_{n,m_n}\right) < \varepsilon \qquad (m_1 < m_2 < \cdots \ldots \ldots)$$

Proof: Since for any $B_{n,m} \searrow \emptyset \quad (m \to \infty), n = 1,2, \ldots\ldots$, for given $\varepsilon > 0$, using the continuity from above of fuzzy measures, we have

$$\lim_{m\to\infty} \mu(B_{1,m}) = 0.$$

Therefore there exists m_1 such that $\mu(B_{1,m_1}) < \frac{\varepsilon}{2}$ for this m_1,

$$(B_{1,m_1} \cup B_{2,m_2}) \searrow B_{1,m_1} \qquad \text{as} \quad m \to \infty$$

Therefore it follows from the continuity from above of μ, that

$$\lim_{m\to\infty} \mu(B_{1,m_1} \cup B_{2,m_2}) = \mu(B_{1,m_1}).$$

Thus there exists $m_2 > m_1$, such that $\mu(B_{1,m_1} \cup B_{2,m_2}) < \frac{\varepsilon}{2}$. Generally, there exists $m_1, m_2, \ldots \ldots\ldots, m_i$ such that

$$\mu(B_{1,m_1} \cup B_{2,m_2} \cup \ldots \ldots \ldots \cup B_{i,m_i}) < \frac{\varepsilon}{2}$$

Hence we obtain a sequence $\{m_n\}$ $n = 1,2, \ldots \ldots\ldots \infty$ of numbers and a sequence $\{B_{n,m_n}\}$ of sets. By using the monotonicity and the continuity from below of μ, we have

$$\mu\left(\bigcup_{n=1}^{\infty} B_{n,m_n}\right) \leq \frac{\varepsilon}{2} < \varepsilon.$$

This gives the proof of the theorem. □

Theorem 4.3: If μ is null-additive, then μ is strongly regular.

Proof: Let $A \in \mathfrak{B}$ and given $\varepsilon > 0$, from theorem 4.5 we know that μ is regular. Therefore, there exists a sequence $\{S_n\}$, $n=1,2,\ldots.$ of closed sets and a sequence $\{T_n\}$ of open sets such that for every $n = 1,2, \ldots\ldots$, $S_n \subset A \subset T_n \quad \mu(T_n - S_n) < \frac{1}{n}$ without loss of generality, we can assume that the sequence $\{S_n\}$ is increasing in n and the sequence $\{T_n\}$ is decreasing in n. Thus $\{T_n - S_n\}$ is a decreasing sequence of sets with respect to n and as $n \to \infty$.

$$(T_n - S_n) \searrow \left(\bigcap_{n=1}^{\infty} (T_n - S_n)\right).$$

$$Let\ F_1 = \bigcap_{n=1}^{\infty} (T_n - S_n)$$

And noting that $\mu(F_1) \leq \mu(T_n - S_n) < \frac{1}{n} \qquad n = 1,2, \ldots \ldots\ldots \quad then\ \mu(F_1) = 0.$

On the other hand, from theorem 4.2 there exists a sequence $\{K_n\}$ of compact subsets in X, such that for every $n = 1,2, \ldots\ldots$, $\mu(X - K_n) < \frac{1}{n}$ and we can assume that $\{K_n\}$ is decreasing in n. Therefore as $n \to \infty$

$$(X - K_n) \searrow \bigcap_{n=1}^{\infty} (X - K_n).$$

$$Let\ F_2 = \bigcap_{n=1}^{\infty} (X - K_n), \qquad then\ \mu(F_2) = 0.$$

Thus we have, $[(X - K_n) \cup (T_n - S_n)] \searrow (F_1 \cup F_2) \quad as\ n \to \infty$, noting that $\mu(F_1 \cup F_2) = 0$ by the continuity of μ, then

$$\lim_{n\to\infty} \mu((X - K_n) \cup (T_n - S_n)) = 0$$

Therefore there exists n_0 such that $\mu\left(\left(X-K_{n_0}\right)\cup\left(T_{n_0}-S_{n_0}\right)\right)<\varepsilon$.
Let $K_\varepsilon = K_{n_0}\cap S_{n_0}$ and $T_\varepsilon = T_{n_0}$ $then\, K_\varepsilon$ is a compact set and T_ε is an open set, and $K_\varepsilon \subset A \subset T_\varepsilon$. Since $(T_\varepsilon - K_\varepsilon)\subset\left(\left(X-K_{n_0}\right)\cup\left(T_{n_0}-S_{n_0}\right)\right)<\varepsilon$
This shows that μ is strongly regular. □

Let f and f_n be real-valued $\mathfrak{B}-$ measurable function on X. It is well known that if $f_n \to f$ every where on a finite measure space $(X,\mathfrak{B},m)$, then for any $\varepsilon>0$ there exists a subset E such that $m(X-E)<\varepsilon$ and $\{f_n\}$ converges to f uniformly on E. This Egoroff's theorem was extended to a finite fuzzy measure space [4], [5]. The following is a further generalization of the theorem to a fuzzy measure space which is not necessarily finite.

Theorem 4.4: Let a fuzzy measure μ be exhaustive and auto-continuous from above on $\mathfrak{B}$. If $f_n \to f$ every where on X, then for any $\varepsilon>0$ there exists a closed subset S_ε such that $\mu(X-S_\varepsilon)<\varepsilon$ and $\{f_n\}$ converges to f uniformly on S_ε.

Proof: If a fuzzy measure μ is auto-continuous, then for any $\varepsilon>0$ there exists $\delta>0$ such that $\mu(A)\vee\mu(B)\le \delta \;\Rightarrow\; \mu(A\cup B)\le\varepsilon$. Put

$$E_{n,k}=\bigcap_{i=n}^{\infty}\left\{x\colon |f_i(x)-f(x)|<\frac{1}{k}\right\}\quad k=1,2,\ldots$$

then $E_{n,k}$ is increasing in n for each fixed k. The set of all those x's, for which $f_n\to f$ is

$$\bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}E_{n,k}.$$

Since $f_n\to f$ every where, we have $E_{n,k}\nearrow X \;\; as\, n\to\infty$ for any $k\ge 1$ or equivalently $(X-E_{n,k})\searrow\emptyset$ $as\, n\to\infty$ for any $k\ge1$. For the $\delta>0$ given above, we may take a sub-sequence $\{E_{n_k,k}\}$ of $\{E_{n,k}\colon n\ge1, k\ge 1$ such that

$$\mu\left(\bigcup_{k=1}^{\infty}\left(X-E_{n_k,k}\right)\right)\le\delta.\;\; Put\;\; S=\bigcap_{k=1}^{\infty}E_{n_k,k}\;\;, then\, \mu(X-S)\le\delta$$

$and\,\{f_n\}\, converges\, to\, f$ uniformly on S. Now take any closed subset S_ε of S such that $\mu(S-S_\varepsilon)\le\delta$. Then $\mu(X-S_\varepsilon)\le\varepsilon$ and $\{f_n\}$ converges to f uniformly on S_ε. □

Similarly we can prove the following theorem. Let the metric space X be complete and separable, and a fuzzy measure μ be exhaustive and auto-continuous from above. If $f_n\to f$ every where on X, then for any $\varepsilon>0$ there exist a compact subset K_ε such that $\mu(X-K_\varepsilon)<\varepsilon$ and $\{f_n\}$ converges to f uniformly on K_ε.
Lusin's Theorem is also important in the real analysis. We generalized Lusin's theorem from a classical measure space to a finite auto-continuous fuzzy measure space [8], [9]. We extend the result of [8] to a $\sigma-$ finite fuzzy measure space.

Theorem 4.5: Let a fuzzy measure μ be $\sigma-$ finite, exhaustive and auto-continuous from above on $\mathfrak{B}$.If f is a real measurable function on X, then for each $\varepsilon>0$ there exist a closed subset $S_\varepsilon\in\mathcal{C}$ such that f is continuous on S_ε and $\mu(X-S_\varepsilon)\le\varepsilon$.

Proof: If a fuzzy measure μ is auto-continuous from above, then for any $\varepsilon>0$ there exists $\delta>0$ such that $\mu(A)\vee\mu(B)\le\delta \;\Rightarrow\; \mu(A\cup B)\le\varepsilon$. To prove this theorem we use three steps in different situations where $\varepsilon>0$ is fixed.
(1) Suppose that f is a simple function, i.e.

$$f(x)=\sum_{i=1}^{n}a_i\,X_{A_i}(x)$$

$$Where\; a_i\ne a_j,\qquad A_i\cap A_j=\emptyset\,,(i\ne j),\qquad A_i\in\mathfrak{B}\; and\;\; X=\bigcup_{i=1}^{n}A_i.$$

By the regularity of μ, for any $\varepsilon>0$ there exists some closed subsets $S_i\subset A_i$ $(1\le i\le n)$ such that

$$\mu\left(\bigcup_{i=1}^{n}(A_i - S_i)\right) \leq \varepsilon.$$

$$Put\ S_\varepsilon = \bigcup_{i=1}^{n} S_i\ , then\ f\ is\ continuous\ on\ the\ closed\ subset\ S_\varepsilon\ of\ X, and$$

$$\mu(X - S_\varepsilon) \leq \mu\left(\bigcup_{i=1}^{n}(A_i - S_i)\right) \leq \varepsilon$$

since the distance of two disjoint closed set is greater than zero, it is obvious that f is continuous on S_ε.
(2) Let f be a non-negative measurable function. Then f is a limit of an increasing sequence $\{f_n\}$ of simple functions. Let $\delta_1 > 0$ and $\delta_2 > 0$ to satisfy
$\mu(A) \vee \mu(B) \leq \delta_i \Rightarrow \mu(A \cup B) \leq \delta_{i-1}$
where $\delta_0 = \delta$. By the result of case (1) there exist a sequence $\{S_n\}$ in $\mathcal{C}$ such that

$$\mu\left(\bigcup_{n=1}^{\infty}(X - S_n)\right) \leq \delta_2$$

and f_n is continuous on S_n. If we put

$$S_0 = \bigcap_{n=1}^{\infty} S_n$$

then $\mu(X - S_0) \leq \delta_2$ and f_n is continuous of S_0.
On the other hand, since μ is $\sigma -$ finite, there exist a sequence $\{X_m\}$ such that $X_m \nearrow X\ and$ $\mu(X_m) < \infty$ where $m = 1,2, \ldots$..Then there exists m_0 such that $\mu(X - X_{m_0}) \leq \delta_2$ and thus $\mu\left(X - \left(S_0 \cap X_{m_0}\right)\right) \leq \delta_1$. We have a subset F of $S_0 \cap X_{m_0}$ such that $\{f_n\}$ converges to f uniformly on F and $\mu\left(\left(S_0 \cap X_{m_0}\right) - F\right) \leq \delta_2$ [4]. By the regularity of μ, there is a closed subset S_ε of F such that $\mu(F - S_\varepsilon) \leq \delta_2$, and then $\mu\left(\left(S_0 \cap X_{m_0}\right) - S_\varepsilon\right) \leq \delta_1$. Thus f is continuous on S_ε, and we have

$$\mu(X - S_\varepsilon) \leq \mu\left(\left(X - \left(S_0 \cap X_{m_0}\right)\right) \cup \left(\left(S_0 \cap X_{m_0}\right) - S_\varepsilon\right)\right) \leq \varepsilon.$$

(3) Let f be an arbitrary measurable function on X. If we put

$$f^+ = \frac{|f| + f}{2}\ \ and\ \ f^- = \frac{|f| - f}{2},$$

then f^+ and f^- are non-negative measurable functions and $f = f^+ - f^-$. Applying the result of case (2) to f^+ and f^-, we have two closed subsets S^+ and S^- of X such that f^+ and f^- are continuous respectively on S^+ and S^-, and $[\mu(X - S^+) \vee \mu(X - S^-)] \leq \delta$.
Thus f is continuous on the closed set $S_\varepsilon = S^+ \cap S^-$ and
$\mu(X - S_\varepsilon) = \mu\big((X - S^+) \cup (X - S^-)\big) \leq \varepsilon$. Thus the theorem is proved. □

V. Properties Of Inner\Outer Regularity Of Fuzzy Measure:

We assume that X is a metric space and Let $\mathfrak{B}$ be the Borel $\sigma -$ algebra and $\mathcal{O}$ is the class of all open sets belonging to $\mathfrak{B}$. Let $\mathcal{C}$ be the class of all closed sets belonging to $\mathfrak{B}$.
In the following we present some properties of the inner regularity and outer regularity of fuzzy measure.

Theorem 5.1: If μ is an auto-continuous fuzzy measure, then
(1) A finite union of inner regular sets is inner regular.
(2) A finite intersection of outer regular sets is outer regular.
(3) A finite union of outer regular sets is outer regular.

Proof: (1) Let $\{A_1, A_2, \ldots \ldots, A_n\}$ be a finite class of inner regular sets, then for any $\varepsilon > 0$, $\delta > 0$ and for every $A_i\ \ (i = 1,2, \ldots., n)$, there exists a set S_i in $\mathcal{C}$ such that $S_i \subset A_i$ and $\mu(A_i - S_i) < \delta$.

$$Let\ S = \bigcup_{i=1}^{n} S_i\ \ \ and\ \ \ A = \bigcup_{i=1}^{n} A_i\ \ \ it\ is\ clear\ S \subset A\ \ in\ \mathcal{C}\ \ and\ since$$

$$A - S = \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} (A_i - S_j) \subset \bigcup_{i=1}^{n} (A_i - S_i),$$

Then by μ is an auto-continuous we have

$$\mu(A - S) \leq \mu\left(\bigcup_{i=1}^{n} (A_i - S_i)\right) < \varepsilon.$$

$$i.e.\ A = \bigcup_{i=1}^{n} A_i \ \text{ is inner regular.}$$

(2) Let $\{B_1, B_2, \dots, B_n\}$ be a finite class of outer regular sets. Then for any $\varepsilon > 0$, $\delta > 0$ and for every B_i $(i = 1,2, \dots., n)$, there exists a set T_i in $\mathcal{O}$, such that $B_i \subset T_i$ and $\mu(T_i - B_i) < \delta$.

$$Let\ T = \bigcap_{i=1}^{n} T_i\,,\ \ B = \bigcap_{i=1}^{n} B_i\,, \qquad obviously, B \subset T\ in\ \mathcal{O}\ and\ since$$

$$T - B = \bigcap_{i=1}^{n} T_i - \bigcap_{i=1}^{n} B_i = \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} (T_i - B_j) \subset \bigcup_{i=1}^{n} (T_i - B_i),$$

Then by μ is an auto-continuous we have,

$$\mu(T - B) \leq \mu\left(\bigcup_{i=1}^{n} (T_i - B_i)\right) < \varepsilon$$

$$Therefore\ \ B = \bigcap_{i=1}^{n} B_i \ \text{ is outer regular.}$$

(3) Let $\{B_1, B_2, \dots \dots \dots, B_n\}$ be a finite class of outer regular sets. Then for any $\varepsilon > 0$, $\delta > 0$ and for every B_i $(i = 1,2, \dots., n)$, there exists a open set T_i in $\mathcal{O}$, such that $B_i \subset T_i$ and $\mu(T_i - B_i) < \delta$.

$$Let\ T = \bigcup_{i=1}^{n} T_i\,,\ \ B = \bigcup_{i=1}^{n} B_i\,, \qquad obviously, B \subset T\ in\ \mathcal{O}\ and\ since$$

$$T - B = \bigcup_{i=1}^{n} T_i - \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} (T_i - B_j) \subset \bigcup_{i=1}^{n} (T_i - B_i),$$

Then by μ is an auto-continuous we have,

$$\mu(T - B) \leq \mu\left(\bigcup_{i=1}^{n} (T_i - B_i)\right) < \varepsilon$$

$$Therefore\ \ B = \bigcup_{i=1}^{n} B_i \ \text{ is outer regular.}$$ □

Theorem 5.2: If μ is uniform auto-continuous fuzzy measure then
(1) The union of a sequence of outer regular set is outer regular.
(2) The intersection of a sequence of inner regular sets is inner regular.

Proof: (1) Let $\{B_i\}$ $i = 1,2, \dots \dots . \infty$ be a sequence of outer regular sets, then for any $\varepsilon > 0$, $\delta > 0$ and for every B_i $(i = 1,2, \dots \dots)$, there exists a open set T_i in $\mathcal{O}$, such that $B_i \subset T_i$ and $\mu(T_i - B_i) < \delta$.

$$Let\ T = \bigcup_{i=1}^{\infty} T_i\,,\ \ B = \bigcup_{i=1}^{\infty} B_i\,, \qquad obviously, B \subset T\ in\ \mathcal{O}\ and\ then$$

$$T - B = \bigcup_{i=1}^{\infty} T_i - \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} (T_i - B_j) \subset \bigcup_{i=1}^{\infty} (T_i - B_i),$$

Since μ is a uniform auto-continuous fuzzy measure, we have,

$$\mu(T - B) \leq \mu\left(\bigcup_{i=1}^{\infty} (T_i - B_i)\right) < \varepsilon$$

$$Therefore\ B = \bigcup_{i=1}^{\infty} B_i\ \ is\ outer\ regular.$$

(2) Let $\{A_i\}$ $i = 1,2, \ldots \ldots . \infty$ be a sequence of inner regular sets, then for any $\varepsilon > 0$, $\delta > 0$ and for every A_i $(i = 1,2, \ldots \ldots)$, there exists a closed set S_i in $\mathcal{C}$ such that $S_i \subset A_i$ and $\mu(A_i - S_i) < \delta$.

$$Let\ S = \bigcap_{i=1}^{\infty} S_i\ \ and\ \ A = \bigcap_{i=1}^{\infty} A_i\ \ it\ is\ clear\ S \subset A\ \ in\ \mathcal{C}\ \ and\ then$$

$$A - S = \bigcap_{i=1}^{\infty} A_i - \bigcap_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} (A_i - S_j) \subset \bigcup_{i=1}^{\infty} (A_i - S_i),$$

since μ is an uniform auto-continuous fuzzy measure we have,

$$\mu(A - S) \leq \mu\left(\bigcup_{i=1}^{\infty} (A_i - S_i)\right) < \varepsilon.$$

$$Therefore\ A = \bigcap_{i=1}^{\infty} A_i\ \ is\ inner\ regular.$$ □

Theorem 5.3: If μ is an auto-continuous fuzzy measure, then a necessary and sufficient condition that every set in $\mathcal{C}$ be outer regular is that every bounded set in $\mathcal{O}$ be inner regular.

Proof: Necessary Condition:

Let us suppose that every set in $\mathcal{C}$ is outer regular and let T be a bounded set in $\mathcal{O}$ and $\varepsilon > 0$. Let S be a set in $\mathcal{C}$ such that, $T \subset S$, since $S - T$ is closed and $S - T \in \mathfrak{B}$, then $\mathrm{S} - \mathrm{T} \in \mathcal{C}$. Thus $\mathrm{S} - \mathrm{T}$ is outer regular. Therefore there is a set H in $\mathcal{O}$, such that $\mathrm{S} - \mathrm{T} \subset \mathrm{H}$ and

$$\mu\big(\mathrm{H} - (\mathrm{S} - \mathrm{T})\big) < \varepsilon.$$

Since $\mathrm{T} = [\mathrm{S} - (\mathrm{S} - \mathrm{T})] \supset (\mathrm{S} - \mathrm{H}) \in \mathcal{C}$ then

$$\mu\big(\mathrm{T} - (\mathrm{S} - \mathrm{H})\big) = \mu(\mathrm{T} \cap \mathrm{H}) \leq \mu\big(\mathrm{H} - (\mathrm{S} - \mathrm{T})\big) < \varepsilon$$

hence T is inner regular.

Sufficient Condition:

Let every bounded set in $\mathcal{O}$ is inner regular. If S be a set in $\mathcal{C}$, ε be a positive number, T be a bounded set in $\mathcal{O}$, then there is a set $\mathrm{G} \in \mathcal{C}$, such that $\mathrm{G} \subset (\mathrm{T} - \mathrm{S})$ and $\mu\big((\mathrm{T} - \mathrm{S}) - \mathrm{G}\big) < \varepsilon$. Since $\mathrm{S} = \mathrm{T} - (\mathrm{T} - \mathrm{S}) \subset (\mathrm{T} - \mathrm{G}) \in \mathcal{O}$, thus

$$\mu\big((\mathrm{T} - \mathrm{G}) - \mathrm{S}\big) = \mu\big((\mathrm{T} - \mathrm{S}) - \mathrm{G}\big) < \varepsilon.$$

Hence S is outer regular. □

The other properties of the inner\outer regularity of fuzzy measure are defined, which can be proved easily by the help of above theorems are,

(a) If μ is weekly null-additive and strongly order continuous, then both outer and inner regularity imply regularity.

(b) If μ be null-additive fuzzy measure then,

(i) If μ is continuous from below, then inner regularity implies

$\mu(A) = \sup\{\mu(S): S \subset A, S \in \mathcal{C}\}$ for all $A \in \mathfrak{B}$.
(ii) If μ is continuous from above, then outer regularity implies
$\mu(A) = \inf\{\mu(T): A \subset T, T \in \mathcal{O}\}$ for all $A \in \mathfrak{B}$.
(c) If μ be converse null-additive fuzzy measure,
(i) If μ is continuous from below and strongly order continuous and for any $A \in \mathfrak{B}$,
$\mu(A) = \sup\{\mu(S): S \subset A, S \in \mathcal{C}\}$
then μ is inner regular.
(ii) If μ is continuous from above and for any $A \in \mathfrak{B}$,
$\mu(A) = \inf\{\mu(T): A \subset T, T \in \mathcal{O}\}$
then μ is outer regular.

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