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Orthogonal Generalized Derivations of Semiprime Semirings

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Abstract: *Motivated by some results on Orthogonal Generalized Derivations ofSemiprime Rings, in [2], the authors defined the notion of derivations and generalized derivations on semirings and investigated some results on the derivations in semirings. In this paper, we also introduce the notion of orthogonal generalized derivations of Semiprimesemirings and derived some interesting results.*
keywords: *Semirings, Derivations, Orthogonal derivations, Generalized orthogonalderivations, Centralizer*

I. Introduction

This paper has been inspired by the work of Argac, Nakajima and Albas and alsoMehsinJabelAtteya[8] and [7], Throughout this paper S will represent a Semiring withthe center Z(S). Bresar and vukman[1] introduced the notation of Orthogonality fora pair d,g of derivations on a Semiprime Ring, and they contributed several necessaryand sufficient conditions for d and g to be Orthogonal. Argac, Nakajima and Albas[5]introduced the notation of Orthogonality generalized for a pair D,G of derivations on2-torsionfree Semiprime Ring, and extended the results of Orthogonal derivations toOrthogonal Generalized derivations. Majeed and Mehsin[6] proved the following resultin his paper, if R is a 2-torsionfree Semiprime ring , (D, d) and (G, g) are generalizedderivations of R such that R admists to satisfy $[d(x), g(x)] = 0$, for all $x \epsilon R$ and d acts as aleft centralizer (resp g acts as a left centralizer), then (D, d) and (G, g) are OrthogonalGeneralized derivations of R. In this paper we study and investigate some interestingresults concerning a non-zero generalized derivations with left cancellation property onsemiprimesemiring S, when the non-zero additive mapping acts as a left centralizer of S.

II. Preliminaries

Definition 2.1

A **Semiring** $(S,+,\cdot)$ is a non-empty set S together with two binary operations, + and $\cdot$ such that
(1). $(S,+)$ and $(S,\cdot)$ are a Semigroup.
(2). For all $a, b, c \epsilon S$, $a\cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b\cdot a + c \cdot a$

Definition2.2: A semiring S is said to be **2-torsionfree** if $2x = 0 \Rightarrow x = 0, \forall x \epsilon S$.

Definition 2.3

A semiring S is **Prime** if $xSy = 0 \Rightarrow x = 0$ or $y = 0, \forall x, y \epsilon S$ and S is **Semi Prime** if $xSx = 0 \Rightarrow x = 0, \forall x \epsilon S$.

Definition 2.4: An additive map $d : S \rightarrow S$ is called a **derivation** if $d(xy) = d(x)y+xd(y), \forall x, y \epsilon S$

Definition 2.5: An additive map $d : S\rightarrow S$ is called **left centralizer** if $d(xy) = d(x)y, \forall x, y \epsilon S$.

Definition 2.6: Let d, g be two additive maps from S to S. They are said to be **Orthogonal** if $d(x)Sg(y) = 0 = g(y)Sd(x), \forall x, y \epsilon S$.

Definition 2.7

An additive mapping $D : S \rightarrow S$ is called a **generalized derivation**if there exists a derivation $d : S \rightarrow S$ such that $D(xy) = D(x)y + xd(y), \forall x, y \epsilon S$.

Definition 2.8

Two generalized derivations (D, d) and (G, g) of S are called **Orthogonal** if $D(x) S G(y) = 0 = G(y) S D(x), \forall x, y \epsilon S$
We write $[x, y] = x y – yx$ and note that important identity $[xy, z] = x[y, z]+[x, z]y$ and
$[x,yz] = y[x, z] + [x, y]z$

III. Orthogonal Generalized Derivations In Semirings

Lemma: 3.1

Let S be a 2- torsion free SemiprimeSemiring. Let D and G be two generalized derivations of S. If D and G are Orthogonal to g and d respectively, then (i) dg = 0 and DG is a left centralizer of S. (ii) gd = 0 and GD is a left centralizer of S.

Proof:

(i) Since D and g are Orthogonal, we get D(x)sg(y) = 0,∀ x, y, s ϵ S.
Replacing x by xr we get D(xr)sg(y) = 0,∀ x, y, r, s ϵ S.
⇒[D(x)r + xd(r)] sg(y) = 0 ⇒ D(x)rsg(y) + xd(r)sg(y) = 0
⇒xd(r)sg(y) = 0 (∵ D and g are Orthogonal)
⇒d(r)sg(y) = 0 (∵ S is Semiprime)
⇒g(y)sd(x) = 0.
∴d and g are Orthogonal.
Therefore dg = 0. Now we prove that DG is a left centralizer of S.
Since D is Orthogonal to g and G is Orthogonal to d we get, D(x)sg(y) = 0 andG(x) s d(y)=0so Dg = 0 and Gd = 0,∀x, y, s ϵ S (1)
Now DG(xy) = D[G(xy)] = D[G(x)y + xg(y)] = D[G(x)y] + D[xg(y)]
= DG(x)y + G(x)d(y) + D(x)g(y) + xdg(y) = DG(x)y (∵ by(1))
∴DG is a left centralizer of the Semiring. Similarly we shall prove (ii)

Lemma: 3.2

Let S be a SemiprimeSemiring, d a non-zero derivation of S, and U a non-zero left ideal of S. If for some positive integers t0,t1,…tn, and all x ϵ U, the identity $[[\dots [[d(x^{t_0}),x^{t_1}] ,x^{t_2}],\dots] x^{t_n} = 0$ holds, then either d(U) = 0 or else d(U) and d(S)U are contained in a non-zero central ideal of S. In particuar, if S is a prime semiring, then S is commutative

Theorem: 3.3

Let S be a SemiprimeSemiring with left Cancellation property, (D, d) and (G, g) be a non-zero generalized derivations of S, and U a non-zero ideal of S. If S admits to satisfy [d(x), g(x)] = 0,∀xϵUand a non-zero d acts as a left centralizer (resp a non-zero g acts as a left centralizer), then S contains a non-zero central ideal.

Proof: We have [d(x), g(x)] = 0 ,∀xϵ S
Replacing x by xy, [d(x)y,g(xy)] + [xd(y), g(xy)] = 0
⇒d(x)[y, g(xy)] + [d(x), g(xy)]y + x[d(y), g(xy)] + [x,g(xy)]d(y) = 0, ∀x,y ϵ S
⇒d(x)[y, g(x)y] + d(x)[y, xg(y)] + [d(x), g(x)y]y + [d(x),xg(y)]y + x[d(y), g(x)y] + x[d(y),xg(y)] + [x, g(x)y]d(y) + [x,xg(y)]d(y) = 0,∀x,y ϵ S
⇒ d(x)g(x)[y, y] + d(x)[y, g(x)]y + d(x)x[y, g(y)] + d(x)[y, x]g(y) + g(x)[d(x), y]y + [d(x), g(x)]y^2+ x[d(x), g(y)]y + [d(x), x]g(y)y + xg(x)[d(y), y] + x[d(y), g(x)]y + x^2[d(y), g(y)] + x[d(y), x]g(y) + g(x)[x, y]d(y) + [x, g(x)]yd(y) + x[x, g(y)]d(y) + [x,x]g(y)d(y) = 0,∀x,y ϵ U
Replacing y by x and according to the relation [d(x), g(x)] = 0
d(x)[x, g(x)]x + d(x)x[x, g(x)] + g(x)[d(x), x]x + [d(x), x]g(x)x + xg(x)[d(x), x] + x[d(x), x]g(x) + [x, g(x)]xd(x) + x[x, g(x)]d(x) = 0,∀xϵ U
⇒ d(x)xg(x)x - d(x)g(x)x^2+ d(x)x^2g(x) - d(x)xg(x)x + g(x)d(x)x^2 - g(x)xd(x)x + d(x)xg(x)x -xd(x)g(x)x + xg(x)d(x)x -xg(x)xd(x) + xd(x)xg(x) -x^2d(x)g(x) + xg(x)xd(x) - g(x)x^2d(x) + x^2g(x)d(x) - xg(x)xd(x) = 0,∀xϵ U
⇒ d(x)x^2g(x) - d(x)g(x)x^2+ g(x)d(x)x^2- g(x)xd(x)x + d(x)xg(x)x - xd(x)g(x)x + xg(x)d(x)x + xd(x)xg(x) - x^2d(x)g(x) - g(x)x^2d(x) + x^2g(x)d(x) -xg(x)xd(x) = 0
Since d(x)g(x) = g(x)d(x), then above equation become d(x)x^2g(x) - g(x)xd(x)x + d(x)xg(x)x + xd(x)xg(x) - g(x)x^2d(x) -xg(x)xd(x) = 0,∀xϵ U (1)
Since d acts as a left centralizer, d(x .x^2)g(x)-g(x)xd(x.x)+d(x.x)g(x)x+xd(x.x)g(x)-g(x)x^2d(x) -xg(x)xd(x) = 0,∀xϵ U
d(x)x^2g(x) + xd(x^2)g(x) - g(x)xd(x)x - g(x)x^2d(x) + d(x)xg(x)x + xd(x)g(x)x + xd(x)xg(x) + x^2d(x)g(x) - g(x)x^2d(x) -xg(x)xd(x) = 0
⇒ xd(x^2)g(x) + xd(x)g(x)x + x^2d(x)g(x) - g(x)x^2d(x) = 0,∀xϵ U [∵ by(1)] (2)
⇒ xd(x)xg(x) +x^2d(x)g(x) + xd(x)g(x)x + x^2d(x)g(x) - g(x)x^2d(x) = 0

$\Rightarrow xd(x)xg(x)+2x^2d(x)g(x)+xd(x)g(x)x-g(x)x^2d(x) = 0, \forall x\epsilon U$ (3)
Since $d(x)g(x) = g(x)d(x)$,
$xd(x)xg(x)+ 2x^2g(x)d(x) + xg(x)d(x)x - g(x)x^2d(x) = 0$ (4)
d acts as a left centralizer, $xd(x.x)g(x) + 2x^2g(x)d(x) + xg(x)d(x.x) - g(x)x^2d(x) = 0$
$xd(x)xg(x) + x^2d(x)g(x) + 2x^2g(x)d(x) + xg(x)d(x)x + xg(x)xd(x) - g(x)x^2d(x) = 0$
Using (4) in the above equation, $x^2d(x)g(x)+xg(x)xd(x) = 0, \forall x\epsilon U$ (5)
Since d acts as a left centralizer, $x^2d(xg(x)) + xg(x)xd(x) = 0, \forall x\epsilon U$
$\Rightarrow x^2d(x)g(x)+x^3d(g(x))+xg(x)xd(x) = 0, \forall x\epsilon U$(6)
Using (5) in (6) we get, $x^3d(g(x)) = 0, \forall x\epsilon U$(7)
Right multiplying by r and d acts as a left centralizer, $x^3d(g(x)r) = 0$
$\Rightarrow x^3d(g(x))r + x^3g(x)d(x) = 0, \forall x,r\epsilon U$
Using (7) we get, $x^3g(x)d(x) = 0, \forall x\epsilon U, r\epsilon S$
Using left cancellation property of $x^3g(x)$ this relation reduces to $d(r) = 0, \forall r\epsilon S$
Left multiplying by x, $xd(r) = 0, \forall x\epsilon U, r\epsilon S$
Again right multiplying the same relation by x, $d(r)x = 0, \forall x\epsilon U, r\epsilon S$
Subtracting these relations with replacing r by x and using lemma 3.2, we obtain S contains a non-zero central ideal

Theorem3.4

Let S be a SemiprimeSemiring with left Cancellation property, (D, d) and (G, g) be twonon-zero generalized derivations of S, and U a non-zero ideal of S. If S admits to satisfy $[D(x),G(x)] = 0, \forall x\epsilon$ Uand a non-zero d acts as a left centralizer (resp a non-zero g acts as a left centralizer), then S contains a non-zero central ideal.

Proof: The theorem is nothing to prove if we replace d by D and g by G and use the generalization property in the above theorem

Theorem 3.5

Let S be a SemiprimeSemiring with left Cancellation property, (D, d) and (G, g) be two non-zero generalized derivations of S, and U a non-zero ideal of S. If S admits to satisfy $[D(x),G(x)] = [d(x), g(x)], \forall x\epsilon U$ and a non-zero d acts as a left centralizer (resp a non-zero g acts as a left centralizer), then S contains a non-zero central ideal.

Proof: Given $[D(x),G(x)] = [d(x), g(x)], \forall x\epsilon U$ (1)
Then $[D(x),G(x)] = d(x)g(x) - g(x)d(x), \forall x\epsilon U$
d acts as a left centralizer, $[D(x),G(x)] = d(xg(x)) - g(x)d(x), \forall x\epsilon U$
$= d(x)g(x) + xd(g(x)) - g(x)d(x), \forall x\epsilon S$
$[D(x),G(x)] = [d(x), g(x)]+xd(g(x))$, Using (1), $xd(g(x)) = 0, \forall x\epsilon U$(2)
Right multiplying by y and since d acts as a left centralizer, $xd(g(x), y) = 0, \forall x,y\epsilon U$
$xd(g(x))y + xg(x)d(y) = 0, \forall x,y\epsilon U$, By (2), $xg(x)d(y) = 0, \forall x,y\epsilon U$
Using left cancellation property of xg(x) this relation reduces to $d(y) = 0, \forall y\epsilon U$
By the same method in Theorem 3:3, we complete our proof.

Theorem 3.6

Let S be a SemiprimeSemiring with left Cancellation property, (D, d) and (G, g) be two non-zero generalized derivations of S, and U a non-zero ideal of S. If S admits to satisfy $[D(x),G(x)] = [d(x), g(x)], \forall x\epsilon U$ and a non-zero D acts as a left centralizer (resp a non-zero G acts as a left centralizer), then S contains a non-zero central ideal.

Proof: Given $[D(x),G(x)] = [d(x), g(x)], \forall x\epsilon U$
Replacing x by xy, $D(xy)G(xy) - G(xy)D(xy) = [d(xy), g(xy)], \forall x,y\epsilon U$
$D(xy)G(x)y + D(xy)xg(y) - G(x)yD(xy) -xg(y)D(xy) = [d(xy), g(xy)], \forall x,y\epsilon S$
$D(x)yG(x)y + D(x)yxg(y) - G(x)yD(x)y -xg(y)D(x)y = [d(xy), g(xy)], \forall x,y \epsilon U$
$[D(x)y,G(x)y] + D(x)yxg(y) -xg(y)D(x)y = [d(xy), g(xy)], \forall x,y \epsilon U$
$G(x)[D(x)y, y]+[D(x)y,G(x)]y +D(x)yxg(y)-xg(y)D(x)y = [d(xy, g(xy)], \forall x,y \epsilon U$
$G(x)D(x)y2-G(x)yD(x)y+D(x)yG(x)y-G(x)D(x)y2+D(x)yxg(y)-xg(y)D(x)y =$
$[d(xy), g(xy)], \forall x,y \epsilon U$
$D(x)yG(x)y - G(x)yD(x)y + D(x)yxg(y) -xg(y)D(x)y = [d(xy), g(xy)], \forall x,y \epsilon U$ (1)
Since D acts as a left centralizer

$D(xy)G(x)y - G(x)yD(xy) + D(xyx)g(y) - xg(y)D(xy) = [d(xy), g(xy)], \forall x,y \in U$
$D(x)yG(x)y - G(x)yD(x)y + D(xy)xg(y) - xg(y)D(xy) = [d(xy), g(xy)], \forall x,y \in U$
$D(x)yG(x)y - G(x)yD(x)y + D(x)yxg(y) + xd(y)xg(y) - xg(y)D(xy) = [d(xy), g(xy)]$
Using (1), $xd(y)xg(y) = 0, \forall x,y \in U$
By using left Cancellation property of xd(y)x, this relation reduces to $g(y) = 0, \forall x \in U$
By the same arguement used in theorem 3.3, we complete our proof.

Theorem 3.7

Let S be a SemiprimeSemiring with Cancellation property, (D, d) and (G, g) be two generalized derivations of S, and U a non-zero ideal of S. If S admits to satisfy $[D(x),G(x)] = [d(x), g(x)], \forall x \in U$ and D and a non-zero g act as a left centralizers (resp a G and a non-zero d acts as a left centralizers), then S contains a non-zero central ideal.

Proof: Given $[D(x),G(x)] = [d(x), g(x)], \forall x \in U$
Replacing x by xy and Since D and g acts as a left centralizers, we obtain
$D(x)[y,G(xy)] + [D(x),G(xy)]y = g(x)[d(xy), y] + [d(xy), g(x)]y, \forall x,y \in U$
$\Rightarrow D(x)[y,G(x)y]+D(x)[y,xg(y)]+[D(x),G(x)y]y +[D(x),xg(y)]y = g(x)[d(x)y, y]+ g(x)[xd(y), y] + [d(x)y, g(x)]y + [xd(y), g(x)]y, \forall x,y \in U$
$\Rightarrow D(x)[y,G(x)]y+D(x)x[y, g(y)]+D(x)[y, x]g(y)+G(x)[D(x), y]y+[D(x),G(x)]y^2+ x[D(x), g(y)]y + [D(x), x]g(y)y = g(x)[d(x), y]y + g(x)x[d(y), y] + g(x)[x, y]d(y) + d(x)[y, g(x)]y + [d(x), g(x)]y^2+ x[d(y), g(x)]y + [x, g(x)]d(y)y, \forall x,y \in U$
Replacing y by x and according to $[D(x),G(x)] = [d(x), g(x)]$ we obtain
$D(x)[x,G(x)] + D(x)x[x, g(x)] + G(x)[D(x), x]x + x[D(x), g(x)]x + [D(x), x]g(x)x = g(x)[d(x), x]x + g(x)x[d(x), x] + d(x)[x, g(x)]x + x[d(x), g(x)]x + [x, g(x)]d(x)x, \forall x \in U$
$\Rightarrow D(x)xG(x)x - D(x)G(x)x2 + D(x)x2g(x) - D(x)xg(x)x + G(x)D(x)x2 - G(x)xD(x)x + xD(x)g(x)x - xg(x)D(x)x + D(x)xg(x)x - xD(x)g(x)x = g(x)d(x)x2 - g(x)xd(x)x + g(x)xd(x)x - g(x)x2d(x) + d(x)xg(x)x - d(x)g(x)x2 + xd(x)g(x)x - xg(x)d(x)x + xg(x)d(x)x - g(x)xd(x)x$
According to $[D(x),G(x)] = [d(x), g(x)]$
$D(x)xG(x)+D(x)x^2g(x)-G(x)xD(x)x-xg(x)D(x)x+[d(x), g(x)]x^2= [g(x), d(x)]x^2- g(x)x^2d(x) + d(x)xg(x)x + xd(x)g(x)x - g(x)xd(x)x$
$\Rightarrow D(x)xG(x^2)+(x)x^2g(x)-G(x^2)D(x)x-xg(x)D(x)x = -g(x)x^2d(x)+d(x)xg(x)x+ xd(x)g(x)x - g(x)xd(x)x$ (1)
Since D and g acts as a left centralizers, $D(x^2)G(x^2) - G(x^2)D(x^2) = -g(x^3)d(x) + d(x)xg(x^2) + xd(x)g(x^2) - g(x^2)d(x)x, \forall x \in U$
Since D acts as a left centralizer, $D(x)xG(x^2)-G(x^2)D(x)x = -g(x^3)d(x)+d(x)xg()+ xd(x)g(x^2) - g(x^2)d(x)x, \forall x \in U$ (2)
Since g acts as a left centralizer, (2) becomes
$D(x)xG(x^2) - G(x^2)D(x)x = -g(x^2)xd(x) + d(x)xg(x)x + xd(x)g(x)x - g(x)xd(x)x, \forall x \in U$ (3)
Substituting (1) in (3), $- g(x)x^2d(x) = -g(x^2)xd(x) \forall x \in U$
Then $xg(x)xd(x) = 0, \forall x \in U$
By using left cancellation property of xg(x)x this relation reduces to $d(x) = 0, \forall x \in U$
By the same arguement used in theorem 3:3, we complete our proof.

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