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Stability analysis of two predator-one stage-structured prey model incorporating a prey refuge

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Abstract: *In this paper, a food web model consisting of two predator-one stage structured prey involving Lotka-Voltera type of functional response and a prey refuge , is proposed and analyzed. It is assumed that the prey growth logistically in the absence of predator. The role of prey refuges in predator-prey model is investigated. The existence , uniqueness and boundedness of the solution are studied. The existence and the stability analysis of all possible equilibrium points are studied. Suitable Lyapunov functions are used to study the global dynamics of the proposed model. Numerical simulation for different sets of parameter value and for different sets of initial conditions are carried out to investigate the influence of parameters on the dynamical behavior of the model and to support the obtained analytical results of the model .*
Keywords: *food web , Lyapunov function , refuge , stability analysis , stage-structure .*

I. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Over the past decades, Mathematics has made a considerable impact as a tool to model and understand biological phenomena. In return, biologists have confronted the mathematics with variety of challenging problems , which have simulated developments in the theory of nonlinear differential equations. Such differential equations have long played important role in the field of theoretical population dynamics, and they will, no doubt, continue to serve as indispensable tools in future investigations. Differential equation models for interactions between species are one of the classical applications of mathematics to biology. The development and use of analytical techniques and the growth of computer power have progressively improved our understanding of these types of models. Although the predator-prey theory has been much progress, many long standing mathematical and ecological problems remain open , [1] . Food chains and food webs depict the network of feeding relationship within ecological communities. During the last few decades, a large number of food-chain and food-web systems have been proposed to describe the food transition patterns and processes $[2-4]$ .Living organisms enter into a variety of relationships, such as Prey-Predator, Competition, Mutualism, Commensalism and so on, among themselves according to the needs of individuals as well as those of species groups. Food webs are one example of interactions that go beyond feeding relationships. Recently ,number of researchers have been proposed and studied the dynamics of food webs involving some types of these relationships, for example see $[5-8]$ and the references their in .The study of the consequences of hiding behavior of prey on the dynamics of predator prey interactions can be recognized as a major issue in applied mathematics and theoretical ecology [9,10] . Some of the empirical and theoretical work have investigated the effect of prey refuges and drawn a conclusion that the refuges used by prey have a stabilizing effect on the considered interactions and prey extinction can be prevented by the addition of refuges [11,12] .In fact, the effect of prey refuges on the population dynamics are very complex in nature, but for modeling purposes, it can be considered as constituted by two components, the first effects which affect positively the growth of prey and negatively that of predators. Comprise the reduction of prey mortality due to decrease in predation success. The second one may be the trad-offs and by-products of the hiding behavior of prey which could be advantageous or detrimental for all the interacting populations. A classic secondary effect is the reduction in the birth rate of prey population, because refuges are safe but rarely offer feeding or mating opportunities , [13,14] .Z. Ma and et.al [15] derived a predator-prey model with Lotka-Voltera functional response incorporating prey refuges, the refuges are considered as two types: a constant proportion of prey and a fixed number of prey using refuges. They evaluate the effect with regard to the stability of the interior equilibrium. The results show that the refuges used by prey can increase the equilibrium density of the prey population while decrease that of predator .On the other hand , it is well known that, the age factor is importance for the dynamics and evolution of many mammals. The rate of survival, growth and reproduction almost depend on age or development stage and it has been noticed that the life history of many species is composed of at least two stages, immature and mature, and the species in the first stage may often neither interact with other species nor reproduce, being raised by their mature parents. Most of classical prey-predator models of two species in the literature assumed that all predators are able to attack their prey and reproduce ignoring the fact that the life cycle of most animals consists of at least two stages (immature and mature) . Recently, several of the prey-predator models with stage-structure of species with or without time delays are proposed and analyzed $[16-19]$. In this paper the food web prey-predator model involving prey's refuges is proposed and analyzed , so that the prey growing logistically in the absence of predators. The effect of prey's refuges and prey stage-structure on the dynamical behavior of the food web model is investigated theoretically as well as numerically.

II. The mathematical model

Consider the food web model consisting of two predators-stage structure prey in which the prey species growth logistically in the absence of predation, while the predators decay exponentially in the absence of prey species. It is assumed that the prey population divides into two compartments: immature prey population N_1(t) that represents the population size at time t and mature prey population N_2(t) which denotes to population size at time t . Furthermore the population size of the first predator at time t is denoted by N_3(t) , while N_4(t) represents the population size of second predator at time t . Now in order to formulate the dynamics of such system the following assumptions are considered :

1. The immature prey depends completely in its feeding on the mature prey that growth logistically with intrinsic growth rate $\alpha > 0$ and carrying capacity $k > 0$. The immature prey individuals grown up and becomes mature prey individuals with grown up rate $\beta > 0$. However the mature prey facing death with natural death rate $d_1 > 0$.
2. There is type of protection of the prey species from facing predation by first and second predators with refuge rate constant $m \in [\,0\,,1\,)$.
3. The first and second predators consumed the mature prey individuals only according to the Lotka-Voltera type of functional response with predation rates $c_1 > 0$ *and* $c_2 > 0$ respectively and contribute a portion of such food with conversion rates $0 < e_1 < 1$ *and* $0 < e_2 < 1$ respectively. Moreover, there is an enter specific competition between these two predators with competition force rate $c_3 > 0$ *and* $c_4 > 0$ respectively. Finally in the absence of food the first and second predators facing death with natural death rate $d_2 > 0$ *and* $d_3 > 0$.

Therefore the dynamics of this model can be represented by the set of first order nonlinear differential equations:

$$\begin{aligned}
\frac{dN_1}{dt} &= \alpha\, N_2 \left(1 - \frac{N_2}{K}\right) - \beta\, N_1 \\
\frac{dN_2}{dt} &= \beta\, N_1 - d_1\, N_2 - c_1\,(1-m)\, N_2\, N_3 - c_2\,(1-m)\, N_2\, N_4 \\
\frac{dN_3}{dt} &= -d_2\, N_3 + e_1\, c_1\,(1-m)\, N_2\, N_3 - c_3\, N_3\, N_4 \qquad (1) \\
\frac{dN_4}{dt} &= -d_3\, N_4 + e_2\, c_2\,(1-m)\, N_2\, N_4 - c_4\, N_3\, N_4
\end{aligned}$$

with initial conditions $N_i(0) \geq 0$. Note that the above proposed model has thirteen parameters in all which make the analysis difficult. So in order to simplify the system , the number of parameters is reduced by using the following dimensionless variables and parameters :

$$t = \alpha\, T\,,\ u_1 = \frac{\beta}{\alpha}\,,\ u_2 = \frac{d_1}{\alpha}\,,\ u_3 = \frac{d_2}{\alpha}\,,\ u_4 = \frac{e_1\, c_1\, K}{\alpha}\ u_5 = \frac{c_3}{c_2}\,,\ u_6 = \frac{d_3}{\alpha}\,,\ u_7 = \frac{e_2\, c_2\, K}{\alpha}\,,\ u_8 = \frac{c_4}{c_1}\,,$$

$$x = \frac{N_1}{K}\,, y = \frac{N_2}{K}\,, z = \frac{c_1 N_3}{\alpha}\,, w = \frac{c_2 N_4}{\alpha}\ .$$

Then the non-dimensional form of system (1) can be written as :

$$\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{y\,(1-y)}{x} - u_1\right] = x\ f_1(\,x\,,y\,,z\,,w\,) \\
\frac{dy}{dt} &= y\left[\frac{u_1\, x}{y} - u_2 - (1-m)\, z - (1-m)\, w\right] = y\ f_2(\,x\,,y\,,z\,,w\,) \\
\frac{dz}{dt} &= z\ [-u_3 + u_4\,(1-m) - u_5\, w\,] = z\ f_3(\,x\,,y\,,z\,,w\,) \qquad (2) \\
\frac{dw}{dt} &= w\ [-u_6 + u_7\,(1-m)\, y - u_8\, z\,] = w\ f_4(\,x\,,y\,,z\,,w\,)
\end{aligned}$$

with $x(0) \geq 0\,, y(0) \geq 0\,, z(0) \geq 0$ *and* $w(0) \geq 0$. It is observed that the number of parameters have been reduced from thirteen in the system (1) to eight in the system (2) .Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the following positive four dimensional space. $R_+^4 = \{\,(\,x\,,y\,,z\,,w\,) \in R^4 : x(0) \geq 0\,, y(0) \geq 0\,, z(0) \geq 0\,, w(0) \geq 0\,\}$. Therefore these functions are

lipschitzian on R_+^4 , and hence the solution of the system (2) exists and is unique . Further, all the solutions of system (2) with non-negative initial conditions are uniformly bounded as shown in the following theorem .
Theorem (1): All the solutions of system (2) which initiate in R_+^4 are uniformly bounded .
Proof: Let $\big(x(t)\,, y(t)\,, z(t)\,, w(t)\big)$ be any solution of the system (2) with non-negative initial condition $(x_0 \,, y_0 \,, z_0 \,, w_0) \in R_+^4$. Now according to the first equation of system (2) we have :
$\frac{dx}{dt} = y\,(1 - y) - u_1\, x$. So, by using the comparison theorem on the above differential inequality with the initial point $x(0) = x_0$ we get : $x(t) \le \frac{1}{4\,u_1} + \left(x_0 - \frac{1}{4\,u_1}\right) e^{-u_1 t}$. Thus, $\lim_{t\to\infty} x(t) \le \frac{1}{4\,u_1}$ and hence $\sup\, x(t) \le \frac{1}{4\,u_1}$, $\forall\, t > 0$, $\forall\, u_1 > 0$. Now define the function : $M(t) = x(t) + y(t) + \frac{1}{u_1} z(t) + \frac{1}{u_7} w(t)$ and then taken the time derivative of $M(t)$ along the solution of the system (2.2) we get :
$\frac{dM}{dt} \le \frac{1}{4} + x - s\,M \quad where \quad s = \min\,\{ 1\,, u_2\,, u_3\,, u_6 \}$. Then $\frac{dM}{dt} + s\,M \le H \quad where \quad H = \frac{1}{4} + \frac{1}{4\,u_1}$.
Again by solving this differential inequality for the initial value $M(0) = M_0$, we get :
$M(t) \le \frac{H}{s} + \left(M_0 - \frac{H}{s}\right) e^{-st}$. Then, $\lim_{t\to\infty} M(t) \le \frac{H}{s}$ .So, $0 \le M(t) \le \frac{H}{s}$, $\forall\, t > 0$ .Hence all the solutions of system (2) are uniformly bounded and the proof is complete .

III. The existence of equilibrium points

In this section, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2) has at most five equilibrium points, which are mentioned in the following :
The equilibrium point $E_0 = (0\,,0\,,0\,,0)$, which known as the vanishing point is always exists.
The first equilibrium point $E_1 = (\bar{x}\,, \bar{y}\,, 0\,, 0)$ where $\bar{y} = 1 - u_2$, and $\bar{x} = \frac{u_2}{u_1}(1 - u_2)$ exists uniquely in Int. R_+^2 (Interior of R_+^2) of $xy - plane$ under the following necessary and sufficient condition :

$$u_2 < 1 \tag{3}$$

The first three species equilibrium point $E_2 = (\breve{x}\,, \breve{y}\,, \breve{z}\,, 0)$ where $\breve{x} = \frac{u_3}{u_1\,u_4\,(1-m)}\left[\frac{u_4\,(1-m) - u_3}{u_4\,(1-m)}\right]$,
$\breve{y} = \frac{u_3}{u_4\,(1-m)}$ and $\breve{z} = \frac{u_4\,(1-m)\,(1-u_2) - u_3}{u_4\,(1-m)^2}$ exists uniquely in Int. R_+^3 of $xyz - space$under the following necessary and sufficient condition : $u_3 < \min\,\{ u_4\,(1-m)\,,\ u_4\,(1-m)\,(1-u_2) \}$ (4)
The second three species equilibrium point $E_3 = (\tilde{x}\,, \tilde{y}\,, 0\,, \widetilde{w})$ where $\tilde{x} = \frac{u_6}{u_1\,u_7\,(1-m)}\left[\frac{u_7\,(1-m) - u_6}{u_7\,(1-m)}\right]$,
$\tilde{y} = \frac{u_6}{u_7\,(1-m)}$ and $\widetilde{w} = \frac{u_7\,(1-u_2)\,(1-m) - u_6}{u_7\,(1-m)^2}$ exists uniquely in Int.R_+^3 of $xyw - space$ under the following necessary and sufficient condition : $u_6 < \min\,\{ u_7\,(1-m)\,, u_7\,(1-u_2)(1-m) \}$ (5)
Finally, the positive equilibrium point $E_4 = (x^*\,, y^*\,, z^*\,, w^*)$ where $x^* = \frac{y^*}{u_1}(1 - y^*)$,
$y^* = \frac{1 - u_2 + \frac{u_6}{u_8}(1-m) + \frac{u_3}{u_5}(1-m)}{1 + \frac{u_7}{u_8}(1-m)^2 + \frac{u_4}{u_5}(1-m)^2}$, $z^* = \frac{u_7}{u_8}(1-m)\,y^* - \frac{u_6}{u_8}$ and $w^* = \frac{u_4}{u_5}(1-m)\,y^* - \frac{u_3}{u_5}$exists in Int.$R_+^4$ under the following necessary and sufficient conditions :

$$1 + \frac{u_7}{u_8}\,(1-m)^2 + \frac{u_4}{u_5}\,(1-m)^2 > 1 - u_2 + \frac{u_6}{u_8}\,(1-m) + \frac{u_3}{u_5}\,(1-m) \tag{6 a}$$

$$u_7\,(1-m)\left[1 - u_2 + \frac{u_6}{u_8}\,(1-m) + \frac{u_3}{u_5}\,(1-m)\right] > u_6\left[1 + \frac{u_7}{u_8}\,(1-m)^2 + \frac{u_4}{u_5}\,(1-m)^2\right] \tag{6 b}$$

$$u_4(1-m)\left[1 - u_2 + \frac{u_6}{u_8}(1-m) + \frac{u_3}{u_5}(1-m)\right] > u_3\left[1 + \frac{u_7}{u_8}(1-m)^2 + \frac{u_4}{u_5}(1-m)^2\right] \tag{6 c}$$

IV. Local stability analysis

In this section, the local stability analysis of system (2) around each of the above equilibrium points are discussed through computing the Jacobian matrix $J(x\,, y\,, z\,, w)$ of system (2) at each of them which given by :
$J = [\,a_{ij}\,]_{4\times 4} \quad where$

$a_{11} = -u_1$, $a_{12} = 1 - 2\,y$, $a_{13} = 0$, $a_{14} = 0$, $a_{21} = u_1$, $a_{22} = -u_2 - (1-m)\,z - (1-m)\,w$,
$a_{23} = -(1-m)y$, $a_{24} = -(1-m)y$, $a_{31} = 0$, $a_{32} = u_4\,(1-m)z$, $a_{33} = -u_3 + u_4\,(1-m)y - u_5\,w$,
$a_{34} = -u_5\,z$, $a_{41} = 0$, $a_{42} = u_7(1-m)\,w$, $a_{43} = -u_8\,w$, $a_{44} = -u_6 + u_7\,(1-m)\,y - u_8\,z$.

The Local stability analysis at E_0 : The Jacobian matrix of system (2) at E_0 can be written as :

$$J(E_0) = \begin{bmatrix} -u_1 & 1 & 0 & 0 \\ u_1 & -u_2 & 0 & 0 \\ 0 & 0 & -u_3 & 0 \\ 0 & 0 & 0 & -u_6 \end{bmatrix}$$

Then the characteristic equation of $J(E_0)$ is given by : $[\,\lambda^2 + A_1\,\lambda + A_2\,]\,(-u_3 - \lambda)\,(-u_6 - \lambda) = 0$

where $A_1 = u_1 + u_2$, $A_2 = u_1 (u_2 - 1)$.
So, either $(-u_3 - \lambda) (u_6 - \lambda) = 0$ (7 a)
which gives two of the eigenvalues of $J(E_0)$ by : $\lambda_{0z} = -u_3 < 0$ and $\lambda_{0w} = -u_6 < 0$
Or $\lambda^2 + A_1 \lambda + A_2 = 0$ (7 b)
which gives the other two eigenvalues of $J(E_0)$ by :

$$\lambda_{0x} = \frac{-A_1}{2} + \frac{1}{2}\sqrt{{A_1}^2 - 4A_2} \quad \text{and} \quad \lambda_{0y} = \frac{-A_1}{2} - \frac{1}{2}\sqrt{{A_1}^2 - 4A_2}$$

Therefore, if the following condition holds : $u_2 > 1$ (7 c)
E_0 is locally asymptotically stable in the R_+^4 . However it is a saddle point otherwise .
The Local stability analysis at E_1 : The Jacobian matrix of system (2) at E_1 can be written as :

$$J(E_1) = \begin{bmatrix} -u_1 & 1 - 2\bar{y} & 0 & 0 \\ u_1 & -u_2 & -(1-m)\bar{y} & -(1-m)\bar{y} \\ 0 & 0 & -u_3 + u_4(1-m)\bar{y} & 0 \\ 0 & 0 & 0 & -u_6 + u_7(1-m)\bar{y} \end{bmatrix}$$

Then the characteristic equation of $J(E_1)$ is given by :
$[\lambda^2 + \bar{A}_1 \lambda + \bar{A}_2] (-u_3 + u_4(1-m)\bar{y} - \lambda) (-u_6 + u_7(1-m)\bar{y} - \lambda) = 0$ where
$\bar{A}_1 = u_1 + u_2$ and $\bar{A}_2 = u_1 (u_2 + 2\bar{y} - 1)$
So , either $(-u_3 + u_4 (1-m) \bar{y} - \lambda) (-u_6 + u_7 (1-m)\bar{y} - \lambda) = 0$ (8 a)
which gives two of the eigenvalues of $J(E_1)$ by :
$\lambda_{1z} = -u_3 + u_4 (1-m) \bar{y}$ and $\lambda_{1w} = -u_6 + u_7 (1-m) \bar{y}$
Or $\lambda^2 + \bar{A}_1 \lambda + \bar{A}_2 = 0$ (8 b)
which gives the other two eigenvalues of $J(E_1)$ by :
$\lambda_{1x} = -\frac{\bar{A}_1}{2} + \frac{1}{2}\sqrt{{\bar{A}_1}^2 - 4\bar{A}_2}$ and $\lambda_{1y} = -\frac{\bar{A}_1}{2} - \frac{1}{2}\sqrt{{\bar{A}_1}^2 - 4\bar{A}_2}$ .Therefore , if the following condition holds : $\frac{1-u_2}{2} < \bar{y} < \min\left\{\frac{u_3}{u_4(1-m)}, \frac{u_6}{u_7(1-m)}\right\}$ (8 c)
Under condition (3) . E_1 is locally asymptotically stable in the R_+^4 . However, it is a saddle point otherwise .
The Local stability analysis at E_2 :
The Jacobian matrix of system (2) at E_2 can be written as : $J(E_2) = [b_{ij}]_{4\times4}$ where :

$b_{11} = -u_1 < 0$, $b_{12} = 1 - 2\breve{y}$, $b_{13} = 0$, $b_{14} = 0$, $b_{21} = u_1 > 0$, $b_{22} = -u_2 - (1-m)\breve{z} < 0$,
$b_{23} = -(1-m)\breve{y} < 0, b_{24} = -(1-m)\breve{y} < 0, b_{31} = 0, b_{32} = u_4(1-m)\breve{y} > 0$,
$b_{33} = -u_3 + u_4(1-m)\breve{y}, b_{34} = -u_5\breve{z} < 0 , b_{41} = 0, b_{42} = 0, b_{43} = 0, b_{44} = -u_6 + u_7(1-m)\breve{y} - u_8\breve{z}$.

Then the characteristic equation of $J(E_2)$ is given by :
$[\lambda^3 + \breve{A}_1 \lambda^2 + \breve{A}_2 \lambda + \breve{A}_3] (-u_6 + u_7 (1-m) \breve{y} - u_8 \breve{z} - \lambda) = 0$ where ,

$$\breve{A}_1 = 1 + u_1 - \frac{u_3}{u_4(1-m)} ,$$

$$\breve{A}_2 = \frac{u_1 [u_4 (u_2 - 1) (1-m) + 2u_3]}{u_4 (1-m)} + \frac{(u_1 + u_3) [u_4 (1 - u_2) (1-m) - u_3]}{u_4 (1-m)} ,$$

$$\breve{A}_3 = \frac{u_1 u_3 [u_4 (1-m) (1 - u_2) - u_3]}{u_4 (1-m)} .$$

So , either $-u_6 + u_7 (1-m) \breve{y} - u_8 \breve{z} - \lambda = 0$ (9 a)
Or $\lambda^3 + \breve{A}_1 \lambda^2 + \breve{A}_2 \lambda + \breve{A}_3 = 0$ (9 b)
Hence from equation (2.9 a) we obtain that : $\lambda_{2w} = -u_6 + u_7 (1-m) \breve{y} - u_8 \breve{z}$
which is negative provided that : $u_7 (1-m) \breve{y} < u_6 + u_8 \breve{z}$ (9 c)
On the other hand by using Routh-Hawirtiz criterion equation (9 b) has roots (eigenvalues) with negative real parts if and only if : $\breve{A}_1 > 0$, $\breve{A}_3 > 0$ *and* $\Delta = \breve{A}_1 \breve{A}_2 - \breve{A}_3 > 0$. Now direct computation gives that :

$$\Delta = \frac{u_3 [u_4 (1-m) [u_2 u_3 + u_1(1+u_1) + u_4 (1-u_2) (1-m)] + {u_3}^2]}{{u_4}^2 (1-m)^2} - \frac{{u_3}^2 [u_1 + 2u_4 (1-m)]}{{u_4}^2 (1-m)^2}$$

So , $\Delta > 0$ under the following condition :

$$u_4 (1-m) [u_2 u_3 + u_1 (1+u_1) + u_4 (1-u_2) (1-m)] + {u_3}^2 > u_3 [u_1 + 2u_4 (1-m)] \qquad (10)$$

Now it is easy to verify that $\breve{A}_1 > 0$ and $\breve{A}_3 > 0$ under condition (4).Then all the eigenvalues λ_{2x} , λ_{2y} and λ_{2z} of equation (9 b) have negative real parts. So, E_2 is locally asymptotically stable if and only if conditions (9 c) *and* (10) are hold. However , it is a saddle point otherwise .
The Local stability analysis at E_3 :
The Jacobian matrix of system (2) at E_3 can be written as : $J(E_3) = [c_{ij}]_{4\times4}$ where ,

$c_{11} = -u_1 < 0$, $c_{12} = 1 - 2\tilde{y}$, $c_{13} = 0$, $c_{14} = 0$, $c_{21} = u_1 > 0$, $c_{22} = -u_2 - (1-m)\tilde{w} > 0$,
$c_{23} = -(1-m)\tilde{y} > 0$, $c_{24} = -(1-m)\tilde{y} < 0$, $c_{31} = 0$, $c_{32} = 0$, $c_{33} = -u_3 + u_4(1-m)\tilde{y} - u_5\tilde{w}$,

$c_{34} = 0$, $c_{41} = 0$, $c_{42} = u_7(1-m)\,\tilde{w} > 0$, $c_{43} = -u_8\,\tilde{w} < 0$, $c_{44} = -u_6 + u_7\,(1-m)\,\tilde{y}$.
Then the characteristic equation of $J(E_3)$ is given by :
$[\,\lambda^3 + B_1\,\lambda^2 + B_2\,\lambda + B_3\,]\,(-u_3 + u_4\,(1-m)\,\tilde{y} - u_5\,\tilde{w} - \lambda\,) = 0$ where

$$B_1 = 1 + u_1 - \frac{u_6}{u_7\,(1-m)} \;,$$

$$B_2 = \frac{u_1\,[\,u_7\,(u_2-1)\,(1-m) + 2\,u_2\,]}{u_7\,(1-m)} + \frac{(\,u_6 - u_1\,)\,[\,u_7\,(1-u_2)\,(1-m) - u_6\,]}{u_7\,(1-m)} \;,$$

$$B_3 = \frac{u_1 u_6\,[\,u_7(1-u_2)(1-m) - u_6\,]}{u_7\,(1-m)} - \frac{u_1(1+u_2)\,[\,u_7(1-m) - 2u_6\,]}{u_7\,(1-m)} \;.$$

while $\Delta = B_1\,B_2 - B_3$

$$= \frac{u_6 u_7(1-m)[u_1(1+u_1+u_6) + u_7(1-u_2)(1-m)]}{{u_7}^2\,(1-m)^2} - \frac{{u_6}^2\,[u_1 + u_7\,(2+u_1-u_2)(1-m)]}{{u_7}^2\,(1-m)^2}$$

So , either $\quad -u_3 + u_4\,(1-m)\,\tilde{y} - u_5\,\tilde{w} - \lambda = 0 \qquad (11\,a)$

Or $\quad \lambda^3 + B_1\,\lambda^2 + B_2\,\lambda + B_3 = 0 \qquad (11\,b)$

From equation (11 a) we obtain that : $\lambda_{3z} = -u_3 + u_4\,(1-m)\,\tilde{y} - u_5\,\tilde{w}$ which is negative provided that:

$$u_4\,(1-m)\,\tilde{y} < u_3 + u_5\,\tilde{w} \qquad (12)$$

On the other hand it is easy to verify that $B_1 > 0$ and $B_3 > 0$ under condition (5) while $\Delta > 0$ under the following condition :

$$u_7(1-m)\,[u_1(1+u_1+u_6) + u_7(1-u_2)(1-m)] > u_6\,[u_1 + u_7(2+u_1-u_2)(1-m)] \qquad (13)$$

Then all the eigenvalues λ_{3x} , λ_{3y} and λ_{3w} of equation (11 b) have negative real parts. So, E_3 is locally asymptotically stable if and only if conditions (12) and (13)are hold. However , it is a saddle point otherwise.

Theorem (2): Assume that the positive equilibrium point $E_4 = (\,x^*, y^*, z^*, w^*)$ of system (2)exists in $IntR_+^4$. Then it is locally asymptotically stable provided that the following conditions hold :

$$ {P_{12}}^2 < \frac{4}{3}\,P_{11}\,P_{22} \qquad (14\,a)$$

$$ {P_{23}}^2 < \frac{2}{3}\,P_{22}\,P_{33} \qquad (14\,b)$$

$$ {P_{24}}^2 < \frac{2}{3}\,P_{22}\,P_{44} \qquad (14\,c)$$

$$ P_{33}\,P_{44} < {P_{34}}^2 \qquad (14\,d)$$

Proof : It is easy to verify that , the linearized system of system (2) can be written as :
$\frac{dX}{dt} = \frac{dR}{dt} = J(E_4)\,R$, here $X = (\,x, y, z, w\,)$ and $R = (\,r_1, r_2, r_3, r_4\,)^t$ $where$ $r_1 = x_1 - {x_1}^*$,
$r_2 = x_2 - {x_2}^*, r_3 = x_3 - {x_3}^*$ and $r_4 = x_4 - {x_4}^*$. Moreover , $J(E_4) = [\,d_{ij}\,]_{4\times 4}$ $where$
$d_{11} = -u_1, d_{12} = 1 - 2y^*, d_{13} = 0\,, d_{14} = 0, d_{21} = u_1, d_{22} = -u_2 - (1-m)y^* - (1-m)w^*$,
$d_{23} = -(1-m)\,y^*\,, d_{24} = -(1-m)\,y^*\,, d_{31} = 0\,, d_{32} = u_4\,(1-m)y^*$,
$d_{33} = -u_3 + u_4\,(1-m)\,y^* - u_5\,w^*, d_{34} = -u_5\,z^*\,, d_{41} = 0\,, d_{42} = u_7\,(1-m)\,w^*\,, d_{43} = -u_8\,w^*$,
$d_{44} = -u_6 + u_7\,(1-m)\,y^* - u_8\,z^*$.

Now consider the following positive definite function : $V = \frac{{r_1}^2}{2} + \frac{{r_2}^2}{2} + \frac{{r_3}^2}{2} + \frac{{r_4}^2}{2}$. It is clearly that $V : R_+^4 \longrightarrow R$ and is a C^1 positive definite function. Now by differentiating V with respect to time t and doing some algebraic manipulation gives that :

$\frac{dV}{dt} = -P_{11}\,{r_1}^2 + P_{12}\,r_1\,r_2 - P_{22}\,{r_2}^2 + P_{23}\,r_2\,r_3 + P_{24}\,r_2\,r_4 - P_{33}\,{r_3}^2 + P_{34}\,r_3\,r_4 - P_{44}\,{r_4}^2$ where ,

$P_{11} = u_1\,, P_{12} = u_2 + 1 - 2\,y^*, P_{22} = u_2 + (1-m)\,z^* + (1-m)\,w^*, P_{23} = (1-m)(\,u_4\,z^* - y^*\,)$,
$P_{24} = (1-m)(\,u_7\,w^* - y^*\,)\,, P_{33} = u_3 - u_4\,(1-m)\,y^* + u_5\,w^*\,, P_{34} = -u_5\,z^* - u_8\,w^*$,
$P_{44} = u_6 - u_7\,(1-m)\,y^* + u_8\,z^*$.

Now it is easy to verify that the above set of conditions guarantee the quadratic terms give below :

$$\frac{dV}{dt} \le -\left[\sqrt{P_{11}}\;r_1 - \frac{\sqrt{P_{22}}}{\sqrt{3}}\,r_2\right]^2 - \left[\frac{\sqrt{P_{22}}}{\sqrt{3}}\,r_2 - \frac{\sqrt{P_{33}}}{\sqrt{2}}\,r_3\right]^2 - \left[\frac{\sqrt{P_{22}}}{\sqrt{3}}\,r_2 - \frac{\sqrt{P_{44}}}{\sqrt{2}}\,r_4\right]^2 - \left[\frac{\sqrt{P_{33}}}{\sqrt{2}}\,r_3 + \frac{\sqrt{P_{44}}}{\sqrt{2}}\,r_4\right]^2$$

So , $\frac{dV}{dt}$ is negative definite , and hence V is a Lyapunov function .Thus , E_4 is locally asymptotically stable and the proof is complete .

V. Global stability analysis

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2) is studied analytically with the help of Lyapunov method as shown in the following theorems .

Theorem (3): Assume that the equilibrium point $E_0 = (0,0,0,0)$ of system (2) is locally asymptotically stable in the R_+^4 . Then the equilibrium point E_0 of system (2) is globally asymptotically stable .

Proof: Consider the following function : $V_1(x,y,z,w) = c_1 x + c_2 y + c_3 z + c_4 w$, where c_1, c_2, c_3 and c_4 are positive constants to be determine .Clearly $V_1 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation , gives that :

$$\frac{dV_1}{dt} = c_1\, y(1-y) + u_1(c_2 - c_1)x - c_2\, u_2 y + (c_3\, u_4 - c_2)(1-m)yz - c_2\, u_2\, y$$
$$+(c_4\, u_7 - c_2)(1-m)yw - c_3\, u_3\, z - c_4\, u_6\, w - (c_3\, u_5 + c_4\, u_8)\, w\, z\ .$$

By choosing $c_1 = c_2 = 1$, $c_3 = \frac{1}{u_4}$, $c_4 = \frac{1}{u_7}$ we get : $\frac{dV_1}{dt} \le -(u_2 - 1)\, y$. Then we obtain that $\frac{dV_1}{dt}$ is negative definite and hence V_1 is a Lyapunov function. Thus E_0 is globally asymptotically stable and the proof is complete .

Theorem (4) :Assume that the equilibrium point $E_1 = (\bar{x}, \bar{y}, 0, 0)$of system (2) is locally asymptotically stable in the $Int\, R_+^2$, Then E_1 is globally asymptotically stable on any region $\Omega_1 \subset Int\, R_+^2$ that satisfies the following conditions :

$$\frac{1}{x} - \frac{(y+\bar{y})}{\bar{x}} + \frac{u_1}{(1-m)\, y\, \bar{y}} \le 2\sqrt{\frac{u_1}{(1-m)\, x\, y\, \bar{y}}} \quad (15\, a)$$

$$\frac{y+\bar{y}}{\bar{x}} < \frac{1}{x} + \frac{u_1}{(1-m)\, y\, \bar{y}} \quad (15\, b)$$

$$\frac{y^2(x-\bar{x})^2}{x\, \bar{x}} < \left[\sqrt{\frac{(1-y)y}{x\, \bar{x}}}(x-\bar{x}) - \sqrt{\frac{u_1\, \bar{x}}{(1-m)\, y\, \bar{y}^2}}(y-\bar{y})\right]^2 \quad (15\, c)$$

Proof: Consider the following function :

$$V_2(x,y,z,w) = c_1\left(x - \bar{x} - \bar{x}\, \ln\frac{x}{\bar{x}}\right) + c_2\left(y - \bar{y} - \bar{y}\, \ln\frac{y}{\bar{y}}\right) + c_3\, z + c_4\, w$$

where c_1, c_2, c_3 and c_4 are positive constants to be determine .Clearly $V_2 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_2 with respect to time t and doing some algebraic manipulation , gives that :

$$\frac{dV_2}{dt} = -c_1\frac{(x-\bar{x})^2\, \bar{y}}{x\, \bar{x}} + \left[\frac{c_1}{x} - \frac{c_1(y+\bar{y})}{\bar{x}} + \frac{c_2 u_1}{y}\right](x-\bar{x})(y-\bar{y}) - c_1\, \frac{\bar{y}}{x\, \bar{x}}\, (x-\bar{x})^2 - c_2\, \frac{u_1\, \bar{x}}{y\, \bar{y}}(y-\bar{y})^2$$
$$+c_2\frac{u_1}{y}(x-\bar{x})(y-\bar{y}) - c_2\, (1-m)(y-\bar{y})z - c_2(1-m)(y-\bar{y})\, w - c_3\, u_3\, z$$
$$+c_3\, u_4\, (1-m)\, y - c_4\, u_6\, w + c_4\, u_7\, (1-m)\, y\, w - (c_3\, u_5 + c_4\, u_8)\, w\, z\, .$$

By choosing $c_1 = 1$, $c_2 = \frac{1}{(1-m)\bar{y}}$, $c_3 = \frac{1}{u_3}$, $c_4 = \frac{1}{u_6}$ we get :

$$\frac{dV_2}{dt} \le -\left[\sqrt{\frac{\bar{y}}{x\, \bar{x}}}(x-\bar{x}) - \sqrt{\frac{u_1 \bar{x}}{(1-m)\, y\, \bar{y}^2}}(y-\bar{y})\right]^2 + \frac{y^2(x-\bar{x})^2}{x\, \bar{x}} - \left[\frac{u_3 - u_4\, (1-m)\, \bar{y}}{u_3\, \bar{y}}\right] y\, z$$
$$-\left[\frac{u_6 - u_7\, (1-m)\, \bar{y}}{u_6\, \bar{y}}\right] y\, w$$

So , according to condition (8 c) we obtain that : $\frac{dV_2}{dt} \le -\left[\sqrt{\frac{\bar{y}}{x\bar{x}}}(x-\bar{x}) - \sqrt{\frac{u_1\bar{x}}{(1-m)y\bar{y}^2}}(y-\bar{y})\right]^2 + \frac{y^2\, (x-\bar{x})^2}{x\, \bar{x}}$. However, conditions (15 a) and (15 b) guarantee the completeness of the quadratic term between x and y . So, if condition (15 c) holds then we obtain that $\frac{dV_2}{dt}$ is negative definite on the region Ω_1 and hence V_2 is a Lyapunov function defined on the region Ω_1 . Thus E_1 is globally asymptotically stable on the region Ω_1 and the proof is complete .

Theorem (5): Assume that the equilibrium point $E_2 = (\check{x}, \check{y}, \check{z}, 0)$ of system (2) is locally asymptotically stable in $IntR_+^3$. Then E_2 is globally asymptotically stable on any region $\Omega_2 \subset Int\, R_+^3$ that satisfied the following conditions :

$$\frac{1}{x} + \frac{u_1}{y} - \frac{(y+\check{y})}{\check{x}} \le 2\sqrt{\frac{u_1}{x\, y}} \quad (16\, a)$$

$$\frac{y+\check{y}}{\check{x}} < \frac{1}{x} + \frac{u_1}{y} \quad (16\, b)$$

$$\frac{y^2\, (x-\check{x})^2}{x\, \check{x}} < \left[\sqrt{\frac{\check{y}}{x\, \check{x}}}\, (x-\check{x}) - \sqrt{\frac{u_1\, \check{x}}{y\, \check{y}}}\, (y-\check{y})\right]^2 \quad (16\, c)$$

In addition that the following condition holds : $\frac{u_5\,\check{z}}{u_4} + (1-m)\,\check{y} < \frac{u_6}{u_7}$ (16 d)

Proof: Consider the following function :

$$V_3(x, y, z, w) = c_1\left(x - \check{x} - \check{x}\ln\frac{x}{\check{x}}\right) + c_2\left(y - \check{y} - \check{y}\ln\frac{y}{\check{y}}\right) + c_3\left(z - \check{z} - \check{z}\ln\frac{z}{\check{z}}\right) + c_4\,w$$

where c_1, c_2, c_3 and c_4 are positive constants to be determine .Clearly $V_3: R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_3 with respect to time t and doing some algebraic manipulation , gives that :

$$\frac{dV_3}{dt} = -c_1\frac{(x-\check{x})^2}{x} + \left[\frac{c_1}{x} - \frac{c_1(y+\check{y})}{\check{x}} + \frac{c_2u_1}{y}\right](x-\check{x})(y-\check{y}) + c_1\frac{y^2}{x\check{x}}(x-\check{x})^2 - c_2\frac{u_1\check{x}}{y\,\check{y}}(y-\check{y})^2$$
$$-c_2(1-m)(y-\check{y})(z-\check{z}) - c_2(1-m)(y-\check{y})\,w + c_3\,u_4(1-m)(y-\check{y})(z-\check{z})$$
$$-c_3\,u_5\,(z-\check{z})\,w - c_4\,u_6\,w + c_4\,u_7\,(1-m)\,y\,w - c_4\,u_8\,w\,z\ .$$

By choosing $c_1 = 1$, $c_2 = 1$, $c_3 = \frac{1}{u_4}$, $c_4 = \frac{1}{u_7}$ we get :

$$\frac{dV_3}{dt} \le -\left[\sqrt{\frac{\check{y}}{x\,\check{x}}}\,(x-\check{x}) - \sqrt{\frac{u_1\,\check{x}}{y\,\check{y}}}\right]^2 + \frac{y^2}{x\,\check{x}}\,(x-\check{x})^2 - \left[\frac{u_6}{u_7} - \frac{u_5\check{z}}{u_4} - (1-m)\,\check{y}\right]w$$

Now, according to condition (16 d) we obtain that : $\frac{dV_3}{dt} \le -\left[\sqrt{\frac{\check{y}}{x\,\check{x}}}\,(x-\check{x}) - \sqrt{\frac{u_1\,\check{x}}{y\,\check{y}}}\right]^2 + \frac{y^2}{x\,\check{x}}\,(x-\check{x})^2$. However, conditions (16 a) and (16 b) guarantee the completeness of the quadratic term between x and y . So, if condition (16 c) holds . Therefore , $\frac{dV_3}{dt}$ is negative on the region Ω_2 and hence V_3 is a Lyapunov function defined on the region Ω_2 . Thus E_2 is globally asymptotically stable and the proof is complete .

Theorem (6): Assume that the equilibrium point $E_3 = (\tilde{x}, \tilde{y}, 0, \tilde{w})$ of system (2) is locally asymptotically stable in $Int\,R_+^3$ .Then E_3 is globally asymptotically stable on any region $\Omega_3 \subset Int\,R_+^3$ that satisfies the following conditions :

$$\frac{1}{x} - \frac{(y+\tilde{y})}{\tilde{x}} + \frac{u_1}{y} \le 2\sqrt{\frac{u_1}{x\,y}} \qquad (17\,a)$$

$$\frac{y+\tilde{y}}{\tilde{x}} < \frac{1}{x} + \frac{u_1}{y} \qquad (17\,b)$$

$$\frac{y^2}{x\,\tilde{x}}\,(x-\tilde{x})^2 < \left[\sqrt{\frac{\tilde{y}}{x\,\tilde{x}}}\,(x-\tilde{x}) - \sqrt{\frac{u_1\,\tilde{x}}{y\,\tilde{y}^2}}\,(y-\tilde{y})\right]^2 \qquad (17\,c)$$

In addition that the following condition holds : $\frac{u_8}{u_7}\,\tilde{w} + (1-m)\,\tilde{y} < \frac{u_3}{u_4}$ (17 d)

Proof: Consider the following function :

$$V_4(x, y, z, w) = c_1\left(x - \tilde{x} - \tilde{x}\ln\frac{x}{\tilde{x}}\right) + c_2\left(y - \tilde{y} - \tilde{y}\ln\frac{y}{\tilde{y}}\right) + c_3\,z + c_4\left(w - \tilde{w} - \tilde{w}\ln\frac{w}{\tilde{w}}\right)$$

where c_1, c_2, c_3 and c_4 are positive constants to be determine .Clearly $V_4: R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_4 with respect to time t and doing some algebraic manipulation , gives that :

$$\frac{dV_4}{dt} = -c_1\frac{(x-\tilde{x})^2\,\tilde{y}}{x\,\tilde{x}} + \left[\frac{c_1}{x} - \frac{c_1(y+\tilde{y})}{\tilde{x}} + \frac{c_2u_1}{y}\right](x-\tilde{x})(y-\tilde{y}) + c_1\frac{y^2}{x\,\tilde{x}}(x-\tilde{x})^2 - c_2\frac{u_1\tilde{x}}{y\,\tilde{y}}(y-\tilde{y})^2$$
$$-c_2(1-m)(y-\tilde{y})z - c_2(1-m)(y-\tilde{y})(w-\tilde{w}) - c_3\,u_3\,z + c_3\,u_4\,(1-m)\,y\,z$$
$$-c_3u_5\,w\,z + c_4\,u_7\,(1-m)(y-\tilde{y})(w-\tilde{w}) - c_4\,u_8\,(w-\tilde{w})\,z\,.$$

By choosing $c_1 = 1$, $c_2 = 1$, $c_3 = \frac{1}{u_4}$, $c_4 = \frac{1}{u_7}$ we get :

$$\frac{dV_4}{dt} \le -\left[\sqrt{\frac{\tilde{y}}{x\,\tilde{x}}}\,(x-\tilde{x}) - \sqrt{\frac{u_1\,\tilde{x}}{(1-m)\,y\,\tilde{y}^2}}\,(y-\tilde{y})\right]^2 - \left[\frac{u_3}{u_4} - (1-m)\,\tilde{y} - \frac{u_8}{u_7}\,\tilde{w}\right]z + \frac{y^2}{x\,\tilde{x}}\,(x-\tilde{x})^2$$

Now, according to condition (17 d) we obtain that : $\frac{dV_4}{dt} \le -\left[\sqrt{\frac{\tilde{y}}{x\,\tilde{x}}}(x-\tilde{x}) - \sqrt{\frac{u_1\tilde{x}}{(1-m)y\tilde{y}^2}}(y-\tilde{y})\right]^2 + \frac{y^2}{x\,\tilde{x}}(x-\tilde{x})^2$. However ,conditions (17 a) and (17 b) guarantee the completeness of the quadratic term between x and y .So, if condition (17 c) holds then we obtain that $\frac{dV_4}{dt}$ is negative definite on the region Ω_3 and hence V_4 is a Lyapunov function defined on the region Ω_3 . Thus E_3 is globally asymptotically stable on the region Ω_3 and the proof is complete .

Theorem (7): Assume that the equilibrium point $E_4 = (x^* , y^*, z^* , w^*)$ of system (2) is locally asymptotically stable in the R_+^4 . Then E_4 is globally asymptotically stable on any region $\Omega_4 \subset Int\, R_+^4$ that satisfies the following conditions :

$$\frac{1}{x} - \frac{(y + y^*)}{x^*} + \frac{u_1}{y} \leq 2\sqrt{\frac{u_1}{x\,y}} \tag{18 a}$$

$$\frac{y + y^*}{x^*} < \frac{1}{x} + \frac{u_1}{y} \tag{18 b}$$

$$\frac{y^2}{x\,x^*}(x - x^*)^2 + \left(1 + \frac{u_8}{u_7}\right)(w\,z^* + w^* z) < \beta \quad where \tag{18 c}$$

$$\beta = \left[\sqrt{\frac{y^*}{x\,x^*}}\,(x - x^*) - \sqrt{\frac{u_1\,x^*}{y\,y^*}}\,(y - y^*)\right]^2$$

Proof: Consider the following function :

$$V_5(x , y , z , w) = c_1\left(x - x_4 - x_4\ \ln\frac{x}{x_4}\right) + c_2\left(y - y_4 - y_4\ \ln\frac{y}{y_4}\right) + c_3\left(z - z_4 - z_4\ \ln\frac{z}{z_4}\right) + c_4\left(w - w_4 - w_4\ \ln\frac{w}{w_4}\right)$$

where $c_1 , c_2 , c_3\ and\ c_4$ are positive constants to be determine .Clearly $V_5 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_5 with respect to time t and doing some algebraic manipulation , gives that :

$$\frac{dV_5}{dt} = -c_1\frac{y^*}{x\,x^*}(x - x^*)^2 + \left[\frac{c_1}{x} - \frac{c_1(y + y^*)}{x^*} + \frac{c_2 u_1}{y}\right](x - x^*)(y - y^*) - c_2\,\frac{u_1\,x^*}{y\,y^*}\,(y - y^*)^2 + \frac{c_1\,y^2}{x\,x^*}\,(x - x^*)^2 - (c_4\,u_8 + c_3\,u_4)\,(w - w^*)\,(z - z^*) + (c_4\,u_7 - c_2)(1 - m)(y - y^*)(w - w^*) + (c_3\,u_4 - c_2)\,(1 - m)\,(y - y^*)\,(z - z^*)$$

By choosing $c_1 = c_2 = 1$, $c_3 = \frac{1}{u_4}$, $c_4 = \frac{1}{u_7}$ we get :

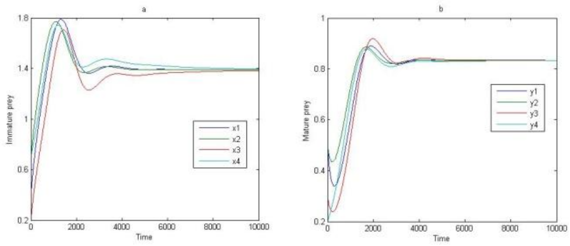
$$\frac{dV_5}{dt} \leq -\left[\sqrt{\frac{y^*}{x\,x^*}}\,(x - x^*) - \sqrt{\frac{u_1\,x^*}{y\,y^*}}\,(y - y^*)\right]^2 + \frac{y^2}{x\,x^*}(x - x^*)^2 + \frac{1}{u_7}\,(u_7 + u_8)\,(w\,z^* + w^* z)$$

However, conditions (18 a) and (18 b) guarantee the completeness of the quadratic term between $x\ and\ y$. So , if condition (18 c) holds then we obtain that $\frac{dV_5}{dt}$ is negative definite on the region Ω_4 and hence V_5 is a Lyapunov function defined on the region Ω_4 . Thus E_4 is globally asymptotically stable on the region Ω_4 and the proof is complete .

VI. Numerical simulation

In this section the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are : first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that , for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Fig. (1) .

$$u_1 = 0.1\ , u_2 = 0.1\ , u_3 = 0.1\ , u_4 = 0.8\ , u_5 = 0.2$$
$$u_6 = 0.1\ , u_7 = 0.8\ , u_8 = 0.2\ , m = 0.8 \tag{19}$$



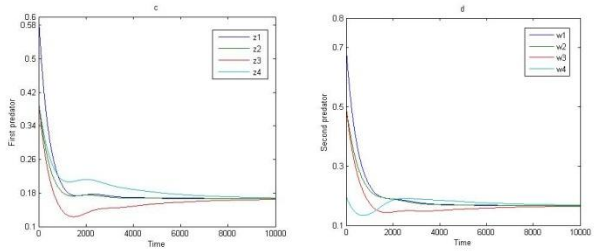


Fig. (1): Time series of the solution of system (2)that started from four different initial points $(0.4\,,0.5\,,0.6\,,0.7)$, $(0.7\,,0.8\,,0.4\,,0.5)$, $(0.2\,,0.3\,,0.4,0.5)$ and $(0.7\,,0.2\,,0.4\,,0.2)$ for the data given by Eq. (2.19) . (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time, (d) trajectories of w as a function of time .

Clearly, Fig. (1) shows that system (2) has a globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4 = (1.39\,,0.83\,,0.17\,,0.17)$ starting from four different initial points and this is confirming our obtained analytical results..Now in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (19) with varying one parameter at each time. It is observed that for the data as given in Eq. (19) with $0.1 \leq u_1 \leq 0.9$, the solution of system (2) approaches asymptotically to the positive equilibrium point as shown in Fig. (2) .

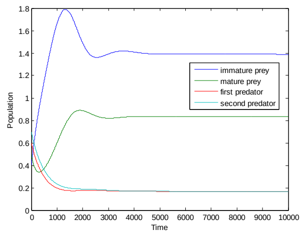


Fig. (2): Time series of the solution of system (2) for the data given by Eq. (19) which approaches to $(1.39\,,0.83\,,0.17\,,0.17)$ in the interior of R_+^4 .

By varying the parameter u_2 and keeping the rest of parameters values as in Eq. (19) , it is observed that for $0.2 \leq u_2 \leq 0.4$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 , while for $0.5 \leq u_2 \leq 0.9$ the solution of system (2) approaches asymptotically to $E_1 = (\,\bar{x}\,,\bar{y}\,,0\,,0\,)$ in the interior of the positive quadrant of $xy - plane$ as shown in Fig (3).

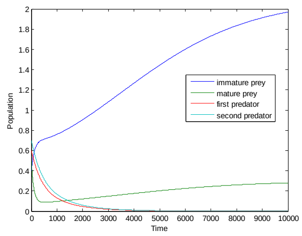


Fig.(3): Times series of the solution of system (2) for the data given by Eq. (19) which approaches to $(1.97\,,0.28\,,0\,,0)$ in the interior of the positive quadrant of $xy - plane$.

On the other hand varying the parameter u_3 keeping the rest of parameters values as in Eq. (19) ,it is observed that for $0.2 \leq u_3 \leq 0.9$ system (2) approaches asymptotically to $E_1 = (\,\bar{x}\,,\bar{y}\,,0\,,0\,)$ in the interior of the positive quadrant of $xy - plane$ .Moreover ,varying the parameter u_4 and keeping the rest of parameters values as in Eq. (19) , it is observed that for $0.1 \leq u_4 \leq 0.4$ the solution of system (2) approaches asymptotically to $E_1 = (\,\bar{x}\,,\bar{y}\,,0\,,0\,)$ in the interior of the positive quadrant of $xy - plane$, while for $0.5 \leq u_4 \leq 0.9$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 .For the parameter $0.1 \leq u_5 \leq 0.9$ and keeping the rest of parameters values as in Eq. (19) showed that the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 .Moreover ,for the parameters values given in Eq.(19)with$0.2 \leq u_6 \leq 0.9$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 . For the parameters values given in Eq. (19) with $0.1 \leq u_7 \leq 0.9$ the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 . For the parameter $0.1 \leq u_8 \leq 0.9$ and keeping the rest of parameters values as in Eq. (19) showed that the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 .For the parameters value given in Eq. (19) and varying the parameter m in the rang $0.1 \leq m \leq 0.9$ showed that the solution of system (2) approaches asymptotically to a positive equilibrium point E_4 .For the parameters values given in Eq. (19) with $u_4 = 0.9$, $u_5 = 0.1$, $u_7 = 0.1$ *and* $u_8 = 0.8$,the solution of system (2) approaches asymptotically to $E_2 = (\check{x}, \check{y}, \check{z}, 0)$ in the interior of the positive quadrant of $xyz - plane$ as shown in Fig. (4) .

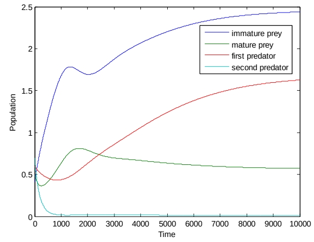


Fig. (**4**): Time series of the solution of system (2) for the data given by Eq. (19) with $u_4 = 0.9, u_5 = 0.1, u_7 = 0.1$ *and* $u_8 = 0.8$ which approaches asymptotically to $(2.44, 0.57, 1.63, 0.01)$ in the interior of the positive quadrant of $xyz - space$.

Moreover , for the parameters values given in Eq. (19) with $u_4 = 0.1, u_5 = 0.8$, $u_6 = 0.01$ the solution of system (2) approaches asymptotically to $E_3 = (\tilde{x}, \tilde{y}, 0, \widetilde{w})$ in the interior of the positive quadrant of $xyw - space$ as shown in Fig. (5).

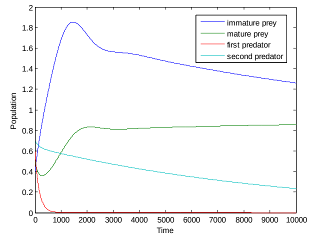


Fig.(**5**): Time series of the solution of system (2) for the data given by Eq. (19) with $u_4 = 0.1, u_5 = 0.8, u_6 = 0.01$,which approaches asymptotically to $(1.26, 0.86, 0, 0.23)$ in the interior of the positive quadrant of $xyw - space$.

Finally, the dynamical behavior at the vanishing equilibrium point $E_0 = (0,0,0,0)$ is investigated by choosing $u_2 = 2$ and keeping other parameters fixed as given in Eq. (19) , and then the solution of system (2) is drawn in Fig. (6).

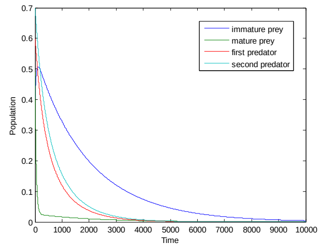


Fig.(6): Time series of the solution of system (2) for the data given by Eq. (19) with $u_2 = 2$, which approaches to $(0, 0, 0, 0)$.

Obviously, Fig. (6) shows clearly the convergence of the solution of system (2) to the vanishing equilibrium point $E_0 = (0,0,0,0)$ when the parameters increase up to a specific values . clearly the used valued in Fig. (6) satisfy the stability condition of the vanishing equilibrium point .

VII. Conclusion and discussion

In this chapter ,we have considered a prey-predator system incorporating a stage structure of prey with refuge . It is assumed that the predator species prey upon the prey according to Lotka-Voltera type of functional response. The existence, uniqueness and boundedness of the solution of the system are discussed. The existence of all possible equilibrium points is studied. The local and global dynamical behaviors of the system are studied analytically as well as numerically. Finally to understand the effect of varying each parameter on the global dynamics of system (2) has been solved numerically for a biological feasible set of hypothetical parameters values and the following results are obtained :

1 . For the set of hypothetical parameters values given in Eq. (19) , the system (2) approaches asymptotically to global stable positive equilibrium point .

2 . It is observed that the system (2) has no effect on the dynamical behavior for the data given in Eq.(19)with varying the parameter value u_1 and the system still approaches to positive equilibrium point.
3 . As the natural death rate of mature prey u_2 increasing in the range $0.2 < u_2 < 0.4$ and keeping other parameters fixed as in Eq. (19) , then again the solution of system (2) approaches asymptotically to the positive equilibrium point .However ,increasing u_2 in the range $0.5 < u_2 < 0.9$ will cause extinction in the predators and the solution of system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$.
4 . As the first predator natural death rate u_3 increasing keeping the rest of parameters as in Eq. (19) , then again the solution of system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$. It is observed that the natural death rate u_6 of the second predator has the same effect as u_1 .
5 . As the predation rate u_4 increasing in the range $0.1 \le u_4 \le 0.7$ causes extinction in the predators and the solution of system (2) approaches asymptotically to $E_1 = (\bar{x}, \bar{y}, 0, 0)$. However increasing u_4 in the range $0.5 \le u_4 \le 0.9$ then again the solution of system (2) approaches asymptotically to the positive equilibrium point .
6 . As the competition rate u_5 increasing keeping the rest of parameters as in Eq. (19) , then again the solution of system (2) approaches asymptotically to the positive equilibrium point . It is observed that the competition rate u_8 and the number of preys inside refuge m have the same effect as u_1 .
7 . As the predation rate u_7 increasing in the range $0.1 \le u_7 \le 0.9$ and keeping the rest of parameters as in Eq.(19), then again the solution of system (2) approaches asymptotically to the positive equilibrium point .

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