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Bounds on generalized fuzzy entropy measure

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Abstract: *In this communication, we propose a new generalized fuzzy entropy measure using segment decomposition and effective range and study its particular cases. Also some fuzzy coding theorems have been established.*
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I. Introduction

In coding theory, it is assumed that Q is a finite set of alphabets and there are D code characters. A codeword is defined as a finite sequence of code characters and a variable length code C of size K is a set of K code words denoted by $c_1, c_2, \dots, c_k$ with lengths $n_1, n_2, \dots, n_k$ respectively. Without loss of generality it may be assumed that $n_1 \le n_2 \le \dots \le n_k$.

The channel, which is considered here, is not noiseless. In other words, the codes considered are error correcting codes. The criterion for error correcting is defined in terms of a mapping α, which depends on the noise characteristics of the channel. This mapping α is called the error admissibility mapping. Given codeword 'c' and error admissibility α, the set of codeword's received over the channel when c was sent, denoted by $\alpha(c)$ is the error range of c.

Various kinds of error pattern can be described in terms of mapping α. In particular α may be defined as (Bernard & Sharma [1988])

$$\alpha_e(c) = \{\underline{u} | w(c - \underline{u}) \le e\},$$

Where e is the random substitution error and $w(c - \underline{u})$ is the Humming weight, i.e. the number of non-zero coordinates of $(c - \underline{u})$. It can be easily verified by Bernard and Sharma [1988] that the number of sequences in $\alpha_e(c)$ denoted as $|\alpha_e(c)|$ is given by

$$|\alpha_e(c)| = \sum_{i=0}^{n} \binom{n}{i} (D-1)^i,$$ where n is the length of cord word c.

We may assume that α_0 corresponds to the noiseless. In other words, if c is sent then c is received w.r.t. α_0. Moreover it is clear that $|\alpha_e(c)|$ depends only on the length n of c when α and D are given. In noiseless coding, the class of uniquely decodable instantaneous codes is studied. It is known that these codes satisfy prefix property (Abramson [1963]).

In the same way Hartnett [1974] studied variables length code over noisy channel, satisfying the prefix property in the range. These codes are called α-prompt codes. Such codes have the property that they can decode promptly.

Further, Burnard and Sharma [1988] gave a combinational information inequality that must necessarily be satisfied by code word lengths of prompt code codes. Two useful concepts, namely, segment decomposition and the effective range $r_\alpha(c_i)$ of code words c_i of length n_i under error mapping α as the Cartesian product of ranges of the segment are also given by Bernard and Sharma [1988]. The numbers of sequences in effective range of c_i denoted by $|r_\alpha|_{n_i}$ depends only on α and n_i. It is given that

$$|r_\alpha|_{n_i} = |\alpha|_{n_1} |\alpha|_{n_2} \dots |\alpha|_{n_i - n_{i-1}}.$$

Also, we adopt the notion $|\alpha|_0 = 1$. Moreover, Bernard and Sharma [1988] obtained the following inequality

Lemma 1.1: For any set of length $n_1 \le n_2 \le \dots \le n_k$

$$|r_\alpha|_{n_i} = |r_\alpha|_{n_{i-1}} . |r_\alpha|_{n_i - n_{i-1}}$$

Proof: The proof easily follows from the definition of the effective range.
We have

$$|r_\alpha|_{n_i} = |\alpha|_{n_1} . |\alpha|_{n_2 - n_1} \dots |\alpha|_{n_i - n_{i-1}}$$

And $$|r_\alpha|_{n_{i-1}} = |\alpha|_{n_1} . |\alpha|_{n_2 - 1} \dots |\alpha|_{n_{i-1} - n_{i-2}}$$

Therefore $$|r_\alpha|_{n_i} = |r_\alpha|_{n_{i-1}} . |r_\alpha|_{n_i - n_{i-1}}$$

Theorem 1.1: An α-prompt code with k code words of length $n_1 \le n_2 \le \dots \le n_k$, satisfies the following inequality

$$\sum_{i=1}^{k} |r_\alpha|_{n_i} D^{-n_i} \le 1$$

(1.1)

Proof: Let N_i denote the number of code words of length i in the code. Then, since the range of the word of length one has to be disjoint, we have

$$N_1 \le \frac{q}{|r_\alpha|_1} = \frac{q}{|\alpha|_1} = \frac{q}{q} = 1$$

Next, we know that for a code to be α-prompt, no sequence in the range of a code word can be prefix of any sequence in the range of another code word. Since $N_1 \le 1$, if there are more than one code word and some noise effect is there, then we will not able to get any word of length one and we will have to consider words of length 2 or more only.

The first digit will be one of the code symbols, i.e. for forming words of larger than $N_1 = 0$ and the first position can be filled in just one way for purpose of uniformity of arguments at larger stages. We will say that the first position can be filled in $\left[\frac{D}{|r_\alpha|_1} - N_1\right]$ ways.

The number of symbols that may be added at the second position is at most $\frac{D}{|\alpha|_1}$ which is equivalent to $D\frac{|r_\alpha|_1}{|r_\alpha|_2}$ from Lemma 1.1 Thus, we will have

$$N_2 \le \left[\frac{D}{|r_\alpha|_1} - N_1\right]\left[D.\frac{|r_\alpha|_1}{|r_\alpha|_2}\right]$$

$$= \frac{D^2}{|r_\alpha|_2} - N_1.D\frac{|r_\alpha|_1}{|r_\alpha|_2}$$

Now to form words of length 3, only those sequences of length 2 which are not code words can be accepted as permissible prefix. Their number is

$$\frac{D^2}{|r_\alpha|_2} - N_1.D\frac{|r_\alpha|_1}{|r_\alpha|_2} - N_2.$$

Once again, the number of symbols that may be added in the third position is $\frac{D}{|\alpha|_1}$. From Lemma 1.1, we can take

$$\frac{D}{|\alpha|_1} = D\frac{|r_\alpha|_2}{|r_\alpha|_3}.$$

Thus,

$$N_3 \le \left[\frac{D^2}{|r_\alpha|_2} - N_1 D\frac{|r_\alpha|_1}{|r_\alpha|_2} - N_2\right]\left[D\frac{|r_\alpha|_2}{|r_\alpha|_3}\right]$$

$$= \frac{D^3}{|r_\alpha|_3} - N_1 D^2\frac{|r_\alpha|_1}{|r_\alpha|_3} - N_2 D\frac{|r_\alpha|_2}{|r_\alpha|_3}$$

We may proceed in the same manner to obtain results for various N_i's. For the last length n_k, we will have

$$N_{n_k} \le \frac{D^{n_k}}{|r_\alpha|_{n_k}} - N_1 D^{n_{k-1}}\frac{|r_\alpha|_1}{|r_\alpha|_{n_k}} - N_2 D^{n_{k-2}}\frac{|r_\alpha|_2}{|r_\alpha|_{n_k}} \dots N_{n_{k-1}} D\frac{|r_\alpha|_{n_{k-1}}}{|r_\alpha|_{n_k}}.$$

This can be written as $\sum_{i=1}^{k} |r_\alpha|_i N_i D^{-i} \le 1$.

Changing the summation from the length $1, 2, \dots, n_k$ to the code word length $n_1, n_2, \dots, n_k$. The above inequality can be equivalently put as $\sum_{i=1}^{k} |r_\alpha|_{n_i} D^{-n_i} \le 1$, which proves the theorem.

Remark 1.1: If the codes of constant length n are taken, then the average inequality (1.1) reduces to Hamming sphere packing bound (Hamming [1950]).

Remark 1.2: If the channel is noiseless, the inequality (1) reduces to the well known Kraft inequality (Kraft [1949]).

Bernard and Sharma [1990] have obtained a lower bound on average code word length for prompt code using a quantity similar to Shannon entropy.

Campbell [1965] considered a code length of order t defined by

$$L(t) = {}^{1}\!/_{t} \log_D \sum_{i=1}^{k} (p_i D^{tn_i})\,;\ (0 < t < \infty) \quad (1.2)$$

An application of L-Hospitals rule shows that

$$L(0) = \lim_{t\to 0} L(t) = \sum_{i=1}^{k} n_i p_i \quad (1.3)$$

For large t, $\sum_{i=1}^{k} p_i D^{tn_i} \cong p_j D^{tn_j}$, where n_j is the largest of the number $n_1, n_2, \dots, n_k$. Moreover, $L(t)$ is a monotonic non-decreasing function of t (Beckenbach and Bellman [1961]).Thus $L(0)$ is the conventional measure of mean length and $L(\infty)$ is the measure which would be used if the maximum length were of prime importance.

Definition: Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets were introduced by Lotfi A. Zadeh in (1965) as an extension of classical notion of set. In classical set theory, the membership of the elements in a set is assessed in binary terms according to a bivalent condition—an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of the membership function valued in the real unit interval [0,1] expressed as $\mu_A(x_i) : U \rightarrow [0,1]$, where U is universe of discourse which represents the grade of membership of $x \in U$ in A as follows

$$\mu_A(x_i) = \begin{cases} 0, & \text{if } x \notin A \text{ and there is no ambiguity} \\ 1, & \text{if } x \in A \text{ and there is no ambiguity} \\ 0, & \text{if maximum ambiguity, i.e. } x \in A \text{ or } x \notin A \end{cases}$$

Let $A = \{x_i : 0 < \mu_A(x_i) < 1, \forall i = 1, 2, \dots, n\}$

$B = \{x_i : 0 < \mu_B(x_i) < 1, \forall i = 1, 2, \dots, n\}$

and $U = \{u_i : u_i > 0, \forall i = 1, 2, \dots, n\}$

be two fuzzy sets and U, the set of utilities corresponding to fuzzy membership function $\mu_A(x_i)$ for any event E. Corresponding to the above membership functions, we have the following fuzzy information scheme.

$$F.S. = \begin{bmatrix} E_1 & E_2 \dots & E_n \\ \mu_A(x_1) & \mu_A(x_2) \dots & \mu_A(x_n) \\ \mu_B(x_1) & \mu_B(x_2) \dots & \mu_B(x_n) \\ u_1 & u_2 \dots & u_n \end{bmatrix}$$

2.1 Lower Bound on Code Word Length t

Suppose that a person believe that the degree of membership of ith event is $\mu_B(x_i)$ and the code with code length n_i has been constructed accordingly. But contrary to his belief the true degree of membership is $\mu_A(x_i)$.

We will now obtain a lower bound of mean length $L(t)$ under the condition

$$\sum_{i=1}^{k} \left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{-1}\right) |r_\alpha|_{n_i} D^{-n_i} \leq 1 \qquad (2.1)$$

Remark 2.1: For a noiseless channel $|r_\alpha|_{n_i} = 1 \ \forall \ i = 1, 2, \dots, k$. The inequality (2.1) reduces to the fuzzy Inequality corresponding to Autar and Soni [1975]

$$\sum_{i=1}^{k} \left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{-1}\right) D^{-n_i} \leq 1 \qquad (2.2)$$

Remark 2.2: Moreover, if $\mu_A(x_i) + \left(1 - \mu_A(x_i)\right) = \mu_B(x_i) + \left(1 - \mu_B(x_i)\right)$ for each i, (2.2) reduces to Kraft [1949] inequality

$$\sum_{i=1}^{k} D^{-n_i} \leq 1 \qquad (2.3)$$

Theorem 2.1: Let a source S have k messages symbols $S_1, S_2, \dots, S_k$ with message degree of membership $\mu_A(x_1),\ \mu_A(x_2), \dots,\ \mu_A(x_k);\ \mu_A(x_i) \geq 0$. Let an α-prompt code encode these messages into a code alphabet of D symbols and let the length of the code word corresponding to the messages S_i be n_i. Then the code length of order t, $L(t)$, shall satisfy the inequality

$$L(t) \geq \frac{1}{1-\beta} \log_D \sum_{i=1}^{k} \left(\mu_A^{\beta}(x_i) + \left(1 - \mu_A(x_i)\right)^{\beta}\right)\left(|r_\alpha|_{n_i}\right)^{1-\beta} \qquad (2.4)$$

Proof: In the Holder's inequality

$$\left[\sum_{i=1}^{k} x_i^p\right]^{1/p}\left[\sum_{i=1}^{k} y_i^q\right]^{1/q} \leq \sum_{i=1}^{k} x_i y_i \qquad (2.5)$$

With the equality if and only if $x_i = cy_i$, where c is a positive number, $1/p + 1/q = 1$ and $p < 1$. We note the direction of Holder's inequality is the reverse of the usual one as $p < 1$ (Backenbach and Bellman [1961]).

Substituting

$$p = -t, q = 1 - \beta, x = \left(\mu_A^{\frac{-1}{t}}(x_i) + \left(1 - \mu_A(x_i)\right)^{\frac{-1}{t}}\right) D^{-n_i} \text{ and } y_i = \left(\mu_A^{\frac{1}{t}}(x_i) + \left(1 - \mu_A(x_i)\right)^{\frac{1}{t}}\right) |r_\alpha|_{n_i},$$

we get

$$\left\{\sum_{i=1}^{k} \left[\left(\mu_A^{\frac{-1}{t}}(x_i) + \left(1 - \mu_A(x_i)\right)^{\frac{-1}{t}}\right) D^{-n_i}\right]^{-t}\right\}^{-1/t} \left\{\sum_{i=1}^{k} \left[\left(\mu_A^{\frac{1}{t}}(x_i) + \left(1 - \mu_A(x_i)\right)^{\frac{1}{t}}\right) |r_\alpha|_{n_i}\right]^{1-\beta}\right\}^{1/(1-\beta)} \leq \sum_{i=1}^{k} D^{-n_i} |r_\alpha|_{n_i}$$

or

$$\left\{\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{tn_i}\right\}^{-1/t}\left\{\sum_{i=1}^{k}\left(\mu_A^{\frac{1-\beta}{t}}(x_i)+\left(1-\mu_A(x_i)\right)^{\frac{1-\beta}{t}}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\}^{1/(1-\beta)} \le \sum_{i=1}^{k} D^{-n_i}\,|r_\alpha|_{n_i}$$

Moreover, $1/p+1/q=1, \Rightarrow \beta=(1+t)^{-1}$, with this substitution the above inequality reduces to

$$\left\{\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{tn_i}\right\}^{-1/t}\left\{\sum_{i=1}^{k}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\}^{1/(1-\beta)} \le \sum_{i=1}^{k} D^{-n_i}\,|r_\alpha|_{n_i}$$

Using inequality of Bernard and Sharma [1988], viz. $\sum_{i=1}^{k} D^{-tn_i}\,|r_\alpha|_{n_i} \le 1$

Which gives

$$\left\{\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{tn_i}\right\}^{1/t} \ge \left\{\sum_{i=1}^{k}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\}^{1/(1-\beta)}$$

or

$$\frac{1}{t}\log_D\left\{\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{tn_i}\right\} \ge \frac{1}{1-\beta}\log_D\left\{\sum_{i=1}^{k}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\}$$

Hence

$$L(t) \ge \frac{1}{1-\beta}\log_D\left\{\sum_{i=1}^{k}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\} \qquad (2.6)$$

The quantity $\frac{1}{1-\beta}\log_D\left\{\sum_{i=1}^{k}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\}$ is similar to fuzzy entropy corresponding to Renyi's entropy of order β [1961].

It can be easily verified that the quantity in (2.4) hold if and only if

$$n_i = -\beta\log_D\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)+\log_D\left\{\sum_{i=1}^{k}\left(\mu_A^{\beta}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\}$$

Particular Cases:

a) For $t=0$ and $\beta=1$, the inequality (2.4) reduces to the fuzzy inequality corresponding to the Bernard and Sharma [1990]

$$\bar{n} \ge \sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\log_D\left[\frac{\left[|r_\alpha|_{n_i}\right]}{\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right]^{1-\beta}$$

b) For noiseless channel, $|r_\alpha|_{n_{i-1}}$ ∀i, the inequality (2.4) reduces to the fuzzy inequality corresponding to the Campbell [1965]

$$L(t) \ge H_\beta(A),$$

where $H_\beta(A)$ is the fuzzy entropy corresponding to the Renyi's entropy of order β

c) If the channel is noiseless and $t=0$, $\beta=1$, then the inequality reduces the fuzzy entropy corresponding to the well known Shannon's [1948] inequality $\bar{n} \ge H(A)$, where $H(A)$ is the fuzzy entropy corresponding to the Shannon's entropy.

Theorem 2.2: . Let an α-prompt code encode the K messages $S_1, S_2, \ldots, S_k$ into a code alphabet of D symbols and let the length of the corresponding encoded messages S_i be n_i. Then the code length of order t, $L(t)$ shall satisfy the inequality.

$$L(t) \ge \frac{1}{1-\beta}\log_D\left\{\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i)+\left(1-\mu_B(x_i)\right)^{\beta-1}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}\right\} \qquad (2.7)$$

With equality if and only if

$$n_i = -\log\left(|r_\alpha|_{n_i}\right)^{-\beta}\left(\mu_B^{\beta}(x_i)+\left(1-\mu_B(x_i)\right)^{\beta}\right) + \log_D\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i)+\left(1-\mu_B(x_i)\right)^{\beta-1}\right)\left[|r_\alpha|_{n_i}\right]^{1-\beta}$$

where $L(t) = \frac{1}{t}\log_D\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{tn_i}$.

Proof: In the Holder's inequality

$$\left[\sum_{i=1}^{k} x_i^p\right]^{1/p}\left[\sum_{i=1}^{k} y_i^q\right]^{1/q} \le \sum_{i=1}^{k} x_i y_i$$

With the equality if and only if

$x_i^p = cy_i^q$, where c is a positive number, $1/p + 1/q = 1$ and $p < 1$. We note that direction of Holder's inequality is the reverse of the usual one as $p < 1$ (Beckenbach and Bellman [1961]).
substituting

$$p = -t, q = t\beta,\ x_i = \left(\mu_A^{\frac{-1}{t}}(x_i) + \left(1 - \mu_A(x_i)\right)^{\frac{-1}{t}}\right) D^{-n_i}$$

and $$y_i = \left(\mu_A^{\frac{1}{t\beta}}(x_i) + \left(1 - \mu_A(x_i)\right)^{\frac{1}{t\beta}}\right)\left(\mu_B^{-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{-1}\right) |r_\alpha|_{n_i},$$

We get

$$\left(\sum_{i=1}^{k}\left(\mu_A(x_i) + \right.\right.$$
1−μAxiDtni−1ti=1kμAxi+1−μAxiμB−tβxi+1−μBxi−tβrαnitβ1tβ

$$\leq \sum_{i=1}^{n}\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{-1}\right) |r_\alpha|_{n_i} D^{-n_i}$$

Moreover, $1/p + 1/q = 1, \Rightarrow \beta = (1 + t)^{-1}$, with this substitution the above inequality reduces to

$$\left(\sum_{i=1}^{k}\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right) D^{tn_i}\right)^{\frac{-1}{t}} \left(\sum_{i=1}^{k}\left\{\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{\beta-1}\right)\left(|r_\alpha|_{n_i}\right)^{1-\beta}\right\}\right)^{\frac{1}{1-\beta}}$$

$$\leq \sum_{i=1}^{n}\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{-1}\right) |r_\alpha|_{n_i} D^{-n_i}$$

this gives

$$\left(\sum_{i=1}^{k}\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right) D^{tn_i}\right)^{\frac{1}{t}} \geq$$

$$\left(\sum_{i=1}^{k}\left\{\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{\beta-1}\right)\left(|r_\alpha|_{n_i}\right)^{1-\beta}\right\}\right)^{\frac{1}{1-\beta}}$$

or $$\frac{1}{t}\log_D\left(\sum_{i=1}^{k}\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right) D^{tn_i}\right) \geq$$

$$\frac{1}{1-\beta} log_D\left(\sum_{i=1}^{k}\left\{\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{\beta-1}\right)\left(|r_\alpha|_{n_i}\right)^{1-\beta}\right\}\right)$$

Hence,

$$L(t) \geq \frac{1}{1-\beta} log_D\left(\sum_{i=1}^{k}\left\{\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i) + \left(1 - \right.\right.\right.\right.$$

μBxiβ−1*rαni*1−*β*

The quantity $\frac{1}{1-\beta} log_D\left(\sum_{i=1}^{k}\left\{\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right)\left(\mu_B^{\beta-1}(x_i) + \left(1 - \mu_B(x_i)\right)^{\beta-1}\right)\left(|r_\alpha|_{n_i}\right)^{1-\beta}\right\}\right)$ is equivalent to fuzzy inaccuracy corresponding to Nath's inaccuracy [1970] of order β.

Particular Cases:

For $t = 0$ *and* $\beta \to 1$, the inequality (2.7) reduces to $\bar{n} \geq \sum_{i=1}^{k}\left(\mu_A(x_i) + \left(1 - \right.\right.$
μAxilogDrαniμBxi+1−*μBxi* (2.8)

For noiseless channel, $\left(|r_\alpha|_{n_i}\right) = 1;\ \forall i$, the inequality (2.8) reduces to

$$\bar{n} \geq \sum_{i=1}^{k}\left(\mu_A(x_i) + \left(1 - \mu_A(x_i)\right)\right) log_D\left(\mu_B(x_i) + \left(1 - \mu_B(x_i)\right)\right) = H\left(\mu_A(x_i), (x_i)\right) \qquad (2.9)$$

Where $H\left(\mu_A(x_i), (x_i)\right)$ is a fuzzy measure of inaccuracy corresponding to Kerridge [1961] measure of inaccuracy.

a) When $\mu_A(x_i) = \mu_B(x_i)$, then the R.H.S. of (2.9) reduces to the fuzzy inequality corresponding to the Shannon [1948] measure of inaccuracy.

b) For noiseless channel $\left(|r_\alpha|_{n_i}\right) = 1;\ \forall i$, the inequality (2.7) reduces to fuzzy inequality corresponding to Autar and Soni [1975]

$L(t) \geq H_\beta\left(\mu_A(x_i), \mu_B(x_i)\right)$ (2.10)

Where $H_\beta\left(\mu_A(x_i), \mu_B(x_i)\right)$ is fuzzy measure of inaccuracy corresponding to Nath [1970] of orderβ.

2.2 β- measure of Uncertainty Involving Utilities

Consider a fuzzy function corresponding to Gill et.al [1989] as

$$H_k^\beta(A,U) = \frac{\sum_{i=1}^k\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left[\frac{u_i}{\sum_{i=1}^k u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)_i}\right]^{1-\beta}-1}{1-2^{1-\beta}}; \quad \beta > 0(\neq 1) \tag{2.11}$$

Which is β-measure of uncertainty involving utilities.

Remark: When the utility aspect of the scheme is considered (i.e. $u_i = 1, i = 1, 2, 3, \ldots, k$ as well as $\beta \to 1$, the measure (2.11) becomes fuzzy information measure corresponding to Shannon's [1948] measure of information. Further, define a parametric mean length credited with utilities and membership function $\mu_A(x_i)$ as

$$L(U^\beta) = \frac{\left[\sum_{i=1}^k u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^\beta - 1}{1-2^{1-\beta}} \tag{2.12}$$

Where $\beta > 0\ (\neq 1)$, $\mu_A(x_i) \geq 0, i = 1, 2, \ldots, k$ and $\sum_{i=1}^k \mu_A(x_i) = 1$ which is a generalization fuzzy mean length corresponding to Campbell [1965], and for $\beta \to 1$, it reduces to fuzzy mean code word length corresponding to Shannon [1948] measure and gave a characterization of $HU_K^\beta(A;U)$ under the condition

$$\sum_{i=1}^k u_i\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right)D^{-n_i} \leq u_i\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right) \tag{2.13}$$

Theorem 2.2: Suppose $n_1, n_2, \ldots, n_k$ are the lengths of uniquely decodable code words satisfying (2.13), then the average code length satisfies

$$L(U^\beta) \geq H_k^\beta(A,U) \tag{2.14}$$

With the equality in (2.14) if and only if

$$n_i = \beta\, log_D\left[\frac{u_i}{\sum_{i=1}^k u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right] + log_D\left[\sum_{i=1}^k\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^k u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}\right] \tag{2.15}$$

Proof: In the Holder's inequality (Beckenback et.al [1961])

$$\left[\sum_{i=1}^k x_i^p\right]^{1/p}\left[\sum_{i=1}^k y_i^q\right]^{1/q} \leq \sum_{i=1}^k x_i y_i \tag{2.16}$$

For all $x_i > 0, y_i > 0, i = 1, 2, \ldots, k\ and\ p < 1, where\ \frac{1}{p} + \frac{1}{q} = 1$ with the equality in (2.16) if and only if there exists a positive number c such that

$$x_i^p = c y_i^q \tag{2.17}$$

We substitute

$$x_i = \left(\mu_B^{\frac{\beta}{\beta-1}}(x_i) + \left(1-\mu_B(x_i)\right)^{\frac{\beta}{\beta-1}}D^{-n_i}\right);\ y_{i=}\left(\mu_B^{(1-\beta)^{-1}}(x_i) +\right.$$

1−𝜇𝐵𝑥𝑖1−𝛽−1𝐷−𝑛𝑖𝑢𝑖𝑖=1𝑘𝑢𝑖𝜇𝐴𝑥𝑖+1−𝜇𝐴𝑥𝑖; ∀𝑖

$p = (1-\beta^{-1})\ and\ \ q = 1-\beta$, we get

$$\left[\sum_{i=1}^k\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^{\frac{\beta}{\beta-1}}\left[\sum_{i=1}^k\left(\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^k u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}\right]^{(1-\beta)^{-1}}$$

$$\leq \sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)D^{-n_i}$$

Using the inequality (2.13), the above inequality can be written as

$$\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^{\beta} \geq \left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}\right]$$

$$\frac{\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^{\beta}-1}{1-2^{1-\beta}} \geq \frac{\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}\right]-1}{1-2^{1-\beta}}$$

Hence, $\quad L\left(U^{\beta}\right) \geq H_k^{\beta}(A,U).$

Theorem 2.3: Let $n_1, n_2, \ldots, n_k$ are the lengths of uniquely decodable code words, then the average code length $L\left(U^{\beta}\right)$ can be made to satisfy the inequality

$$H_k^{\beta}(A,U) \leq L\left(U^{\beta}\right) \leq D.H_k^{\beta}(A,U) + \frac{D-1}{1-2^{1-\beta}} \qquad (2.18)$$

Proof: Suppose

$$n_i = \beta\, log_D\left[\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right] + log_D\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}\right] \qquad (2.19)$$

Clearly, $\tilde{n}_i$ and $\tilde{n}_{i+1}$ satisfy the inequality in Holder's inequality. Moreover $\tilde{n}_i$ satisfy the inequality (2.13).

Let n_i be the (unique) integer between $\tilde{n}_i$ and $\tilde{n}_{i+1}$. Since$\beta > 0\ (\neq 1)$, we have

$$\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)\tilde{n}_i}\right]^{\beta} \leq \left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^{\beta}$$

$$< D\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)\tilde{n}_i}\right]^{\beta} \qquad (2.20)$$

We know

$$\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta} = \left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^{\beta}$$

Hence, (2.20) can be expressed as

$$\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}$$

$$\leq \left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)D^{(\beta^{-1}-1)n_i}\right]^{\beta}$$

$$< D\left[\sum_{i=1}^{k}\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)\left(\frac{u_i}{\sum_{i=1}^{k}u_i\left(\mu_A(x_i)+\left(1-\mu_A(x_i)\right)\right)}\right)^{1-\beta}\right]$$

Thus, $\quad H_k^{\beta}(A,U) \leq L\left(U^{\beta}\right) \leq D.H_k^{\beta}(A,U) + \frac{D-1}{1-2^{1-\beta}}$

References

[1]. **Abramson, N. [1963]:** Information theory and coding; McGraw Hill, New York.

[2]. **Autar, R and Soni, R.S. [1975]:** Inaccuracy and coding theorem; Journ. Appl. Prob. Vol.12, pp. 845-851.

[3]. **Beckenback, E.F. and Bellman, R. [1961]:** Inequalities; Springer, Berlin.

[4]. **Bernard, M.A. and Sharma, B.D. [1988]:** Some combinatorial results on variable length error correcting codes; ARS combinatorial, 25B, pp. 181-194.

[5]. **Bernard, M.A. and Sharma, B.D. [1990]:** A lower bound on average code word length of variable length error correcting code; EEE trns. Information theory, 36(6), pp. 1474-1475.

[6]. **Campbell, L.L. [1965]:** A coding theorem and Renyi's entropy; Information and control, vol. 8, pp.423-429.

[7]. **Gill, M.A., Perez, R. and Gill, P. [1989]:** A family of measure of uncertainty involving utilities; Definition, properties, application and statistical inference, Metrika, vol. 36, pp.129-147.

[8]. **Hamming, R.W. [1950]:** Error detecting and error correcting codes; Bell System Tech. Jour. Vol. 29, pp.147-160.

[9]. **Hartnett, W.E. [1974]:** Foundation of coding theory; D. Riddling Publishing Co., Dordietcht, Holland.

[10]. **Kerridge, D.F. [1961]:** Inaccuracy and Inference, J.R. Statist. Soc. B, Vol. 23, pp. 184-194.

[11]. **Kraft, K.J. [1949]:** A device for quantizing grouping and coding aptitude modulated pulses; M.S. thesis, Electrical Engg. Deppt. Massach. Inst. Of Tech.

[12]. **Nath, P. [1970]:** An axiomatic characterization of inaccuracy for discrete generalized probability distributions; Op Search, Vol. 7, pp.115-133. .

[13]. **Reniy's, A. [1961]:** On measure entropy and information; Prac. Fourth Berkeley Symp.Math. Stats. And Prob., University Of California Press,Vol. 1, pp. 547-561.

[14]. **Shannon, C.E. [1948]:** A mathematical theory of communication; Bell Syst. Tech. Jour., Vol. 27, pp. 379-423, 623-659.

[15]. **Zadeh, L.A.[1965]:** Fuzzy Sets; Inform, and Control, Vol. 8, pp. 338-353.