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Unsteady Flow in a Double-Sided Symmetric Thin Liquid Films

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Abstract:*In this paper, we considerthe unsteady flow within a double-sided symmetric thin liquid film with negligible inertia. We apply the Navier-Stokes equations in two dimensional flows for incompressible fluid.The similarity method is used in which the explicit time dependence can be isolated and thus the shape of the film will depend on one variable only.The differential equation of the film thickness is obtained analytically and the solution of equation thatrepresents the film thickness is obtained numerically by using Rung-Kutta method.*
Keywords:*Thin Liquid Films, Navier-Stokes equations, continuity equation.*

I. Introduction

This paper is concerned with a dynamical system in such a way that no solid boundaries exist.It is more convened to assume that any solid boundaries which exist are at sufficient distance to produce or have no significant direct mechanical effect on that part of the system which is under consideration that are imposed in this work.The purpose of studying such liquid systems is to determine the global effect of varying the boundary conditions at liquid surfaces, in cases where the physical relevance of these boundary conditions appears to be a matter of doubt. However, in the case of motion in thin films, for instance, it has been suggested [1] that the surfaces of the thin film behave more like inextensible solids rather than ordinary fluid surfaces. [2],considered the thinning process of an inclined thin liquid film over a solid boundary with an inclination angle θ to the horizontal in gravity driven flow. They assumed that the fluid thickness is constant far behind the front and they neglected the thickness of the film at the beginning of the motion.The drainage of thin liquid films on an inclined solid surface is considered by [3], and the equation that governs the film thickness is obtained analytically by using the similarity method.The dynamic rupture process of a thin liquid film on a cylinder is investigated numerically; a nonlinear differential equation that describes the long-wave evolution of the interface shape. The tendency of acceleration becomes more explicit in the case of large surface tension [4]. The contact line induced instabilities for a thin film of fluid under destabilizing gravitational force in three dimensional setting. The instabilities in the setting vary in the transverse direction. It is argued that the flow pattern strongly depends on the inclination angle and the viscosity gradient [5]. A mathematical model is constructed by [6] to describe a two dimensional flow for an inclined thin liquid films with an inclination angle α to the horizontal under the action of gravity. An asymptotic analysis is employed by using the lubrication approximation.

II. Formulation and Governing Equations

We consider the flow of a viscousliquid within ahorizontal double-sided symmetric thin liquid film with zero shear stress on their bounding surfaces in two-dimensions.We take the Cartesian coordinates X and Y in which the X-axis is the axis of symmetric, and the flowis predominantly in the Xdirection, and the Y-axis is perpendicular to the plane of the film, as shown in Figure(1), and the equation of the bounding surfaces of the film is given by $= \pm H(X,T)$ .Let $q = (U,V)$denotes the fluid velocity, where U and V are the velocity components of the velocity field qinX and Ydirections respectively.Here, it is assumed that two dimensional incompressible flow governed by the Navier-Stokes equations of motion in X and Ydirections which are the longitudinal and transverse momentum equations for unsteady flow which are given in differential form respectively as follows:

$$\rho\left[\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right] = -\frac{\partial P}{\partial X} + \mu\left[\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right] \quad (1)$$

and

$$\rho\left[\frac{\partial V}{\partial T} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right] = -\frac{\partial P}{\partial Y} + \mu\left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right] \quad (2)$$

whereT is the time,Pis the pressure, μ is the dynamic viscosity of the liquid andρ is the density of the liquid.
The continuity equation is given by:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \quad (3)$$

In lubrication theory, the inertia terms can be neglected and the Navier-Stokes equations (1) and (2) become as follows:

$\frac{\partial P}{\partial X} = \mu\left[\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right]$ (4)

and

$\frac{\partial P}{\partial Y} = \mu\left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right]$ (5)

For thin liquid films the slope of the boundary free surfaces is so small and thus

$\frac{\partial H}{\partial X} \ll 1.$ (6)

Now, we consider the class of unsteady flows in which the asymptotic conditions as $X \to -\infty$ are uniform and thus, we assume that

$H(X) \to \overline{H}\ , \quad U(X,Y) \to \overline{U}\ , \quad V(X,Y) \to 0$ (7)

where $\overline{H}$ and $\overline{U} > 0$ are constants.

Now from the conservation of mass and since the free surface is a stream line, the material or the substantial derivative $\frac{DF}{DT} = 0$ must be vanished on $Y = H(X,T)$ and thus, we have

$\frac{DF}{DT} = \frac{\partial F}{\partial T} + U\frac{\partial F}{\partial X} + V\frac{\partial F}{\partial Y} = 0$ (8)

where $F(X,Y,T) = Y - H(X,T)$ (9)

From equations (8) and (9), we obtain

$\frac{\partial H}{\partial T} + U\frac{\partial H}{\partial X} - V = 0$ (10)

Now, by integrating the continuity equation (3), with respect to Y, we get $V = -Y\frac{\partial U}{\partial X}$, on the surface of the film $Y = H(X,T)$, we have

$V = -H\frac{\partial U}{\partial X}$ (11)

By substituting equation (11) into equation (10), we obtain

$\frac{\partial H}{\partial T} + U\frac{\partial H}{\partial X} + H\frac{\partial U}{\partial X} = 0$ (12)

At the surface of the free film, we have the following boundary conditions. The conditions on normal and tangential stress at the surface of the film however, they require further analysis. To apply these, we require the Cartesian components of the unit normal vectorand the unit tangent vector, which are given respectively by

$$\left.\begin{aligned} n_x = -t_y &= -\frac{\partial H}{\partial X}\left[1 + \left(\frac{\partial H}{\partial X}\right)^2\right]^{-\frac{1}{2}} \\ n_y = t_x &= \left[1 + \left(\frac{\partial H}{\partial X}\right)^2\right]^{-\frac{1}{2}} \end{aligned}\right] \qquad (13)$$

and the curvature, for the free surface is given by

$\frac{\partial n_i}{\partial x_i} = \frac{\partial^2 H}{\partial X^2}\left[1 + \left(\frac{\partial H}{\partial X}\right)^2\right]^{-\frac{3}{2}}$ (14)

Also, we require the Cartesian components of Stokes' formula for the stress, which are given by

$$\left.\begin{aligned} P_{XX} &= -P + 2\mu\frac{\partial U}{\partial X} \\ P_{XY} &= \mu\left[\frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y}\right] \\ P_{YY} &= -P + 2\mu\frac{\partial V}{\partial Y} \end{aligned}\right] \qquad (15)$$

III. Dimensional Analysis

To express the Navier-Stokes equations, the equation of continuity and the associated boundary conditions into non-dimensional form, we now introduce the following non-dimensional variable as follows [7],

$x = \frac{k\epsilon X}{\overline{H}}, \quad y = \frac{Y}{\overline{H}}, \quad h = \frac{H}{\overline{H}}, \quad u = \frac{U}{\overline{U}}$

$v = \frac{V}{\epsilon\overline{U}}, p = \frac{\overline{H}P}{\epsilon^2\sigma}, t = \frac{\epsilon\overline{U}T}{\overline{H}}$ (16)

Where $\overline{H}$ and $\overline{U}$ are the characteristics. The Capillary number is defined as

$\epsilon = \frac{\mu\overline{U}}{\sigma}$ (17)

Also the Reynolds number is defined as

$k\,\epsilon = \frac{\rho\overline{U}\,\overline{H}}{\mu}$ (18)

where ϵ and k are non-dimensional constants.

Substituting equation (17) into equation (18), we obtain

$$k = \frac{\rho\sigma\bar{H}}{\mu^2} \tag{19}$$

By using the dimensionless variables given by equation (16), into equations (3)-(5), the continuity equation gives

$$\frac{k\bar{U}\epsilon}{\bar{H}}\frac{\partial u}{\partial x} + \frac{\bar{U}\epsilon}{\bar{H}}\frac{\partial v}{\partial y} = 0 \tag{20}$$

andthe Navier-Stoke equation in x and y directions respectively give

$$\frac{k\epsilon^3\sigma}{\bar{H}^2}\frac{\partial p}{\partial x} = \mu\left[\frac{k^2\epsilon^2\bar{U}}{\bar{H}^2}\frac{\partial^2 u}{\partial x^2} + \frac{\bar{U}}{\bar{H}^2}\frac{\partial^2 u}{\partial y^2}\right] \tag{21}$$

$$\frac{\epsilon^2\sigma}{\bar{H}^2}\frac{\partial p}{\partial y} = \mu\left[\frac{k^2\epsilon^3\bar{U}}{\bar{H}^2}\frac{\partial^2 v}{\partial x^2} + \frac{\epsilon\bar{U}}{\bar{H}^2}\frac{\partial^2 v}{\partial y^2}\right] \tag{22}$$

The variables in (7) in non-dimensional formbecome

$$h(-\infty) = 1\ , \qquad u(-\infty, y) = 1\ , \qquad v(-\infty, y) = 0 \tag{23}$$

Now, equation (12) in non-dimensional is given by

$$\frac{\epsilon\bar{U}\bar{H}}{\bar{H}}\frac{\partial h}{\partial t} + \frac{k\epsilon\bar{U}\bar{H}}{\bar{H}}u\frac{\partial h}{\partial x} + \frac{k\epsilon\bar{U}\bar{H}}{\bar{H}}h\frac{\partial u}{\partial x} = 0 \tag{24}$$

After simplifying equations (20)-(22) and (24), and by using equations (17)-(19), respectively, we obtain

$$k\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{25}$$

$$\frac{\partial^2 u}{\partial y^2} = \epsilon^2\left[k\frac{\partial p}{\partial x} - k^2\frac{\partial^2 u}{\partial x^2}\right] \tag{26}$$

$$\frac{\partial^2 v}{\partial y^2} = \left[\frac{\partial p}{\partial y} - k^2\,\epsilon^2\,\frac{\partial^2 v}{\partial x^2}\right] \tag{27}$$

$$\frac{\partial h}{\partial t} + ku\frac{\partial h}{\partial x} + kh\frac{\partial u}{\partial x} \tag{28}$$

Also equations (13)-(15) in non-dimensional form become

$$\left.\begin{aligned} n_x &= -t_y = -k\,\epsilon\, h'\left[1 + k^2\,\epsilon^2\,h'^2\right]^{-\frac{1}{2}} \\ n_y &= t_x = \left[1 + k^2\,\epsilon^2\,h'^2\right]^{-\frac{1}{2}} \end{aligned}\right] \tag{29}$$

$$\frac{\partial n_i}{\partial x_i} = \frac{k^2\epsilon^2}{\bar{H}}h''\left[1 + k^2\,\epsilon^2\,h'^2\right]^{-\frac{3}{2}} \tag{30}$$

and

$$\left.\begin{aligned} P_{xx} &= \frac{\epsilon^2\sigma}{\bar{H}}\left[-p + 2k\frac{\partial u}{\partial x}\right] \\ P_{xy} &= \frac{\epsilon\sigma}{\bar{H}}\left[\frac{\partial u}{\partial y} + k\,\epsilon^2\,\frac{\partial v}{\partial x}\right] \\ P_{yy} &= \frac{\epsilon^2\sigma}{\bar{H}}\left[-p + 2\frac{\partial v}{\partial y}\right] \end{aligned}\right] \tag{31}$$

The Laplace condition on normal stress at the surface thereforegives

$$n_x n_x P_{xx} + n_y n_y P_{yy} + n_x n_y P_{xy} + n_y n_x P_{yx} = -\sigma\frac{\partial n_i}{\partial x_i} \tag{32}$$

By substituting equations (29)-(31) into the above equation, we obtain

$$\left[1 + k^2\,\epsilon^2\,h'^2\right]p_s + 2k\left[1 - k^2\,\epsilon^2\,h'^2\right]\left[\frac{\partial u}{\partial x}\right]_s + 2kh'\left[\frac{\partial u}{\partial y} + k\,\epsilon^2\,\frac{\partial v}{\partial x}\right]_s = -k^2 h''\left[1 + k^2\,\epsilon^2\,h'^2\right]^{-\frac{1}{2}} \tag{33}$$

where the subscript s denotes values at the surface of the film y=h. The condition on tangential stressand by using equations (29) and (16),we obtain

$$\left[1 - k^2\,\epsilon^2\,h'^2\right]\left[\frac{\partial u}{\partial y} + k\,\epsilon^2\,\frac{\partial v}{\partial x}\right]_s - 2k^2\,\epsilon^2\,h'\left[\frac{\partial u}{\partial x}\right]_s + 2k\,\epsilon^2\,h'\frac{\partial v}{\partial y} = 0 \tag{34}$$

Substituting the continuity equation (25) into the above equation, we obtain

$$\left[1 - k^2\,\epsilon^2\,h'^2\right]\left[\frac{\partial u}{\partial y} + k\,\epsilon^2\,\frac{\partial v}{\partial x}\right]_s - 4k^2\,\epsilon^2\,h'\left[\frac{\partial u}{\partial x}\right]_s = 0 \tag{35}$$

Now, substituting equation (35) into equation (33), we obtain

$$p_s + 2k\left[\frac{1 + k^2\epsilon^2 h'^2}{1 - k^2\epsilon^2 h'^2}\right]\left[\frac{\partial u}{\partial x}\right]_s + \frac{k^2 h''}{\left[1 + k^2\epsilon^2 h'^2\right]^{\frac{3}{2}}} = 0 \tag{36}$$

IV. Similarity Method

We assumed through this work that the film thickness is constant at $x \to -\infty$and this assumption is true since the fluid thins out there and so we apply the similarity method to see how this process evolves in time. For self-similar solution, we assume that

$$h(x, t) = h_1(t)H_1(\eta) \tag{37}$$

Where η is the self-similar variable:

$$\eta = \frac{x}{x_1(t)}, \quad (38)$$

This method is to isolate the explicit time dependence and then the shape of the film will depend on the variable η only.

Now let

$$\left.\begin{array}{l} h_1(t) = D(t)^{\delta} \\ \text{and} \\ x_1(t) = A(t)^{\alpha} \end{array}\right] \quad (39)$$

Where D and A are the self-similar exponents and δ andα are constants.

V. Perturbation Method

We now develop formal expansions as follows

$$\left.\begin{array}{l} u = u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \quad \dots\dots, \\ v = v_0 + \epsilon^2 v_2 + \epsilon^4 v_4 + \quad \dots\dots, \\ p = p_0 + \epsilon^2 p_2 + \epsilon^4 p_4 + \quad \dots\dots, \\ h = h_0 + \epsilon^2 h_2 + \epsilon^4 h_4 + \quad \dots\dots, \end{array}\right] \quad (40)$$

where the functional coefficients of $\epsilon^{2n}, n = 0,1,2,3,4,\dots\dots,$ are independent of ∈. Further, we assert that all these coefficients are bounded functions with bounded derivatives of all order with respect to x and $y$and this assertion leads to a self-consistent and non-trivial solution for the shape of the film. We proceed as follows

a) The zero-order problem

It is a simple matter to see that the zero- order problem, obtained by setting $\epsilon = 0,$ in all the preceding equations, is degenerated in the sense that a solution exists for every arbitrary assigned function and we obtain

$$u = u_0, \quad v = v_0, \quad p = p_0 \quad \text{and} \quad h = h_0 \quad (41)$$

For k=1, we get $\overline{H} = \frac{\mu^2}{\rho\sigma}$ and this ensures the balance between viscous forces and surface tension forces and then $\overline{H}$ can be determined from the physical parameters for each liquid. Then, for other values of k, we can proceed in a similar way.

Now from equation (28) and using equations (37), (38)and (41), we obtain

$$D\delta t^{\delta-1} H_1(\eta) - D\alpha t^{\delta-1} \eta \frac{\partial H_1}{\partial \eta} + u_0 \frac{D}{A} t^{\delta-\alpha} \frac{\partial H_1}{\partial \eta} + D t^{\delta} H_1(\eta) \frac{\partial u_0}{\partial x} = 0 \quad (42)$$

The fourth term is incomparable with the other terms and it is so small for high viscosity ofliquid since the velocity u_0 is small, thus its derivative becomes very small and we neglect it. Thus, equation (42) isreduced to give

$$D\delta t^{\delta-1} H_1(\eta) - D\alpha t^{\delta-1} \eta \frac{\partial H_1}{\partial \eta} + u_0 \frac{D}{A} t^{\delta-\alpha} \frac{\partial H_1}{\partial \eta} = 0 \quad (43)$$

By equating the exponents of t in equation (43), we get

$$\delta - 1 = \delta - \alpha$$

or

$\alpha = 1$and, we can choose $\delta = 1$for the balance of equation (43).

and thus equation (43) is reduced to give

$$u_0(\eta, t) = -\frac{A H_1}{\frac{\partial H_1}{\partial \eta}} + A\eta \quad (44)$$

Now from continuity equation (25) and equations (37), (38) and (41), we obtain

$$\frac{\partial v_0}{\partial y} = -\frac{H_1 \frac{\partial^2 H_1}{\partial \eta^2}}{t\left(\frac{\partial H_1}{\partial \eta}\right)^2} \quad (45)$$

Integrating equation (45), with respect to y, we obtain

$$v_0(\eta, t) = -\frac{H_1 \frac{\partial^2 H_1}{\partial \eta^2}}{t\left(\frac{\partial H_1}{\partial \eta}\right)^2} y + f(\eta, t) \quad (46)$$

Since the film is symmetric, we have $v_0(\eta, t) = 0$ at $y = 0$,therefor$f(\eta, t) = 0$ and equation (46) on the surface film $(y = h_0)$ become

$$v_0(\eta, t) = -\frac{D H_1{}^2 \frac{\partial^2 H_1}{\partial \eta^2}}{\left(\frac{\partial H_1}{\partial \eta}\right)^2} \quad (47)$$

From equations (36), (37), (38) and (41), we obtain

$$p_0(\eta, t) = -\frac{D \frac{\partial^2 H_1}{\partial \eta^2}}{A^2 t} - \frac{2 H_1 \frac{\partial^2 H_1}{\partial \eta^2}}{t\left(\frac{\partial H_1}{\partial \eta}\right)^2} \quad (48)$$

b) The second-order problem

It is a simple matter to see that the second- order problem, obtained by setting $\in^4 = 0$, in all the preceding equations, is degenerate in the sense that a solution exists for every arbitrary assigned function and we obtain

$$u = u_0 + \in^2 u_2 ,\ v = v_0 + \in^2 v_2, \quad p = p_0 + \in^2 p_2 \text{ and } h = h_0 + \in^2 h_2 \tag{49}$$

Now substituting equations (44),(47) and (48) into equation (49), we obtain

$$u(\eta, t) = -\frac{AH_1}{\frac{\partial H_1}{\partial \eta}} + A\eta + \in^2 u_2 \tag{50}$$

$$v(\eta, t) = -\frac{DH_1{}^2\frac{\partial^2 H_1}{\partial \eta^2}}{\left(\frac{\partial H_1}{\partial \eta}\right)^2} + \in^2 v_2 \tag{51}$$

and

$$p(\eta, t) = -\frac{D\frac{\partial^2 H_1}{\partial \eta^2}}{A^2 t} - \frac{2H_1\frac{\partial^2 H_1}{\partial \eta^2}}{t\left(\frac{\partial H_1}{\partial \eta}\right)^2} + \in^2 p_2 \tag{52}$$

Now, by substituting equations (50) and (52), into equation (26), we obtain

$$\frac{\partial^2 u_2}{\partial y^2} = -\frac{D\frac{\partial^3 H_1}{\partial \eta^3}}{A^3 t^2} - \frac{3\frac{\partial^2 H_1}{\partial \eta^2}}{At^2\frac{\partial H_1}{\partial \eta}} - \frac{3H_1\frac{\partial^3 H_1}{\partial \eta^3}}{At^2\left(\frac{\partial H_1}{\partial \eta}\right)^2} + \frac{6H_1\left(\frac{\partial^2 H_1}{\partial \eta^2}\right)^2}{At^2\left(\frac{\partial H_1}{\partial \eta}\right)^3} \tag{53}$$

Integration equation (53), with respect to y, we get

$$\frac{\partial u_2}{\partial y} = yf(\eta, t) + G(\eta, t) \tag{54}$$

where $f(\eta, t) = -\frac{D\frac{\partial^3 H_1}{\partial \eta^3}}{A^3 t^2} - \frac{3\frac{\partial^2 H_1}{\partial \eta^2}}{At^2\frac{\partial H_1}{\partial \eta}} - \frac{3H_1\frac{\partial^3 H_1}{\partial \eta^3}}{At^2\left(\frac{\partial H_1}{\partial \eta}\right)^2} + \frac{6H_1\left(\frac{\partial^2 H_1}{\partial \eta^2}\right)^2}{At^2\left(\frac{\partial H_1}{\partial \eta}\right)^3}$

and $G(\eta, t)$ is an arbitrary function of integration. Since u_2 must be an even function of y in a symmetric film, then

$$\frac{\partial u_2}{\partial y} = 0 \quad \text{where } y = 0 \tag{55}$$

Thus

$G(\eta, t) = 0$

and

$$\frac{\partial u_2}{\partial y} = yf(\eta, t) = y\left[-\frac{D\frac{\partial^3 H_1}{\partial \eta^3}}{A^3 t^2} - \frac{3\frac{\partial^2 H_1}{\partial \eta^2}}{At^2\frac{\partial H_1}{\partial \eta}} - \frac{3H_1\frac{\partial^3 H_1}{\partial \eta^3}}{At^2\left(\frac{\partial H_1}{\partial \eta}\right)^2} + \frac{6H_1\left(\frac{\partial^2 H_1}{\partial \eta^2}\right)^2}{At^2\left(\frac{\partial H_1}{\partial \eta}\right)^3}\right] \tag{56}$$

On the surface s, the equations (56), becomes

$$\left[\frac{\partial u_2}{\partial y}\right]_s = h_0 f(\eta, t) = DH_1\left[-\frac{D\frac{\partial^3 H_1}{\partial \eta^3}}{A^3 t} - \frac{3\frac{\partial^2 H_1}{\partial \eta^2}}{At\frac{\partial H_1}{\partial \eta}} - \frac{3H_1\frac{\partial^3 H_1}{\partial \eta^3}}{At\left(\frac{\partial H_1}{\partial \eta}\right)^2} + \frac{6H_1\left(\frac{\partial^2 H_1}{\partial \eta^2}\right)^2}{At\left(\frac{\partial H_1}{\partial \eta}\right)^3}\right] \tag{57}$$

From equations (35), (50) and (51), we have

$$\left[\frac{\partial u_2}{\partial y}\right]_s = DH_1\left[\frac{6\frac{\partial^2 H_1}{\partial \eta^2}}{At\frac{\partial H_1}{\partial \eta}} + \frac{H_1\frac{\partial^3 H_1}{\partial \eta^3}}{At\left(\frac{\partial H_1}{\partial \eta}\right)^2} - \frac{2H_1\left(\frac{\partial^2 H_1}{\partial \eta^2}\right)^2}{At\left(\frac{\partial H_1}{\partial \eta}\right)^3}\right] \tag{58}$$

Thus equations (57) and (58), give the following differential equation

$$D\left(\frac{\partial H_1}{\partial \eta}\right)^3\frac{\partial^3 H_1}{\partial \eta^3} + 9A^2\left(\frac{\partial H_1}{\partial \eta}\right)^2\frac{\partial^2 H_1}{\partial \eta^2} + 4A^2 H_1\frac{\partial H_1}{\partial \eta}\frac{\partial^3 H_1}{\partial \eta^3} - 8A^2 H_1\left(\frac{\partial^2 H_1}{\partial \eta^2}\right)^2 = 0 \tag{59}$$

Equation (59), is the governing equation for the thickness of a symmetric double-sided film for unsteady flow.

VI. Figures And Tables

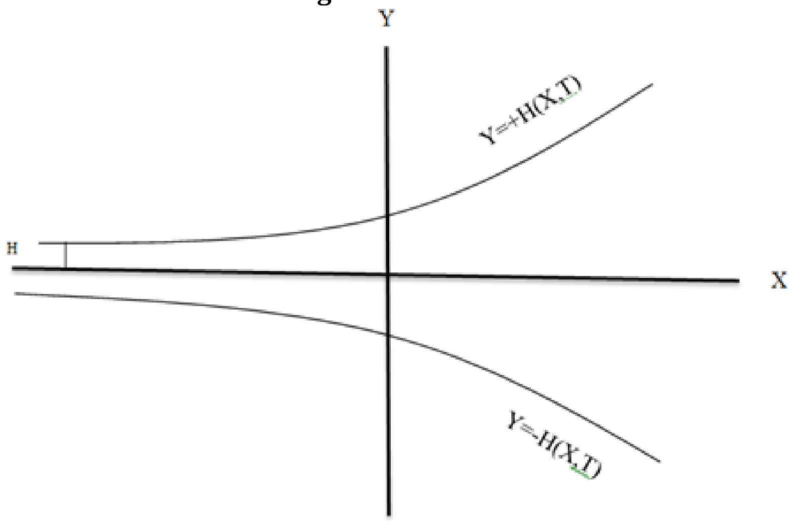


Fig. (1), cross- section of a horizontal symmetric thin liquid film.

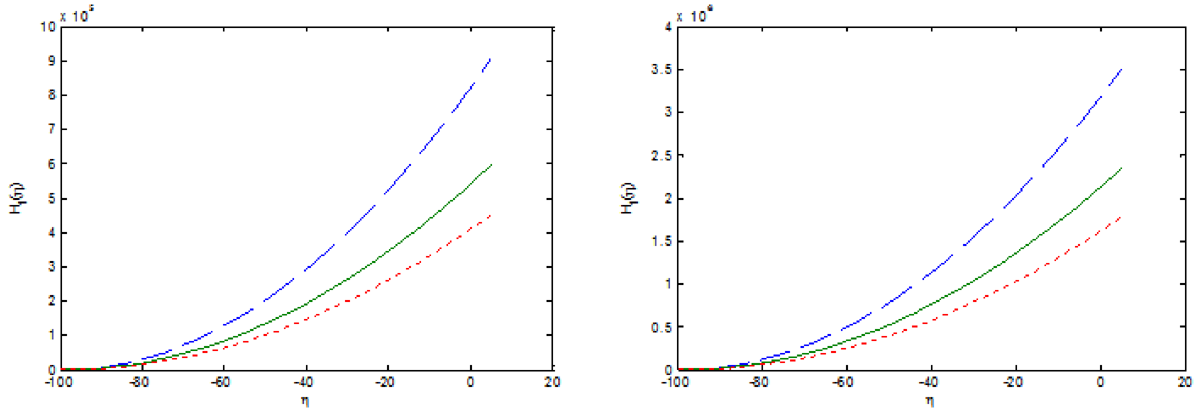


Fig. (2): Solution curves of equation (59) in (η, H_1) –plane for different values of constants--$D = 0.5$, —$D = 0.75$, ... $D = 1$, for $A = 0.5$ and $A = 1$, respectively.

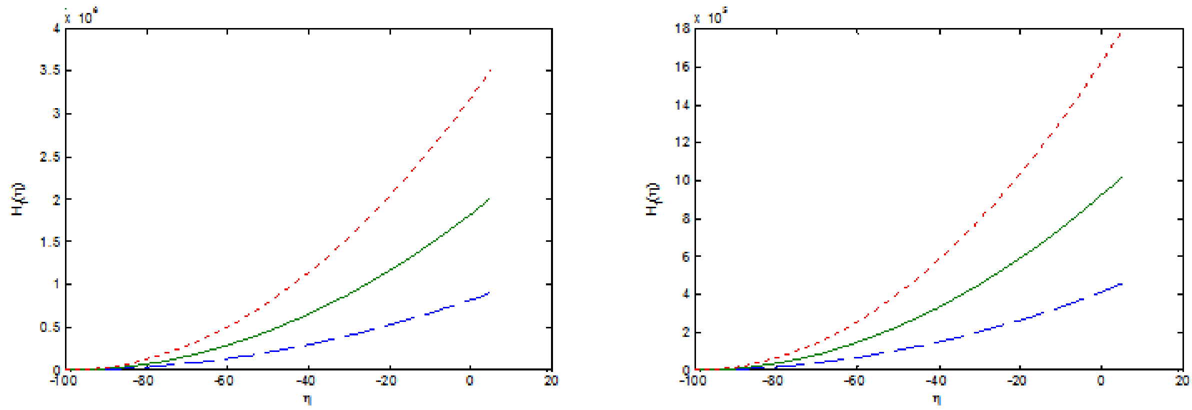


Fig. (3): Solution curves of equation (59) in (η, H_1) –plane for different values of constants--$A = 0.5$, —$A = 0.75$, ...$A = 1$, for $D = 0.5$ and $D = 1$, respectively.

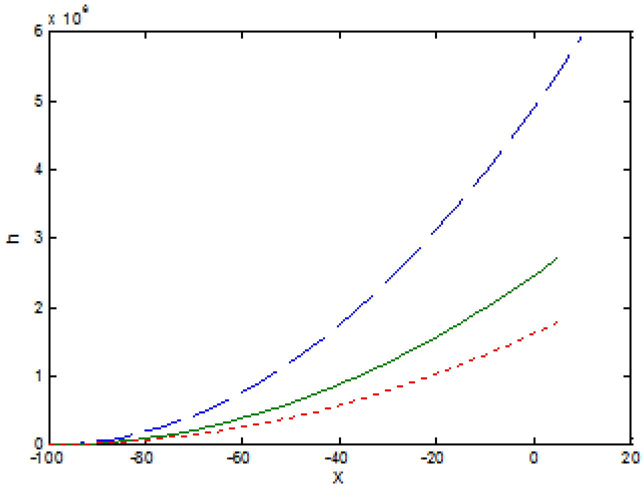
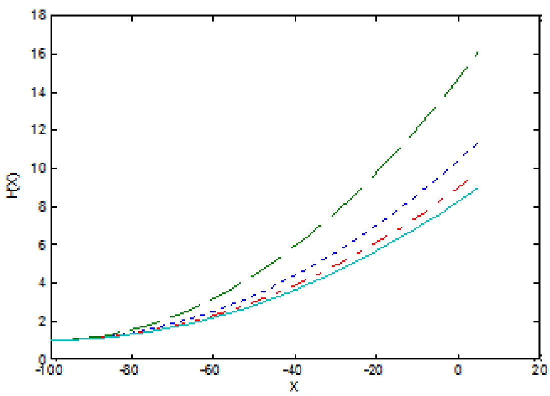


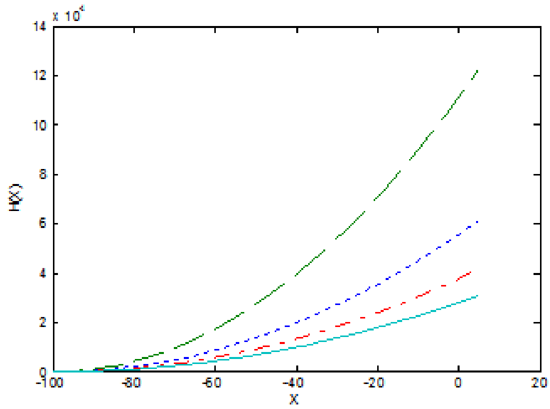
Fig. (4): Solution curves of equation (59) in (x, h) –plane for different valuesof time--$t = 0.25$, —$t = 0.5$,…$t = 0.75$and for constants $A = 0.5$ and $D = -1$.

The following table represents the viscosity, density and surface tension for some liquids.

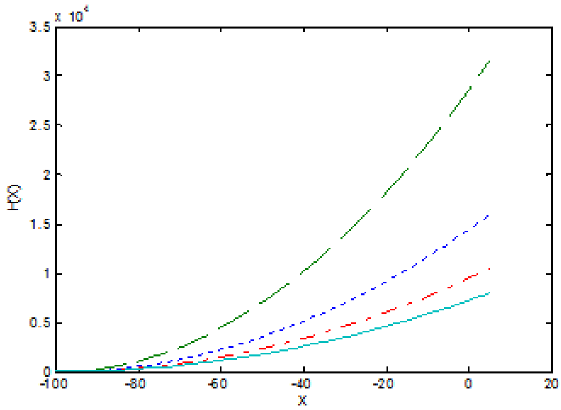
Liquid	μ	ρ	σ	$\bar{H}$
Water	0.0113	0.998	72.97	1.75×10^{-6}
Mercury	0.0155	13.55	510.76	3.47×10^{-8}
Linseed oil	0.4309	0.94	33.57	5.88×10^{-3}
Olive oil	0.8379	0.91	33.56	2.29×10^{-2}
Glycerin	14.9	1.26	62.75	2.8079
Carbon Tetrachloride	0.00974	1.59	26.27	2.27×10^{-6}

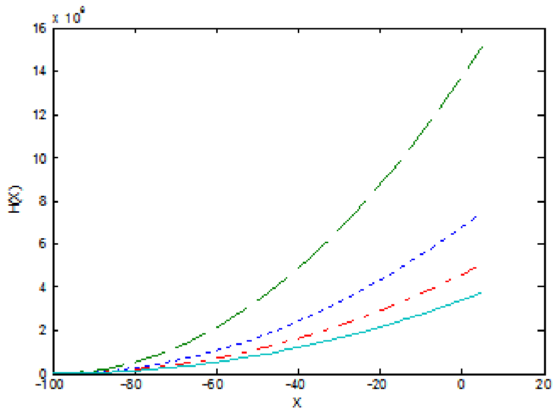
Table(1): Value of $\bar{H}$where $\bar{H} = \frac{\mu^2}{\rho\sigma}$,



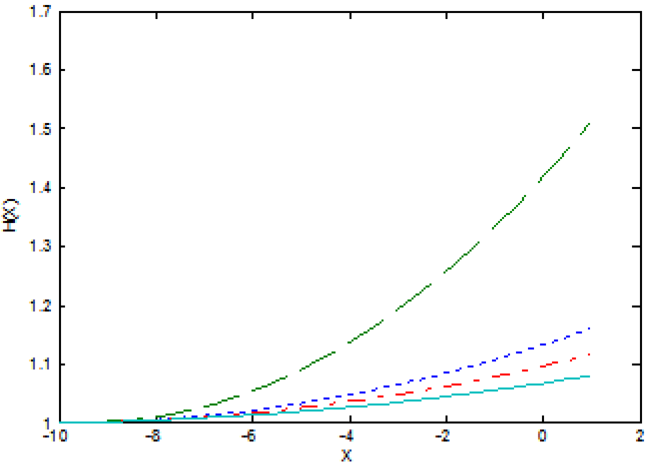


(Water)(Olive oil)





(Linseed oil) **(Glycerin)**



(Carbon tetrachloride)

Fig. (5): Solution curves of equation (59) in (X, H)-plane for different values of time-- $t = 0.25$, ... $t = 0.5$, — · $t = 0.75$, — $t = 1$ and for constants $A = 0.5$ and $D = -1$.

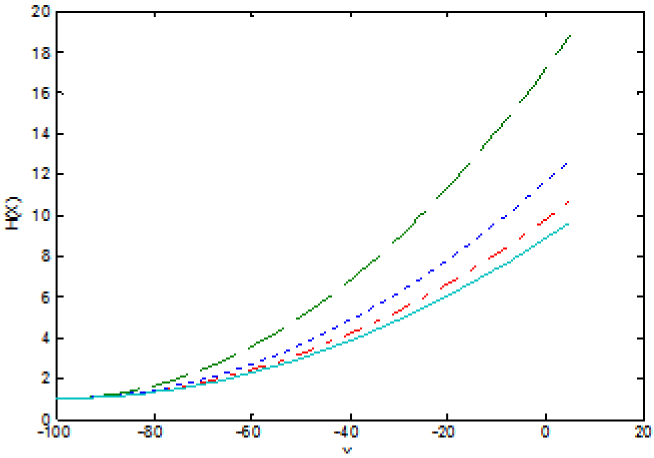


Fig. (6): Solution curves of equation (59) in (X, H)-plane for **mercury** or different values of time-- $t = 0.001$, ... $t = 0.005$, — · $t = 0.01$, — $t = 0.05$ and for constants $A = 0.5$ and $D = -1$.

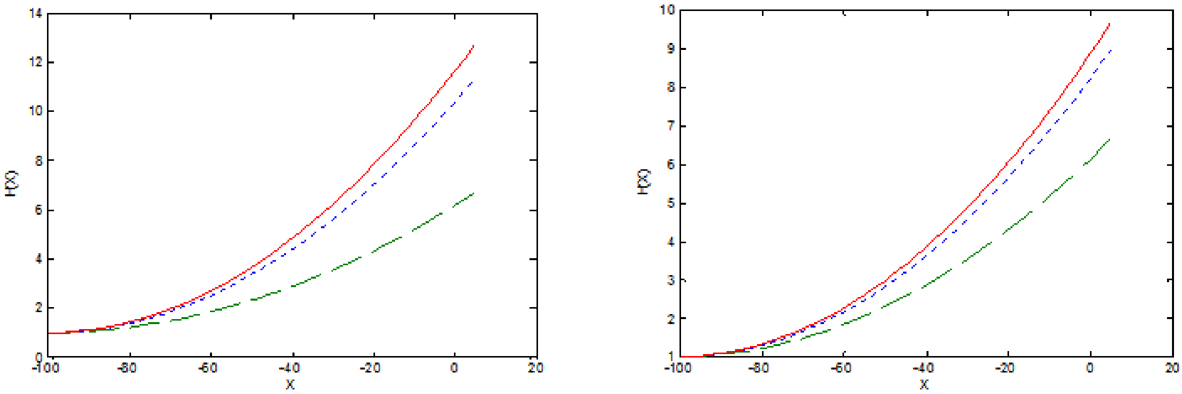


Fig. (7) Solution curves of equation (59) in (X, H) –plane for **—**Carbon tetrachloride, ...water, --mercury and for $t = 0.5$ and $t = 1$ respectively and for constants $A = 0.5$, and $D = -1$.

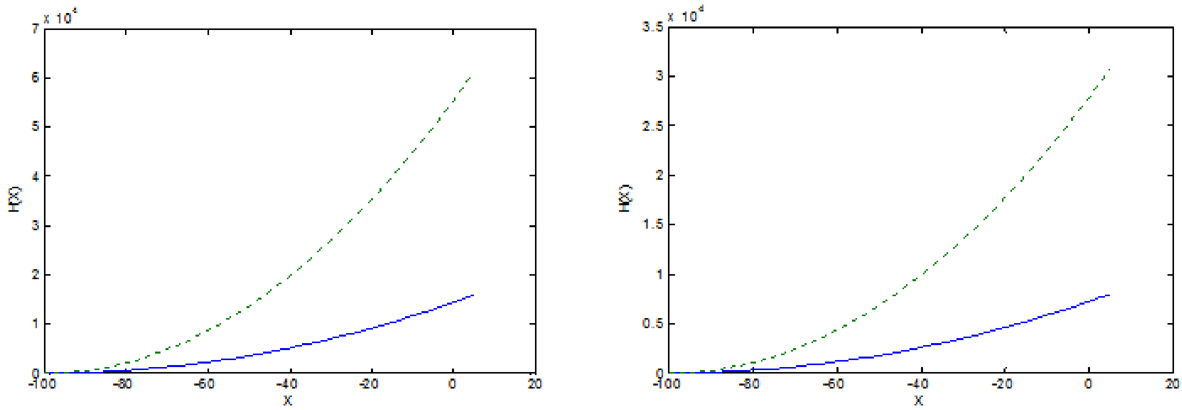


Fig. (8) Solution curves of equation (59) in (X, H) –plane for… Olive oil, – Linseed oil and for time $t = 0.5$ and $t = 1$respectively,and for constants $A = 0.5$, and$D = -1$.

VII. Conclusion

The flow within the horizontal double-sided symmetric thin liquid film with zero shear stress in two dimensional coordinates with negligible inertia is considered. The similarity and perturbation methods are used to obtain a non-linear differential equation that governs such flow for unsteady state in dimensionless form. The solution curves Fig.(2) of the non-linear deferential equation shows that as the constant A increases, the film thickness decreases, for any value of the constant D, the thickness increases as the constant A increases in (η, H_1)-plane as shown in Fig.(3). Fig(4) shows that the film thickness decreases as the time increasesfor all liquid films in (x, h)-plane.For all liquids namely (water, Linseed oil, Olive oil, glycerin and Carbon tetrachloride)the thickness decreases as the time increases as shown in figs. (5) and (6) in(X, H)-plane.. However, in case of mercury and for large values of time the film will be raptured and this is because of the very small value of $\overline{H}$. For fixed time the film thickness for mercury is less than that of water and less than that of Carbon tetrachloride. It is worth mentioning here that from continuum mechanics the inertia term may not be neglected in the case of glycerin.

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