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Describing Pseudospherical Planes and Other Properties of Evolutionary SolitonEquations

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Abstract: *In thispaper we will derive Bäcklund transformations and conservation laws based on geometrical properities of evolution equations with more than two independent variables that describe pseudospherical surfaces.*
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I. Introduction

In this paper we, interest in Bäcklund transformation[12], and its connection with some special equations and their associatedsoliton theory.Under this transformation.an infinite family ofconstant curvature surfaces can be produced from a given one.The notion of a differential equation for a function $u(x,t)$ that describes a pseudospherical surface (P.S.S.) was introduced in [1,6,7], where classifications for some equations oftypes

$$u_{xt} = \psi\left(u, u_x, u_{xx}, \dots\dots \frac{\partial^k u}{\partial x^k}\right) \quad \text{and} \quad u_t = \psi\left(u, u_x, \dots\dots \frac{\partial^k u}{\partial x^k}\right)$$

Were obtained. Furthermorecharacterizations of equations with more than two independent variables of types

$$u_{xt} = \psi\left(u, u_x, \dots\dots \frac{\partial^k u}{\partial x^k}, u_y, \dots, \quad \frac{\partial^{k'} u}{\partial x^{k'}}\right), u_t = \psi\left(u, u_x, \dots\dots \frac{\partial^k u}{\partial x^k}, u_y, \dots, \quad \frac{\partial^{k'} u}{\partial x^{k'}}\right)$$

$$\text{and} u_{tt} = \psi\left(u, u_x, \dots, \frac{\partial^k u}{\partial x^k}, u_y, \dots, \frac{\partial^{k'} u}{\partial y^{k'}}, u_t\right) \text{are given in [2,3,4].}$$

A systematic procedure to determine linear problems associated to non-linear equations of the abovetypes was also introduced in case of two independent variables.

In this work, we consider evolution equations for a function $u(x,y,t)$ that describes an(η,ξ) 3-dim. P.S.P. as given in [2,3,4] and we investigate an analogous method to derive Bäcklundtransformations and conservation laws based on geometrical properties of these 3- dimensionalpseudo spherical planes in R^5.

II. Local theory of constant negative curvature submanifolds of R^{2n-1}

Let M be an n-dimensional Riemannian manifold with constant curvature K isometrically immersed in $\overline{M}^{2n-1}$with constant curvature $\overline{K}$,with $K<\overline{K}$. Let $e_1, e_2, \dots, e_{2n-1}$be a moving orthonormal frame on an open set of $\overline{M}$,so that at points of M,$e_1, e_2, \dots, e_n$are tangents to M.Let $\omega_A$be the dual orthonormal coframe and consider ω_{AB} defined by [2]

$$de_A = \sum_B \omega_{AB} e_B$$

The structure equations of $\overline{M}$are

$$d\omega_A = \sum_B \omega_B \wedge \omega_{BA} \quad , \quad \omega_{AB} + \omega_{BA} = 0 \tag{1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \overline{K}\omega_A \wedge \omega_B \quad \text{with} \quad 1 \le A,B,C \le 2n-1 \tag{2}$$

Restricting these forms to M we have $\omega_\alpha = 0$,so (1) gives with $n+1 \le \alpha,\beta,\gamma \le 2n-1$ and $1 \le I, J, L \le n$,

$$d\omega_\alpha = \sum_I \omega_I \wedge \omega_{I\alpha} = 0 \tag{3}$$

$$d\omega_I = \sum_J \omega_J \wedge \omega_{JI} \tag{4}$$

from (2) we obtain, Gauss equation

$$d\omega_{IJ} = \sum_{L} \omega_{IL} \wedge \omega_{LJ} + \sum_{\alpha} \omega_{I\alpha} \wedge \omega_{\alpha J} - \bar{K}\omega_I \wedge \omega_J \quad (5)$$

andCodazzi equation

$$d\omega_{I\alpha} = \sum_{A} \omega_{IA} \wedge \omega_{A\alpha} \quad (6)$$

M has constant sectional curvature K if and only if

$$\Omega_{IJ} = d\omega_{IJ} - \sum_{L} \omega_{IL} \wedge \omega_{LJ} = -K\, \omega_I \wedge \omega_J \quad (7)$$

$$\sum_{\alpha} \omega_{I\alpha} \wedge \omega_{\alpha J} = (\bar{K} - K)\, \omega_I \wedge \omega_J \quad (8)$$

Also, equation (2) implies that[2]

$$d\omega_{\alpha\beta} = \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}$$

$$\text{With} \qquad \Omega_{\alpha\beta} = \sum_{I} \omega_{\alpha I} \wedge \omega_{I\beta} \quad (9)$$

The forms $\Omega_{\alpha\beta}$ give the normal curvature of M and $I = \sum_I (\omega_I)^2$ is its first fundamental form.
For our purpose in this paper, we write these equations when $\bar{M}$ is taken to be R^5 and M is a 3-dimensional submanifold with constant sectional curvature $K = -1$(i.e. pseudo spherical 3-plane in R^5).
The equations take the forms[2]

$$\left.\begin{aligned} d\omega_1 &= \omega_4 \wedge \omega_2 + \omega_5 \wedge \omega_3 \\ d\omega_2 &= -\omega_4 \wedge \omega_1 + \omega_6 \wedge \omega_3 \\ d\omega_3 &= -\omega_5 \wedge \omega_1 - \omega_6 \wedge \omega_2 \\ d\omega_4 &= \omega_1 \wedge \omega_2 \\ d\omega_5 &= \omega_1 \wedge \omega_3 \\ d\omega_6 &= \omega_2 \wedge \omega_3 \end{aligned}\right\} \quad (10)$$

where we have written

$\omega_4 = \omega_{12}$ $\qquad$ $\omega_5 = \omega_{13}$, and

$\omega_6 = \omega_{23}$ with $\qquad$ $\omega_{ij} = -\omega_{ji}$, $i, j = 1,2,3$, $\omega_{ii} = 0$

We shall recall here the definition of a differential equation to describe a pseudospherical surface, introduced in [1] and modify it in order to suit our purposes here.

Definition 2.1

A differential equation E-for a real function $u(x, y, t)$ describes a 3-dimensional pseudospherical plane in R^5(simply P.S.P.) if it is the necessary and sufficient condition for the existence of differentiable functions $f_{\alpha i}$, $1 \le \alpha \le 6$and$1 \le i \le 3$, depending on u and its derivatives, such that the 1-forms[2,3]

$$\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt \qquad (11)$$

satisfy the structure equations of a 3-plane of constant sectional curvature -1 in R^5 i.e. equations (10).

Definition 2.2

We shall define such 3-dimensional P.S.P to be a two-parameters 3-dimensional P.S.P$f_{31} = f_{41} = \eta$ and $f_{22} = f_{42} = \xi$, with η and ξ constant parameters. In Fact, one can see that when$u(x, y, t)$ is a generic solution of E, it provides a metric defined on an open subset of R^3, whose sectional curvature is -1 and the lengths of the vector fields $\frac{\partial}{\partial x}$and $\frac{\partial}{\partial y}$ satisfy $\left|\frac{\partial}{\partial x}\right|^2 \ge \eta^2$, $\left|\frac{\partial}{\partial y}\right|^2 \ge \xi^2$.[2,3]

III. Generalization of Bäcklund's theorem

In this section, we define a pseudospherical geodesic congruence between two n-dimensional submanifolds M and M' of a space form $\bar{M}_k^{2n-1}$ with constant sectional curvature K. We prove a generalization of Bäcklund's theorem,[12] for such submanifolds and the complete integrability of the differential ideal associated to the existence of a pseudospherical congruence.

In what follows we need the notion of angles between two k-planes in a $2k$-dimensional inner product space.[10]

Definition 3.1

Let $E_1$and E_2be two k-planes in a $2k$-dimensional inner product space$(V, <,>)$and $\pi: V \to E_1$ the orthogonal projection. Define a symmetric bilinear form on E_2 by $(v_1, v_2) = < P(v_1), P(v_2) >$. The k angles between E_1 and $E_2$are defined to be $\theta_1, \ldots, \theta_k$where $\cos^2\theta_1, \ldots \cos^2\theta_k$ are the k-eigenvalues for the self-adjoint operator $A: E_1 \to E_2$ such that $(v_1, v_2) = < Av_1, v_2 >$.[10]

Definition 3.2

Suppose the n angles between E_1 and $E_2$are $\theta_1,\ldots,\theta_n$. Then it follows from the definition that there are two orthonormal bases $e_1,\ldots,e_{2n}$ and $e_1^{'},\ldots,e_{2n}^{'}$ of V such that $e_1^{'},\ldots,e_n^{'}$ are eigenvectors of A with eigenvalues $cos^2\theta_1,\ldots.. cos^2\theta_n$ respectively,$e_1,\ldots,e_n$form a base for E_1, and

$$e_i^{'} = \cos\theta_i e_i + \sin\theta_i e_{n+i-1},$$
$$e_{n+i-1}^{'} = -\sin\theta_i e_i + \cos\theta_i e_{n+i-1}$$

for $1 \le\ i \le n$.[9]

Definition 3.3

A geodesic congruence between two n-dimensional submanifolds M and M' of a ($2n$-l)-dimensional space form $\bar{M}$ is a diffeomorphism $\ell: M \to M'$, such that for $P \in M$ and $P' = \ell(P)$, there exists a unique geodesicγ in $\bar{M}$ joining P and P', whose tangent vectors at P and P' are in $T_P M$ and $T_{P'}M'$ respectively.[10]

Definition 3.4

A geodesic congruence $\ell: M \to M'$ between two n-dimensional submanifolds of $\bar{M}$ is called pseudospherical if:
(1) the distance between P and $P' = \ell(P)$ on $\bar{M}$, is a constant r, independent of P;
(2) the (n-l) angles between $v_P$and $v_{P'}$ are all equal to a constantθ, independent, of P;
(3) the normal bundles v and $v^{'}$are flat ;
(4) the bundle map $\Gamma: v \to v'$given by the orthogonal projection commutes with the normal connections.[10]

Definition 3.5

For given geodesic congruence $\ell: M \to M'$, we remark that, the normal spaces v_P and $v_{P'}$, at corresponding points P and P' are (n-1) dimensional and orthogonal to the plane determined by the position vector X of M and the tangent vector of γ at P. Therefore, $v_P$and $v_{P'}$, lie in a ($2n$-2) dimensional vector space, i.e. there are (n-1) angles between v_P and $v_{P'}$.[10]

Theorem 3.1

Suppose there is a pseudo-spherical congruence $l: M \to M^{'}$of n-manifolds in R^{2n-1} with distance r between corresponding points and angle $\theta \neq 0$ between corresponding normals. Then both M and $M^{'}$ have constant sectional curvature $-sin^2\theta/r^2$. [9]

Proof.

Since v' is flat, we may choose an orthonormal frame $e_{n+1}^{'},\ldots\ldots,e_{2n-1}^{'}$ for v' such that the normal connection

$$\omega_{n+i-1,n+j-1}^{'} = 0 \qquad (12)$$

Here and throughout this section, we shall agree on the index ranges

$$2 \le i,j,k \le n. \qquad (13)$$

If we use condition (2) of the definition of a pseudo-spherical congruence, there is a local orthonormal frame field $e_1,\ldots,e_{2n-1}$ for M such that [9]

$$\left.\begin{aligned} e_{n+i-1}^{'} &= -\sin\theta e_i + \cos\theta e_{n+i-1},\\ \mathrm{e}_1 &= \text{the unit direction of} \overrightarrow{PP'}, \end{aligned}\right\} (14)$$

and$e_1,\ldots,e_n$ form an orthonormal frame for TM. Let

$$\left.\begin{aligned} e_1^{'} &= -e_1,\\ e_i^{'} &= cos\,\theta e_i +\ sin\,\theta e_{n+i-1}; \end{aligned}\right\} (15)$$

then$e_1^{'},\ldots,e_n^{'}$ form an orthonormal frame for $TM^{'}$. Since $\Gamma: v \to v^{'}$ commutes with the normal connections, $\Gamma e_{n+i-1} = e_{n+i-1}^{'}$, and $\omega_{n+i-1,n+j-1}^{'} = 0$, we have

$$\omega_{n+i-1,n+j-1} = 0 \qquad (16)$$

Suppose locally M is given by an immersion $X: U \to R^{2n-1}$, where U is an open subset of R^n, then $M^{'}$ is given by

$$X^{'} = X + re_1. \qquad (17)$$

Taking the differential of (17) gives [9]

$$\left.\begin{aligned} dX^{'} &= dX + rde_1,\\ &= \omega_1\mathrm{e}_1 + \sum_i \omega_i\mathrm{e}_i + r\sum_i \omega_{1i}\mathrm{e}_i + r\sum_i \omega_{1,n+i-1}\mathrm{e}_{n+i-1}\\ &= \omega_1\mathrm{e}_1 + \sum_i(\omega_i + r\omega_{1i})\mathrm{e}_i + r\sum_i \omega_{1,n+i-1}\mathrm{e}_{n+i-1}. \end{aligned}\right\} (18)$$

On the other hand, letting $\omega_1^{'},\ldots,\omega_n^{'}$be the dual coframe of $e_1^{'},\ldots,e_n^{'}$,we have

$$\left.\begin{aligned} dX' &= \omega_1' e_1' + \sum_i \omega_i' e_i' \ , \quad \text{using}(15) \\ &= -\omega_1' \mathrm{e}_1 + \sum_i \omega_i' \left(-\cos\theta e_i + \sin\theta e_{n+i-1}\right). \end{aligned}\right\} (19)$$

Comparing coefficients of $e_1, \ldots, e_{2n-1}$ in (18) and (19), we get

$$\left.\begin{aligned} \omega_1' &= -\omega_1, \\ \cos\theta\omega_i' &= \omega_i + r\omega_{1i}\,, \\ \sin\theta\omega_i' &= r\omega_{1,n+i-1.} \end{aligned}\right\} (20)$$

This gives

$$\omega_i + r\omega_{1i} = r\cot\theta\omega_{1,n+i-1.} (21)$$

Using (12), (14) and (16), we have

$$\left.\begin{aligned} 0 &= \omega'_{n+i-1\,,n+j-1} \\ &= de'_{n+i-1}.\,e'_{n+j-1} \\ &= d(-\sin\theta e_i + \cos\theta e_{n+i-1}).(-\sin\theta e_j + \cos\theta e_{n+j-1}) \\ &= \sin^2\theta de_i.\,e_j - \sin\theta\cos\theta(de_i.\,e_{n+j-1} - de_j.\,e_{n+i-1}) \\ &= \sin^2\theta\omega_{ij} - \sin\theta\cos\theta(\omega_{i,n+j-1} - \omega_{j,n+i-1}). \end{aligned}\right\} (22)$$

Therefore we have

$$\omega_{ij} = \cot\theta\left(\omega_{i,n+j-1} - \omega_{j,n+i-1}\right). \quad (23)$$

In order to find the curvature, we compute the following 1-forms:

$$\left.\begin{aligned} \omega'_{1,n+k-1} &= de_1'.\,e'_{n+k-1}\,, \quad \text{using}(14)\text{and}(15) \\ &= -\sin\theta\omega_{1k} - \cos\theta\omega_{1,n+k-1}\,, \text{using}(21) \\ &= -\frac{\sin\theta}{r}\omega_k\,, \\ \omega'_{i,n+k-1} &= de_i'.\,e'_{n+k-1} \\ &= -\sin\theta\cos\theta\omega_{ik} + \cos^2\theta\omega_{i,n+k-1} + \sin^2\theta\omega_{k,n+i-1}\,, \ \text{using}(23) \\ &= \omega_{k,n+i-1}. \end{aligned}\right\} (24)$$

Hence from equation (9) we have

$$\left.\begin{aligned} \Omega_{1i}' &= -\sum_k \omega'_{1,n+k-1} \wedge \omega'_{i,n+k-1}\,, \quad \text{using}(24) \\ &= \frac{\sin\theta}{r}\sum_k \omega_k \wedge \omega_{k,n+i-1,}\text{using}(1) \\ &= -\frac{\sin\theta}{r}\omega_1 \wedge \omega_{1,n+i-1}, \quad \text{using}(20) \\ &= \frac{\sin^2\theta}{r^2}\omega_1' \wedge \omega_i'\,; \\ \Omega_{ij} &= -\sum_i \omega'_{i,n+k-1} \wedge \omega'_{j,n+k-1}, \quad \text{using}(24) \\ &= -\sum_i \omega_{k,n+i-1} \wedge \omega_{k,n+j-1}. \end{aligned}\right\} (25)$$

Since v is flat and $\omega_{n+i-1,n+j-1} = 0$, we have

$$\begin{aligned} 0 &= -d\omega_{n+i-1,n+j-1}\,, \quad \text{using}(9) \\ &= \omega_{1,n+i-1} \wedge \omega_{1,n+j-1} + \sum_k \omega_{k,n+i-1} \wedge \omega_{k,n+j-1}. \end{aligned}$$

So we have

$$\left.\begin{aligned} \Omega_{1i}' &= \omega_{1,n+i-1} \wedge \omega_{1,n+j-1}, \quad \text{using}(9) \\ &= \frac{\sin^2\theta}{r^2}\omega_i'\omega_j' \end{aligned}\right\} (26)$$

Therefore M' has constant sectional curvature $-sin^2\theta/r^2$. By symmetry, M also has constant sectional curvature $-sin^2\theta/r^2$. [9]

Theorem 3.2

Suppose $M$is a local n-submanifold with constant negative sectional curvature $K = -sin^2\theta/r^2$in $\boldsymbol{R}^{2n-1}$, where $r > 0$ and $\theta$are constants. Let $v_1^0, \ldots, v_n^0$, be an orthonormal base at P_0 consisting of principal curvature vectors, and $v_0 = \sum_{i=1}^{n} c_i v_i^0$ a unit vector with $c_i \neq 0$ for all $1 \leq i \leq n$; then there exists a local n-submanifold M' of $\boldsymbol{R}^{2n-1}$, and a pseudo-spherical congruence $l: M \to M'$such that if $P_0' = l(P_0)$,we have $\overrightarrow{P_0 P'} = rv_0$, and θ is the angle between the normal planes at P_0andP'. [9]

Theorem 3.3

Let $f_{\alpha i}, 1 \leq \alpha \leq 6, 1 \leq i \leq 3$, be differentiable functions of variables x, y and t such that [5,8]

$$\left.\begin{aligned}
&-f_{11,y} + f_{12,x} + \eta f_{52} + \xi f_{21} - \eta f_{22} - \xi f_{51} = 0,\\
&-f_{11,t} + f_{13,x} + \eta f_{53} + f_{43} f_{21} - \eta f_{23} - f_{51} f_{33} = 0,\\
&-f_{12,t} + f_{13,y} + \xi f_{53} + f_{43} f_{22} - f_{52} f_{33} - \xi f_{23} = 0,\\
&-f_{21,y} + f_{22,x} + \eta f_{62} + \eta f_{12} - \xi f_{61} - \xi f_{11} = 0,\\
&-f_{21,t} + f_{23,x} + \eta f_{63} + \eta f_{13} - f_{61} f_{33} - f_{11} f_{43} = 0,\\
&-f_{22,t} + f_{23,y} + \xi f_{63} + \xi f_{13} - f_{62} f_{33} - f_{12} f_{43} = 0,\\
&-f_{31,y} - f_{32,x} + f_{11} f_{52} - f_{12} f_{51} - f_{22} f_{61} + f_{21} f_{62} = 0,\\
&-f_{31,t} + f_{33,x} + f_{23} f_{61} + f_{13} f_{51} - f_{21} f_{63} - f_{11} f_{53} = 0,\\
&-f_{32,t} + f_{33,y} + f_{23} f_{62} + f_{13} f_{52} - f_{22} f_{63} - f_{12} f_{53} = 0,\\
&-f_{41,y} + f_{42,x} + f_{11} f_{22} - f_{12} f_{21} = 0,\\
&-f_{41,t} + f_{43,x} + f_{13} f_{21} - f_{11} f_{23} = 0,\\
&-f_{42,t} + f_{43,y} + f_{13} f_{22} - f_{12} f_{23} = 0,\\
&-f_{51,y} + f_{52,x} + \eta f_{12} - \xi f_{11} = 0,\\
&-f_{51,t} + f_{53,x} + \eta f_{13} - f_{11} f_{33} = 0,\\
&-f_{52,t} + f_{53,y} + \xi f_{13} - f_{12} f_{33} = 0,\\
&-f_{61,y} + f_{62,x} + \eta f_{22} - \xi f_{21} = 0,\\
&-f_{61,t} + f_{63,x} + \eta f_{23} - f_{21} f_{33} = 0,\\
&-f_{62,t} + f_{63,y} + \xi f_{23} - f_{22} f_{33} = 0,
\end{aligned}\right\} (27)$$

and

$$\begin{aligned}
&f_{11} f_{22} - f_{12} f_{21} + \eta f_{12} - \xi f_{11} + \xi f_{51} - \eta f_{52}\\
&\quad + \sin\phi\,[2\eta f_{12} - 2\xi f_{11} + f_{11} f_{52} - f_{12} f_{51} - \eta f_{52} - \xi f_{21} - f_{21} f_{52} + f_{21} f_{12} - f_{11} f_{22} + \xi f_{51}\\
&\quad + \eta f_{22} + f_{22} f_{51}]\\
&\quad + \cos\phi\,[f_{12} f_{51} - f_{11} f_{52} + 2\eta f_{12} - 2\xi f_{11} + \xi f_{21} + f_{21} f_{52} + f_{11} f_{22} - f_{21} f_{12} - \eta f_{52}\\
&\quad - \eta f_{22} - f_{22} f_{51} + \xi f_{51}] = 0,\\
&f_{13} f_{21} - f_{11} f_{23} + \eta f_{13} - f_{11} f_{33} + f_{21} f_{33} + f_{33} f_{51} - \eta f_{53} - f_{21} f_{43}\\
&\quad + \sin\phi\,[2\eta f_{13} - f_{11} f_{43} + f_{11} f_{53} - f_{13} f_{51} - \eta f_{43} - \eta f_{53} - f_{21} f_{43} - f_{21} f_{53} + f_{21} f_{13} + \eta f_{33}\\
&\quad + f_{33} f_{51} - f_{11} f_{33} + \eta f_{23} + f_{23} f_{51} - f_{11} f_{23}]\\
&\quad + \cos\phi\,[f_{13} f_{51} - f_{11} f_{53} + 2\eta f_{13} - f_{11} f_{43} + f_{21} f_{43} + f_{21} f_{53} - f_{21} f_{13} - \eta f_{43} - \eta f_{53}\\
&\quad - \eta f_{23} - f_{23} f_{51} + f_{23} f_{11} + \eta f_{33} + f_{51} f_{33} - f_{11} f_{33}] = 0,\\
&f_{13} f_{22} - f_{12} f_{23} + \xi f_{13} - f_{12} f_{33} + f_{22} f_{33} + f_{52} f_{33} - f_{22} f_{43} - \xi f_{53}\\
&\quad + \sin\phi\,[2\xi f_{13} - f_{12} f_{43} + f_{12} f_{53} - f_{13} f_{52} - \xi f_{43} - \xi f_{53} - f_{22} f_{43} - f_{22} f_{53} + f_{22} f_{13} + \xi f_{33}\\
&\quad + f_{23} f_{52} - f_{23} f_{12} + f_{33} f_{52} - f_{33} f_{12} + \xi f_{23}]\\
&\quad + \cos\phi\,[f_{13} f_{52} - f_{12} f_{53} + 2\xi f_{13} - f_{12} f_{43} + f_{22} f_{43} + f_{22} f_{53} - f_{22} f_{13} - \xi f_{43} - \xi f_{53}\\
&\quad - \xi f_{23} - f_{23} f_{52} + f_{23} f_{12} + \xi f_{33} + f_{52} f_{33} - f_{12} f_{33}] = 0,
\end{aligned}$$

With $f_{31} = \eta = f_{41}$; $f_{32} = \xi = f_{42}$

Then the following statements are valid.

1. The following system is completely integrable for ϕ ;

$$\left.\begin{aligned}
3\phi_x &= f_{41} + f_{51} + f_{61} - f_{11} + (f_{21} - f_{31})\sin\phi + (f_{31} + f_{21})\cos\phi\\
3\phi_y &= f_{42} + f_{52} + f_{62} - f_{12} + (f_{22} - f_{32})\sin\phi + (f_{32} + f_{22})\cos\phi\\
3\phi_t &= f_{43} + f_{53} + f_{63} - f_{13} + (f_{23} - f_{33})\sin\phi + (f_{33} + f_{23})\cos\phi
\end{aligned}\right\} (28)$$

2. For any solution ϕ of(28) the 1- forms

$$\left.\begin{array}{c}\sigma_1 = f_{11}dx + f_{12}dy + f_{13}dt \quad , \qquad \text{and} \\ \sigma_2 = (f_{21}\sin\phi + f_{31}\cos\phi)dx + (f_{22}\sin\phi + f_{32}\cos\phi)dy \\ +(f_{23}\sin\phi + f_{33}\cos\phi)dt\end{array}\right\} (29)$$

Are closed one – forms

3. If $f_{\alpha i}$ are analytic functions of parameters η and ξ at zero, then the solution $\phi(x, y, t, \eta, \xi)$ of (28) and the one-forms (29) are also analytic in η and ξ at zero.

Proof:

With respect to point 1. , it follows from the Frobenius theorem. and from (27) and (28) . straight forward computations show that (27) implies.

$$\phi_{xy} = \phi_{yx} \quad ; \quad \phi_{xt} = \phi_{tx} \quad ; \quad \phi_{yt} = \phi_{ty} \quad .$$

point 2. , can be proved by showing that the systems (27) and (28) imply that exterior differentiation of the forms σ_1 and σ_2 in (29) is zero, which is the case.

In order to prove point 3. , we suppose that functions $f_{\alpha i}$ are analytic functions of parameters η and ξ. Each equation of (28) can be considered as an ordinary differential equation whose right – hand side is an analytic functions of (ϕ, η, ξ), where the solutions of $\phi(x, y, t, \eta, \xi)$ of this equation exist as defined by point 1. It follows from the theory of ordinary differential equations, [11], on the dependence of solutions up on parameters, that $\phi(x, y, t, \eta, \xi)$ is an analytic functions of η and ξ, for η and ξ in an appropriate neighborhood of zero. This completes the proof of the theorem.[5,8]

IV. Derivation of Bäcklund transformations and conservation laws for evolution equations in higher dimensions

In this section, we extend the results obtained in[5], by introducing a new method to derive an infinite set of conservation laws for equations that describes a P.S.P., based on a geometrical property of these planes. So, firstly. We consider M and M' as sub manifolds of R^{2n-1} of dim n, and $l: M \to M$ be a pseudospherical geodesic congruence between M and M', then there exist local orthonormal forms[8,9]$e_1, e_2, \dots\dots, e_n, e_{n+1}, \dots\dots, e_{2n-1}$ and $e_1', e_2' \dots\dots, e_n', \dots\dots, e_{2n-1}'$ for R^{2n-1} with $e_1, e_2, \dots\dots, e_n$ for M and $e_1', e_2' \dots\dots, e_n'$ for M' such that

$$e_1' = \cos\theta e_1 + \sin\theta e_{n+i-1} \quad , \quad 2 \le i \le n$$
$$e_{n+i-1}' = -\sin\theta e_i + \cos\theta e_{n+i-1} \quad , \qquad (30)$$

Are verified, see [10], and $e_1' = -e_1$, where $e_1$at $P \in M$ is the unit vector tangent to the geodesic from P to $P' = l(P)$

In the special case, when $n = 2$, relations (30) become [1]

$$\left.\begin{array}{c}e_1' = \cos\theta e_1 + \sin\theta e_2 \\ e_2' = -\sin\theta e_1 + \cos\theta e_2\end{array}\right\} (31)$$

Where, it is considered that all the (n-1) angles are the same and equal to θ

In our case of evolution equations of three variables, M and M' 3- dimensional Riemannian sub manifolds of R^5; $e_1', e_2', \dots\dots, e_5'ande_1, e_2, \dots\dots, e_5$ are two different erthonormal frames with e_1', e_2', e_3' tangents to M' and e_1, e_2, e_3 tangents to M. While $\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{13}, \omega_{23}$and $\omega_1', \omega_2', \omega_3', \omega_{12}', \omega_{13}', \omega_{23}'$, are the dual coframes and connections forms on M and M' respectively. For $2 \le i \le 3 (n = 3)$, one can write the following relations, with a pseudo spherical line congruence $l: M \to M$,

$$\left.\begin{array}{c}e_1' = -e_1 \\ e_2' = \cos\theta e_2 + \sin\theta e_3 \\ e_3' = -\sin\theta e_2 + \cos\theta e_3\end{array}\right\} (32)$$

From the relations (32), we have

$$\left.\begin{array}{c}\omega_1 = -\omega_1' \\ \omega_2 = \cos\theta\omega_2' + \sin\theta\omega_3' \\ \omega_3 = -\sin\theta\omega_2' + \cos\theta\omega_3'\end{array}\right\} (33)$$

And

$$\left.\begin{array}{c}\omega_1' = -\omega_1 \\ \omega_2' = \cos\theta\omega_2 - \sin\theta\omega_3 \\ \omega_3' = \sin\theta\omega_2 + \cos\theta\omega_3\end{array}\right\} (34)$$

Now, consider a differential equation (E) for $u(x, y, t)$ which describes a two parameters 3-dim. P.S.P. with associated 1 – forms.

$$\left.\begin{aligned}\omega_1 &= f_{11}dx + f_{12}dy + f_{13}dt;\\ \omega_2 &= f_{21}dx + f_{22}dy + f_{23}dt;\\ \omega_3 &= \eta dx + \xi dy + f_{33}dt;\\ \omega_4 &= \omega_{12} = \eta dx + \xi dy + f_{43}dt;\\ \omega_5 &= \omega_{13} = f_{51}dx + f_{52}dy + f_{53}dt;\\ \omega_6 &= \omega_{23} = f_{61}dx + f_{62}dy + f_{63}dt.\end{aligned}\right\} \quad (35)$$

Where $f_{\alpha i}$, are functions of $u(x,y,t)$ and its derivatives (observe that we are denoting ω_4, ω_5 and ω_6 for ω_{12}, ω_{13} and ω_{23} respectively, which are the classical notation for the connection forms). We have the following[8]

Proposition 4.1
Let E be a differential equation which describes a two parameters 3- dim P.S.P. with associated 1- forms (35) . Then, for each solution $u$of E, the system of equations for $\phi(x,y,t)$.[8]

$$\left.\begin{aligned}\omega_{12} - d\phi + \omega_3' &= 0,\\ \omega_{13} - d\phi + \omega_2' &= 0,\\ \omega_{23} - d\phi + \omega_1' &= 0\end{aligned}\right\} \quad (36)$$

Are completely integrable. Moreover, for each solution $u$of E, and corresponding solution ϕ of (36). the forms

$$\left.\begin{aligned}&\sigma_1 = f_{11}dx + f_{12}dy + f_{13}dt\,,\\ &\sigma_2 = (f_{21}\sin\phi + f_{31}\cos\phi)dx + (f_{22}\sin\phi + f_{32}\cos\phi)dy + (f_{23}\sin\phi + f_{33}\cos\phi)dt\end{aligned}\right\}$$

Are closed forms.

Proof
It follows from (33) that u is a solution of E iff.$\omega_{12} - d\phi + \omega_3' = 0$i.e.,

$$\left.\begin{aligned}&\omega_{12} - d\phi + \sin\phi\,\omega_2 + \cos\phi\,\omega_3 = 0,\\ &\omega_{13} - d\phi + \omega_2' = 0\end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned}&\omega_{13} - d\phi + \cos\phi\,\omega_2 + \sin\phi\,\omega_3 = 0,\\ &\omega_{23} - d\phi + \omega_1' = 0\end{aligned}\right\} \quad (38)$$

$$\text{and}\,\omega_{12} - d\phi + \omega_1 = 0 \quad (39)$$

Are completely integrable forϕ. In this case:

$$\omega_1\ ;\text{and}\ \sin\phi\,\omega_2 + \cos\phi\,\omega_3 \quad (40)$$

Are closed forms. Hence, inserting (35) into (37) we obtain equations.

$$(\eta - \phi_x + f_{21}\sin\phi + \eta\cos\phi)dx + (\xi - \phi_y + f_{22}\sin\phi + \xi\cos\phi)dy + (f_{43} - \phi_t + f_{23}\sin\phi + f_{23}\cos\phi)dt = 0$$

$$3\phi_x = f_{41} + f_{51} + f_{61} - f_{11} + (f_{21} - f_{31})\sin\phi + (f_{31} + f_{21})\cos\phi, \quad (41)$$

Are closed forms. Hence, inserting (35) into (38) we obtain equations.

$$(f_{51} - \phi_x + f_{21}\cos\phi - \eta\sin\phi)dx + (f_{52} - \phi_y + f_{22}\cos\phi - \xi\sin\phi)dy + (f_{53} - \phi_t + f_{23}\cos\phi - f_{23}\sin\phi)dt = 0$$

$$3\phi_y = f_{42} + f_{52} + f_{62} - f_{12} + (f_{22} - f_{32})\sin\phi + (f_{32} + f_{22})\cos\phi\,, \quad (42)$$

Are closed forms. Hence, inserting (35) into (39) we obtain equations.

$$(f_{61} - \phi_x - f_{11})dx + \left(f_{62} - \phi_y - f_{12}\right)dy + (f_{63} - \phi_t - f_{13})dt = 0$$

$$3\phi_t = f_{43} + f_{53} + f_{63} - f_{13} + (f_{23} - f_{33})\sin\phi + (f_{33} + f_{23})\cos\phi \quad (43)$$

Whose integrabilty condition is E. also, inserting (35) into (40), One can obtain the closed forms (29).

Now, we note that whenever E does not involve the parameters η, ξ the closed forms (29) may provide an infinite number of conservation laws.[8]

Also, under certain conditions, equations (28) may provide Bäcklund transformations for E and as we know from theorem (3.3), the conditions

$$\phi_{xy} = \phi_{yx} \quad ; \quad \phi_{xt} = \phi_{tx} \quad ; \quad \phi_{yt} = \phi_{ty} \quad .$$

are valid as the complete integrabilty condition for (28). As conservation laws are common features of mathematical physics, where they describe the conservation of fundamental physical quantities, it is worth studying them in this geometric study. Before giving the method for deriving conservation laws for evolution equations that describes 2 – parameters 3-dim. P.S.P., we consider the following:

Definition4.1
We suppose a system of the form

$$u_t = S[u] \quad (44)$$

In the system, when a functional $J[(x,y,t)]$ satisfies.

$$dJ[U(x,y,t)]/dt = 0, \quad (45)$$

The functional is said to be an integral of equation (44). and

$$\frac{\partial}{\partial t}T[u(x,y,t)] + \frac{\partial}{\partial x}Q[u(x,y,t)] + \frac{\partial}{\partial y}R[u(x,y,t)] = 0 \quad (46)$$

Where usually each of $T[u(x,y,t)], Q[u(x,y,t)]$ and $R[u(x,y,t)]$do not involve derivatives with respect to t, is called a conservation law. In particular, if we are to apply this idea to an evolution equations for $u(x,y,t)$, then T, Q and R may depend upon $x, y, t, u, u_x, u_y, u_{xx}, u_{yy}, \dots\dots$, but not on$u_t$. [8]

If we assume the function $u(x,y,t)$ and its derivatives with respect to x and y go to zero sufficiently fast as $|x| \to \infty, |y| \to \infty$, i.e., if T, Q_x *and* R_y are integrable on $(-\infty, \infty)$, so that $Q \to$ constant as $|x| \to \infty, R \to$ constant as $|y| \to \infty$, then equation (46) can be integrated to yield.

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T dx dy = 0 \quad \text{, i.e} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T[u(x,y,t)]\ dx\, dy = J(u) \quad = \text{Constant}$$

The method to derive conservation laws for evolution equations that describe, spherical surfaces (P.S.S.) is introduced by Cavalcante and Tenenblatin [5], for the case of two independent variables.

In this work, we will give a method with argument analogue to that considered in [5,8] to derive conservation laws for evolution equations that describe 2 – parameters 3-dim. P.S.P. This integrated method is based on geometrical properties of these planes.

Here, we suppose the functions $f_{\alpha i}$ to be analytic in each of η and ξ seperatly, and describe the solutions ϕ of (28) as a power series of η andξ . In addition, from relations (29) we obtain a sequence of closed one – forms. So, we suppose

$$f_{\alpha i}(x,y,t,\eta,\xi) = \sum_{r=0}^{\infty} h_{\alpha i}^{r}(x,t)\,\eta^r + g_{\alpha i}^{r}(y,t)\xi^r \quad (47)$$

And the solution ϕ of (28) may have the form

$$\phi(x,y,t,\eta,\xi) = \sum_{i=0}^{\infty}(\phi_i(x,t)\,\eta^i + \psi_i(y,t)\xi^i) \qquad (48)$$

For fixed , y, t , we consider functions of η and ξ respectively as follows:

$$C(\eta,\xi) = \cos\phi = \cos\left[\sum_{i=0}^{\infty}(\phi_i\,\eta^i + \psi_i\xi^i)\right] \quad (49)$$

$$S(\eta,\xi) = \sin\phi = \sin\left[\sum_{i=0}^{\infty}(\phi_i\,\eta^i + \psi_i\xi^i)\right] \quad (50)$$

From relations (49) and (50), we have

$$\left.\begin{aligned} C(0,0) &= \cos(\phi_0 + \psi_0); \\ S(0,0) &= \sin(\phi_0 + \psi_0) \end{aligned}\right\}$$

$$\frac{d^r C}{d\eta^r}(0,\xi) = -(r-1)!\sum_{\alpha=0}^{r-1}\frac{r-\alpha}{\alpha!}\frac{d^\alpha C}{d\eta^\alpha}(0,\xi)\phi_{r-\alpha} \quad (51)$$

$$\frac{d^r}{d\eta^r}(0,\xi) = (r-1)!\sum_{\alpha=0}^{r-1}\frac{r-\alpha}{\alpha!}\frac{d^\alpha C}{d\eta^\alpha}(0,\xi)\phi_{r-\alpha} \quad \text{, for } r \geq 1$$

$$\frac{d^r C}{d\xi^r}(\eta,0) = -(r-1)!\sum_{\alpha=0}^{r-1}\frac{r-\alpha}{\alpha!}\frac{d^\alpha S}{d\xi^\alpha}(\eta,0)\psi_{r-\alpha} \quad (52)$$

$$\frac{d^r S}{d\xi^r}(\eta,0) = (r-1)!\sum_{\alpha=0}^{r-1}\frac{r-\alpha}{\alpha!}\frac{d^\alpha C}{d\xi^\alpha}(\eta,0)\psi_{r-\alpha} \quad \text{, for } r \geq 1$$

Finally, we define the following functions of x, y, t :

$$G_r^{\alpha i} = (h_{2r}^{\alpha} - h_{3r}^{\alpha})\frac{d^{i-\alpha}C}{d\eta^{i-\alpha}}(0,\xi) - (h_{2r}^{\alpha} + h_{3r}^{\alpha})\frac{d^{i-\alpha}S}{d\eta^{i-\alpha}}(0,\xi),$$

$$L_r^{\alpha i} = (h_{2r}^{\alpha} - h_{3r}^{\alpha})\frac{d^{i-\alpha}S}{d\eta^{i-\alpha}}(0,\xi) + (h_{2r}^{\alpha} + h_{3r}^{\alpha})\frac{d^{i-\alpha}C}{d\eta^{i-\alpha}}(0,\xi),$$

$$F_{3r} = (h_{4r}^3 + h_{5r}^3 + h_{6r}^3 - h_{1r}^3) + L_r^{33}, \qquad r = 1,3$$

$$F_{Pr} = (h_{4r}^P + h_{5r}^P + h_{6r}^P - h_{1r}^P) + \sum_{m=1}^{P-1}\frac{P-m}{m!}G_r^{0m}\phi_{P-m} + \sum_{m=1}^{P}\frac{1}{(P-m)!}L_r^{mP}\ , r = 1,3$$

$$'G_r^{\alpha i} = (g_{2r}^{\alpha} - g_{3r}^{\alpha})\frac{d^{i-\alpha}C}{d\xi^{i-\alpha}}(\eta,0) - (g_{2r}^{\alpha} + g_{3r}^{\alpha})\frac{d^{i-\alpha}S}{d\xi^{i-\alpha}}(\eta,0),$$

$$'L_r^{\alpha i} = (g_{2r}^{\alpha} - g_{3r}^{\alpha})\frac{d^{i-\alpha}S}{d\xi^{i-\alpha}}(\eta,0) + (g_{2r}^{\alpha} + g_{3r}^{\alpha})\frac{d^{i-\alpha}C}{d\xi^{i-\alpha}}(\eta,0),$$

$$F_{3r}' = (g_{4r}^3 + g_{5r}^3 + g_{6r}^3 - g_{1r}^3) + L_r^{33}$$

$$F_{qr}' = \left(g_{4r}^q + g_{5r}^q + g_{6r}^q - g_{1r}^q\right) + \sum_{s=1}^{q-1}\frac{q-s}{s!}G_r^{0s}\psi_{q-s} + \sum_{s=1}^{q}\frac{1}{(q-s)!}L_r^{sq}, \qquad r = 2,3$$

Where each of α, i, p, q is non- negative integer such that $i \geq \alpha{:}\, p, q \geq 4$, and $r = 1,2,3$.
It is easy to see that the functions $'G_r^{\alpha i}$ and $'L_r^{\alpha i}$ depend above depend on $\phi_0, \phi_1, \ldots, \phi_{i-\alpha}$. Whears the functions $'G_r^{\alpha i}$ and $'L_r^{\alpha i}$ depend on $\psi_0, \psi_1, \ldots\ldots, \psi_{i-\alpha}$
Also, the functions $(F_{2r}, F_{3r})$and (F_{pr}) depend on$\phi_0$and $\phi_1, \phi_2, \ldots\ldots, \phi_{p-1}$, respectively, but the functions $(F_{2r}', F_{3r}')$and (F_{qr}') depend on ψ_0 and $\psi_1, \psi_2, \ldots\ldots, \psi_{q-1}$, respectively.
Under the above notation, we obtain the following corollary.

Corollary 4.1
Suppose $f_{\alpha i}(x, y, t, \eta, \xi),\ 1 \leq \alpha \leq 6\ ,\ 1 \leq i \leq 3$, be differentiable functions of $x, y, t,$ analytic at $\eta = 0\ , \xi = 0$ that satisfy(27). Then, in view of the above notation, the following statements hold.
(i) The solution ϕ of (28) is analytic at $\eta = 0\ , \xi = 0\ ;\ \phi_0$ andψ_0 are determined by.

And, for $1 \geq i\ , \phi_i$andψ_i are recursively determined by the system

(ii) For any such solution ϕ, equation(48) and any integer $i \geq 0$

Are closed one- forms.
The proof of the corollary follows with somehow straight forward calculations from equations $(48) \rightarrow (54)$ and equations (28) with the introduced notations.
Now, if we consider a non-linear evolution equation for $u(x, y, t)$which describes a 3–dim.
P.S.P., then there exist functions $f_{\alpha i}, 1 \leq \alpha \leq 6\ ,\ 1 \leq i \leq 3$, depending on $u(x, y, t)$ and its derivatives, such that, for any solution u of the evolution equation, $f_{\alpha i}$ satisfy (27). So, it follows from theorem (3) that equations (28) are completely integrablefor . If we consider $f_{\alpha i}$ to be analytic functions of parameters η, ξ then we can find that the solutions ϕ of (28) and the 1-forms given by (29), are analytic in η, ξ where their coefficients $\phi_i\ , \psi_i$ and β^i, as functions of u, are determined by $(55) \rightarrow (59)$. The ciosed 1-forms β^i provide a sequence of conservation laws for the evolution equation, with equations given by.

$$Q_i = \sum_{\alpha=0}^{i}\frac{1}{(i-\alpha)!}\left(G_3^{\alpha i} + {}'G_3^{\alpha i} + {}'G_2^{\alpha i}\right),$$

$$T_i = \sum_{\alpha=0}^{i}\frac{1}{(i-\alpha)!}\left(G_1^{\alpha i} + G_2^{\alpha i}\right),$$

with

$$Q_{i,x} + R_{i,x} + T_{i,1} = 0 \quad , \quad i \geq 0$$

For Bäcklund transformations of the equation E which describes P.S.P., we remark that the angle ϕ of a pseudospherical line congruence is determined by the system of equations (28). If we suppose that (28) is equivalent to a system of the form:

Then given a solution u of E the system (28) is integrable and ϕis a solution of (62), then u defined by the equation (61) will be a solution for$E$. However, it still needs more work to be done.

V. Conclusion

In this paper, we generalized Bäcklund transformations and conservation laws for evolution equations in higher dimensions.

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