IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 3 Ver. V (May - Jun. 2015), PP 48-50
www.iosrjournals.org

Various Reflexivities in Sequence Spaces

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Abstract: *In this paper, we investigate that for each p, $1 < p < \infty$, space ℓ^p, equipped with the normed topology, is both (i) B-reflexive, and (ii) inductively reflexive. We also discuss that the locally convex spaces $\ell^p [\tau_s(\ell^q)]$, where $1 < p < \infty$ and $1/p + 1/q = 1$, are semi-reflexive (and so polar semi-reflexive) and the locally convex space $\ell^1 [\tau_k(c_0)]$ is inductively semi-reflexive.*
Keywords- *Bornological, B-reflexive, inductively reflexive, normed topology, polar reflexive, sequence space.*

I. Introduction

Semi-reflexivity and reflexivity are well known properties in locally convex spaces. There are other types of reflexivity, namely, polar semi-reflexivity and polar reflexivity in [1], inductive semi-reflexivity and inductive reflexivity introduced by I.A. Berezanskij [2] and B-semireflexivity, B-reflexivity in [3]. The notions of p-completeness and p-reflexivity introduced by Kalman Brauner [4] are nothing but polar semi-reflexivity and polar reflexivity, respectively. In this paper we discuss these reflexivities in sequence spaces. For a locally convex space $E[\tau]$, which we always consider Hausdorff, the dual is denoted by E'. The strong dual of $E[\tau]$ is $E'[\tau_b(E)]$ and the bidual of $E[\tau]$ is $E''=(E'[\tau_b(E)])'$. We follow the notion of Köthe [1] for notations and terminology, unless specifically mentioned.

A locally convex space $E[\tau]$ is called semi-reflexive if $E = E''$. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_b(E')$.

Let τ^o be the topology on E' of uniform convergence over the class of τ-precompact sets (in E). We have $\tau^o \leq \tau_b(E)$. The topology on $(E'[\tau^o])'$ of uniform convergence over the class of τ^o -precompact subsets of $E'[\tau^o]$ is denoted as τ^{oo}.

1.1. Definition (Köthe [1]): A locally convex space $E[\tau]$ is called polar semi-reflexive if $E = (E'[\tau^o])'$. Polar semi-reflexive space $E[\tau]$ is called polar reflexive if $\tau = \tau^{oo}$ i.e. $(\tau^o)^o$.

Consider a locally convex space $E[\tau]$ and a base $\{U_\alpha: \alpha \in I\}$ of τ-neighborhoods of 0 consisting of closed absolutely convex neighborhoods. Let U^o_α be the polar of U_α in E' and $E'_{u\alpha}$ be the linear subspace of E' spanned by U^o_α equipped with the norm topology with U^o_α as unit ball. Let $E'[\tau^*]$ be the inductive limit of the system $\{E'_{u\alpha}\}$ and the embeddings: $E'_{u\alpha} \to E'$. Note that τ^* is the finest locally convex topology on E' making all embeddings: $E'_{u\alpha} \to E'$ continuous. Starting from the locally convex space $E'[\tau^*]$, the topology $\tau^{**} = (\tau^*)^*$ is defined on $(E'[\tau^*])'$. The topology τ^* constructed this way is due to [2].

1.2. Definition (Berezanskií [2]): If $(E'[\tau^*])' = E$, then $E[\tau]$ is called inductively semi-reflexive. If, in addition, $\tau = (\tau^*)^*$, then $E[\tau]$ is called inductively reflexive.

Following P.K. Raman [3], we define that an absolutely convex bounded subset B of the dual E' of a l.c. space $E[\tau]$ is called reflective if the span E'_B is a reflexive Banach space with B as unit ball. The class of all reflective sets is denoted by $\mathcal{R}$. The topology on E of uniform convergence over the saturated class of sets generated by $\mathcal{R}$ *is* called the reflective topology of E and is denoted by $\boldsymbol{\tau_r}$. The polars of the sets of $\mathcal{R}$ i.e. the class$\{K_o : K \in \mathcal{R}\}$ forms a base of neighborhoods of the origin **o** for $E[\boldsymbol{\tau_r}]$.

1.3. Definition (Raman [3]): If a locally convex space $E[\tau]$ is barreled and E□ = the completion of $E[\boldsymbol{\tau_r}]$, then $E[\tau]$ is said to be B-semireflexive if E = E□ algebraically.

If, in addition, $\boldsymbol{\tau} = \boldsymbol{\tau_r}$, we say that $E[\tau]$ is B-reflexive.

Let us denote

ℓ^∞ = The set of all bounded sequences $x=\{\xi_k\}$ of real or complex numbers.

c = The set of all convergent sequences x={ξ_k} of real or complex numbers.
c_0 = The set of all sequences x={ξ_k} of real or complex numbers which are convergent to 0.
ℓ^1 = The set of all sequences x={ξ_k} of real or complex numbers with $\sum_{k=1}^{\infty} \xi_k < \infty$.

ℓ^p ,1< p < ∞, = The set of all sequences x={ξ_k} of real or complex numbers for which $\sum_{k=1}^{\infty} |\xi_k|^p$ converges.

If a bounded sequence x={ξ_k}is considered as a coordinate vector x=(ξ_k), then the coordinate-wise addition and scalar multiplication i.e. for all x= {ξ_k}, y={η_k}∈ ℓ^∞ and α ∈K , x+y = {$\xi_k + \eta_k$} and αx = ={$\alpha\xi_k$}, define a vector space structure on ℓ^∞ (and on c, c_0, ℓ^p ,1≤ p < ∞, as well). Such vector spaces are known as sequence spaces. In subspace relationship, we have $\ell^1 \subset c_0 \subset c \subset \ell^\infty$. The (usual) norm on ℓ^∞ is $\| \|_\infty$ which is defined by $\| x \|_\infty$ = sup $|\xi_k|$. On ℓ^p ,1≤ p < ∞, the norm $\| \|_p$ is given by is $\| x \|_p = (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p}$. In particular, on ℓ^1 , the norm $\| \|_1$ is given by is $\| x \|_1 = \sum_{k=1}^{\infty} |\xi_k|$ and on ℓ^2 , the norm $\| \|_2$ is given by is $\| x \|_2 = (\sum_{k=1}^{\infty} |\xi_k|^2)^{1/2}$.

1.4. Following facts are well known:
(i) Each of ℓ^∞ , c, and c_0, equipped with the norm$\| \|_\infty$, is a (B)-space.
(ii) ℓ^p ,1≤ p < ∞,with the norm $\| x \|_p$ are (B)-spaces.

1.5. Further, we have the following dual relationships between ℓ^∞, c, and c_0, $\ell^1$and ℓ^p ,1< p< ∞:
$(\ell^1)' = \ell^\infty$; $(c_0)' = \ell^1$; $(c)' = \ell^1$ and for each p, 1< p< ∞, $(\ell^p)' = \ell^q$ where, 1/p + 1/q =1. For details of these results see [1], §14,7& 8.

1.6. It is observed that $(c_0)'' = (\ell^1)' = \ell^\infty$ therefore c_0 is not reflexive. Similarly $\ell^1$and ℓ^∞ are also nonreflexive. However for each p, 1< p< ∞, $(\ell^p)' = \ell^q$, where, 1/p + 1/q =1, and so, $(\ell^p)'' = (\ell^q)' = \ell^p$ and therefore each ℓ^p is a reflexive (B)-space.

In this paper, we discuss polar reflexivity, B-reflexivity, Inductive reflexivity on these sequence spaces considered with their normed topologies, and sometimes with weak or Mackey topology.

II. Results

We know that (F)-spaces are always polar reflexive ([1], §23,9(5)). So each of the (B)-spaces ℓ^∞ , c, and c_0 (equipped with the norm$\| \|_\infty$) and ℓ^p ,1≤ p < ∞ (equipped with the norm $\| \|_p$) is polar reflexive.

Let τ_p be the usual normed topology on the (B)- space ℓ^p , 1 < p < ∞, with respect to the norm $\| \|_p$ given by $\| x \|_p = (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p}$. Now we have the following assertion:

2.1. Theorem: For each p, 1 < p < ∞, the locally convex space $\ell^p[\tau_p]$ is both (i) B-reflexive, and (ii) inductively reflexive.
Proof: (i) It is already known that $\ell^p[\tau_p]$ is a reflexive (B)-space (see 1.6). So its strong dual ℓ^q **[**τ_b**(**ℓ^p**)]** , where 1/p + 1/q =1, is also a reflexive (B)-space. Thus $\ell^p[\tau_p]$ is a reflexive and its strong dual is bornological. Hence, by [3], theorem 17, $\ell^p[\tau_p]$ is B-semireflexive. Further, the fact that $\ell^p[\tau_p]$ is a reflexive (B)-space implies that for the unit ball S of $\ell^p[\tau_p]$, the polar S^o in the dual ℓ^q is a reflective set. Hence the reflective topology τ_r on ℓ^p is finer than the normed topology τ_p and consequently we have $\tau_p = \tau_r$. Hence $\ell^p[\tau_p]$ is B-reflexive.
(ii) Since B-semireflexivity implies inductive semi-reflexivity ([5], theorem 2.4), $\ell^p[\tau_p]$ is inductively semi-reflexive. Further, $\ell^p[\tau_p]$ is a (B)-space and so it is bornolgical. A locally convex space which is inductively semi-reflexive and bornological is inductively reflexive ([2], theorem 1.7]. Hence $\ell^p[\tau_p]$ is inductively reflexive.
In particular, for p = 2, we have

2.2. Corollary: The (B)-space ℓ^2 is both B-reflexive and inductively reflexive.

2.3. Theorem: The locally convex space ℓ^p $[\tau_s(\ell^q)]$, where $1< p< \infty$ and $1/p + 1/q = 1$, is semi-reflexive.

Proof: Consider the locally convex space ℓ^p $[\tau_s(\ell^q)]$. Its dual is ℓ^q . On this dual, the strong topology $\tau_b(\ell^p)]$ is nothing but the usual normed topology τ_q. Therefore, $(\ell^q [\tau_b(\ell^p)])' = (\ell^q [\tau_q])' = \ell^p$. It means ℓ^p $[\tau_s(\ell^q)]$ is semi-reflexive.

2.4. Corollary: ℓ^p $[\tau_s(\ell^q)]$, where $1< p< \infty$ and $1/p + 1/q = 1$, is polar semi-reflexive.

Proof: It follows from the fact that every semi-reflexive locally convex space is polar semi-reflexive ([1], §23,9(3)).

For $p = 2$, we obtain

2.5. Corollary: The locally convex space ℓ^2 $[\tau_s(\ell^2)]$ is semi-reflexive.

Though the (B)-space ℓ^1(with the norm topology) is nonreflexive, but if we consider the space ℓ^1 with the Mackey topology or the weak topology, then it holds some reflexivities as asserted in the following two theorems-

2.6. Theorem: The locally convex space ℓ^1 $[\tau_k(c_0)]$ is inductively semi-reflexive.

Proof: Consider the locally convex space ℓ^1 $[\tau_k(c_0)]$. Its dual is c_o. On this dual, the topology $(\tau_k(c_0))^*$ is nothing but the usual normed topology and therefore, $(c_o[(\tau_k(c_0))^*)' = (c_o)' = \ell^1$.

Hence ℓ^1 $[\tau_k(c_0)]$ is inductively semi-reflexive.

2.7. Corollary: The locally convex space ℓ^1 $[\tau_k(c_0)]$ is semi-reflexive.

Proof: Inductively semi-reflexive locally convex space is always semi-reflexive, by ([2], (1.6)).

2.8. Theorem: The locally convex space ℓ^1 $[\tau_s(c_0)]$ is semi-reflexive.

Proof: Consider the locally convex space ℓ^1 $[\tau_s(c_0)]$. We have $(\ell^1 [\tau_s(c_0)])' = c_0$. On this dual, the strong topology $\tau_b(\ell^1)$ is its norm topology. Therefore, $(c_0[\tau_b(\ell^1)])' = \ell^1$ (see 1.5). Hence $\ell^1[\tau_s(c_0)]$ is semi-reflexive.

Using the fact that semi-reflexivity implies polar semi-reflexivity, we have

2.9. Corollary: The locally convex space ℓ^1 $[\tau_s(c_0)]$ is polar semi-reflexive.

III. Conclusion

Each of the sequence space ℓ^p , $1< p< \infty$, (and , in particular, ℓ^2) is both B-reflexive and inductively reflexive. On the other hand, on the dual c_0 of $\ell^1[\tau_s(c_0)]$, the polar topology $(\tau_s(c_0))^o$ of uniform convergence on $\tau_s(c_0)$-precompact subsets of ℓ^1 is the usual normed topology. Now, in c_0, the set $S =[1, ½, …, 1/n, …\}$ is precompact for the normed topology and so $(\tau_s(c_0))^o$- precompact. But S is not finite dimensional. It implies that the topology $(\tau_s(c_0))^{oo}$ of uniform convergence on $\tau_s(c_0)^o$-precompact subsets of c_o is strictly finer than $\tau_s(c_0)$. Therefore, ℓ^1 $[\tau_s(c_0)]$ can't be polar reflexive.

References

[1]. G. Köthe, Topological vector spaces I (Springer-Berlin Heidelberg New Yark , 1983).

[2]. I. A. Berezanskij, Inductively reflexive, locally convex spaces, Sov. Math. Dokl. , 9, (1968) 1080-1082; Translation from Dokl. Akad. Nauk SSSR 182, (1968) 20-22.

[3]. P. K. Raman, On a Class of Reflexive Spaces Related to Ulam's conjecture on measurable cardinals, J.Reine Angew. Math. **245**, (1970), 188-200.

[4]. Kalman Brauner, Dual of Fréechet spaces and a generalization of Banach - Dieudonnée theorem, Duke Math J. 40,(1973), 845-855.

[5]. Atarsingh Meena, G.C. Dubey, Various reflexivities in locally convex spaces , Int. J. Pure.Appl. Math.Sci., Vol.8, No.1(2015), 31-36.