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Numerical Evaluation of Integrals with weight function x^k Using Gauss Legendre Quadrature Rules

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Abstract : *In this paper. Gauss Legendre quadrature have been applied for numerical solution of the integral of the form $\int_0^1 x^k\ f(x)dx$, where k is real number. We compare the numerical solutions with J. Ma, V. Rokhlin, S. Wandzura. et al. [8]. The performance of the method is illustrated with numerical examples.*
Key words: *Finite element method , Numerical Integration, Gauss Legendre Quadrature*

I. Introduction

Integration is of fundamental importance in both pure and applied mathematics as well as in several areas of science and engineering. Most such integrals cannot be evaluated explicitly or analytically and, with many others, it is often faster to integrate them numerically rather than evaluate them exactly using the complicated antiderivatives of the integrands. Construction of quadrature rules for numerical integration based on interpolating polynomials is done by many authors that these polynomials are used to find weights corresponding to nodes.

Gauss Legendre quadrature rules are extremely efficient when the functions to be integrated are well approximated by polynomials. However, the integrands in practical situations are not always polynomial but rational or irrational expressions of trigonometric, exponential, logarithmic etc. For which there is no order of Gaussian quadrature rules cannot evaluate these integrals exactly.

From the literate review we may observe that various numerical integration schemes, the use of Gaussian quadrature is attractive and it can evaluate exactly the (2n -1)th order polynomials with n Gauss integration points in [1] . Rational expressions for which there is no order of Gauss quadrature that will evaluate these integrals exactly in [1- 3]

The integration rule proposed in this paper require zeros of $P_{2n}(x)$ and computations of the associated weights. The integration points have to be increased in order to improve the integration accuracy. This method is applicable to a wide class of functions, functions with end point singularities, such as those encountered in the solution of integral equations, complex analysis, potential theory, and several other areas. The remainder of this paper is organized as follows. Section 2. presents the mathematical preliminaries required for understanding the derivation and also calculated the Gaussian Legendre quadrature nodes and weights of order N= 5,10,15,20,50,100. In Section 3. we provide the numerical results, with comparisons, error estimates and some illustrative examples.

II. Formulation of integrals with weight function x^k

2.1 If $k=\frac{1}{3}$ The Numerical integration of an arbitrary function f is given by

$$I_1=\int_0^1 \sqrt[3]{x}\ f(x)\,dx = \int_0^1 3\,t^3 f(t^3)dx = \sum_{k=0}^{m} 3\,W_k\,t_k^3\,f(t_k^3) = \sum_{k=0}^{m} 3\,W_k\,x_k^3\,f(x_k^3) \quad (1)$$

2.2 If $k=-\frac{1}{3}$ The Numerical integration of an arbitrary function f is given by

$$I_2=\int_0^1 \frac{f(x)}{\sqrt[3]{x}}\,dx = \int_0^1 3tf(t^3)dx = \sum_{k=0}^{m} 3\,W_k t_k\,f(t_k^3) = \sum_{k=0}^{m} 3\,W_k x_k\,f(x_k^3) \quad (2)$$

2.3 If $k=\frac{1}{4}$ The Numerical integration of an arbitrary function f is given by

$$I_3=\int_0^1 \sqrt[4]{x}\ f(x)\,dx = \int_0^1 4t^4 f(t^4)dx = \sum_{k=0}^{m} 4\,W_k t_k^4\,f(t_k^4) = \sum_{k=0}^{m} 4\,W_k x_k^4\,f(x_k^4) \quad (3)$$

2.4 If $k=-\frac{1}{4}$ The Numerical integration of an arbitrary function f is given by

$$I_4 = \int_0^1 \frac{f(x)}{\sqrt[4]{x}}\, dx = \int_0^1 4t^2 f(t^4)dx = \sum_{k=0}^{m} 4\, W_k t_k^2\, f(t_k^4) = \sum_{k=0}^{m} 4\, W_k x_k^2\, f(x_k^4) \quad (4)$$

2.5 If $k = \frac{2}{3}$ The Numerical integration of an arbitrary function f is given by

$$I_5 = \int_0^1 \sqrt[3]{x^2}\, f(x)dx = \int_0^1 3t^4 f(t^3)dx = \sum_{k=0}^{m} 3\, W_k t_k^4\, f(t_k^3) = \sum_{k=0}^{m} 3\, W_k x_k^4\, f(x_k^3) \quad (5)$$

2.6 If $k = -\frac{2}{3}$ The Numerical integration of an arbitrary function f is given by

$$I_6 = \int_0^1 \frac{f(x)}{\sqrt[3]{x^2}}\, dx = \int_0^1 3f(t^3)dx = \sum_{k=0}^{m} 3\, W_k\, f(t_k^3) = \sum_{k=0}^{m} 3\, W_k\, f(x_k^3)$$
(6)

We find out new Gaussian points x_k and weights coefficients W_k of various order N = 5,10,15,50,100 by using equation (1) - (6)and tabulated in Table. 6

III. Numerical Results

We have compared the Numerical results obtained using the proposed method with that of the results given in J.Ma et al. [8] and also estimated error of various order of N are tabulated in Table 1 – 5

Table 1: Integration by the Gauss Legendre quadrature for the weight function w(x) = $\sqrt[3]{x}$

$\int_0^1 x^{25} dx$ = 0.0384615384			
N	Computed value	Error	J.Ma et al. [8]
5	0.0261211427	0.0123403957	0.0246599205
10	0.0384554364	0.0000061020	0.0384268064
15	0.0384612467	$2.91700000 \times 10^{-7}$	0.0384615378
20	0.0384612949	$2.43500000 \times 10^{-7}$	0.0384615384
50	0.0384615382	$2.0000000 \times 10^{-10}$	----
100	0.0384615384	0.	----
$\int_0^1 \sin(10x)\, dx$ = 0.1839071529			
N	Computed value	Error	J.Ma et al. [8]
5	0.2114471549	0.0275400019	0.2036270126
10	0.1839087067	0.0000015537	0.1839063190
15	0.1839080319	8.7899999×10^{-7}	0.1839071529
20	0.1839075163	3.6339999×10^{-7}	0.1839071529
50	0.1839071531	$1.9999998 \times 10^{-10}$	----
100	0.1839071529	0.	----
$\int_0^1 x^{25} * \sqrt[3]{x}\, dx$ = 0.0379746835			
N	Computed value	Error	J.Ma et al. [8]
5	0.0254388538	0.0125358297	0.0239835389
10	0.0379676981	0.0000069853	0.0379609560
15	0.0379743917	2.918000×10^{-7}	0.0379746827
20	0.0379744397	2.438000×10^{-7}	0.0379746835
50	0.0379746833	2.00000×10^{-10}	----
100	0.0379746835	0.	----

Table 2: Integration by the Gauss Legendre quadrature for the weight function w(x) =$\frac{1}{\sqrt[3]{x}}$

$\int_0^1 x^{25} dx$ = 0.0384615384			
N	Computed value	Error	J.Ma et al. [8]
5	0.0261211169	0.0123404215	0.0213608283
10	0.0384554367	0.0000061017	0.0384022552
15	0.0384611950	3.434000×10^{-7}	0.0384615369
20	0.0384613204	2.180000×10^{-7}	0.0384615384
50	0.0384615381	3.000000×10^{-10}	----
100	0.0384615384	0.	----
$\int_0^1 \sin(10x)\, dx$ = 0.1839071529			
N	Computed value	Error	J.Ma et al. [8]
5	0.2114472876	0.0275401346	0.1814203018
10	0.1839087065	0.0000015535	0.1839066050
15	0.1839073998	2.469000×10^{-7}	0.1839071529
20	0.1839073771	2.241999×10^{-7}	0.1839071529
50	0.1839071527	2.00000×10^{-10}	----
100	0.1839071529	0	----
$\int_0^1 x^{25} * \frac{1}{\sqrt[3]{x}}\, dx$ = 0.0389610389			
N	Computed value	Error	J.Ma et al. [8]

5	0.0268217965	0.0121392424	0.0220351282
10	0.0389557324	0.0000053065	0.0389074113
15	0.0389606950	3.439000×10^{-7}	0.0389610378
20	0.0389608212	2.176999×10^{-7}	0.0389610389
50	0.0389610393	3.99999×10^{-10}	----
100	0.0389610389	0.	----

Table 3: Integration by the Gauss Legendre quadrature for the weight function w(x) =$\frac{1}{\sqrt[3]{x^2}}$

$\int_0^1 x^{25} dx$ = 0.0384615384			
N	Computed value	Error	J.Ma et al. [8]
5	0.0261211048	0.0123404336	0.0194693376
10	0.0384554363	0.0000061021	0.0383840696
15	0.0384612328	3.05600×10^{-7}	0.0384615361
20	0.0384613348	2.03600×10^{-7}	0.0384615384
50	0.0384615379	4.99999×10^{-10}	----
100	0.0384615384	0.	----
$\int_0^1 sin(10x)\ dx$ = 0.1839071529			
N	Computed value	Error	J.Ma et al. [8]
5	0.2114471346	0.0275399816	0.1561797519
10	0.1839087068	0.0000015538	0.1839074666
15	0.1839081830	0.0000010300	0.1839071529
20	0.1839081299	9.76999×10^{-7}	0.1839071529
50	0.1839071532	2.99999×10^{-10}	----
100	0.1839071529	0.	----
$\int_0^1 x^{25} * \frac{1}{\sqrt[3]{x^2}}\ dx$ = 0.0394736842			
N	Computed value	Error	J.Ma et al. [8]
5	0.0275413649	0.0119323192	0.0208006141
10	0.0394690901	0.0000045940	0.0394099645
15	0.0394733783	3.05899×10^{-7}	0.0394736828
20	0.0394734815	2.02700×10^{-7}	0.0394736842
50	0.0394736837	4.99999×10^{-10}	----
100	0.0394736842	0.	----

Table 4: Integration by the Gauss Legendre quadrature for the weight function w(x) =$\sqrt[4]{x}$

$\int_0^1 x^{25} dx$ = 0.0384615384		
N	Computed value	Error
5	0.0175100266	0.0006734625
10	0.0383607758	0.0001007622
15	0.0384618973	3.588999×10^{-7}
20	0.0384615186	1.9800×10^{-8}
50	0.0384615382	2.0000×10^{-10}
100	0.0384615384	0.
$\int_0^1 sin(10x)\ dx$ = 0.1839071529		
N	Computed value	Error
5	0.1212300990	0.0626770539
10	0.1839092618	0.0000021088
15	0.1839075472	3.942999×10^{-7}
20	0.1839077900	6.37099×10^{-7}
50	0.1839071525	4.00000×10^{-10}
100	0.1839071529	0.
$\int_0^1 x^{25} * \sqrt[4]{x}\ dx$ = 0.0380952380		
N	Computed value	Error
5	0.0170531092	0.0210421287
10	0.0379874689	0.0001077690
15	0.0380955961	3.5810000×10^{-7}
20	0.0380952166	2.139999×10^{-8}
50	0.0380952385	5.000000×10^{-10}
100	0.0380952380	0.

Table 5: Integration by the Gauss Legendre quadrature for the weight function w(x) = $\frac{1}{\sqrt[4]{x}}$

$\int_0^1 x^{25} dx$ = 0.0384615384		
N	Computed value	Error
5	0.0175099994	0.0209515390
10	0.0383607757	0.0001007626
15	0.0384619117	$3.73299999 \times 10^{-7}$
20	0.0384614906	$4.77999999 \times 10^{-8}$
50	0.0384615387	$2.99999997 \times 10^{-10}$
100	0.0384615384	0.
$\int_0^1 sin(10x)\, dx$ = 0.1839071529		
N	Computed value	Error
5	0.1212300204	0.0626771325
10	0.1839092618	0.00000210889
15	0.1839083548	0.00000120189
20	0.1839074737	3.207999×10^{-7}
50	0.1839071527	$2.0000001 \times 10^{-10}$
100	0.1839071529	0.
$\int_0^1 x^{25} * \frac{1}{\sqrt[4]{x}}\ dx$ = 0.0388349514		
N	Computed value	Error
5	0.0179791615	0.0208557898
10	0.0387408879	0.0000940634
15	0.0388353256	$3.74200000 \times 10^{-7}$
20	0.0388349055	4.5899999×10^{-8}
50	0.0388349517	$3.00000004 \times 10^{-10}$
100	0.0388349514	0.

IV. Conclusions

In this paper, numerical integration of the form $\int_0^1 x^k f(x)\, dx$ where $k = \pm\frac{1}{3}\ ,\ \pm\frac{1}{4}\ ,\ \pm\frac{2}{3}$ New Gaussian points and weight coefficients are calculated of order N = 5, 10,15,20,50,100. Most of the results obtained here are exact up to 10 decimal places and the proposed method can be used to integrate a wide class of functions including functions with end point singularities

Table:6	Weight function = $x^{\frac{1}{3}}$		Weight function = $x^{-\frac{1}{3}}$	
Order	x_k	W_k	x_k	W_k
N=5	0.0032995866	0.0029253233	0.0032995866	0.1319879208
	0.0814053362	0.0657592435	0.0814053362	0.3500968678
	0.3136136641	0.2061254307	0.3136136641	0.4465481129
	0.6473568730	0.2902450741	0.6473568730	0.3878546617
	0.9237444270	0.1847618482	0.9237444270	0.1947949724
N=10	0.0004481629	0.0002053752	0.0004481629	0.0350690559
	0.0118189864	0.0052892205	0.0118189864	0.1019384871
	0.0521903874	0.0222481530	0.0521903874	0.1593065437
	0.1333286723	0.0526736139	0.1333286723	0.2018261397
	0.2573246024	0.0912430828	0.2573246024	0.2255342013
	0.4157153163	0.1271217362	0.4157153163	0.2282211019
	0.5908367702	0.1476088835	0.5908367702	0.2096367816
	0.7591356313	0.1427297549	0.7591356313	0.1715148005
	0.8957630827	0.1091077857	0.8957630827	0.1174159156
	0.9795271216	0.0517601931	0.9795271216	0.0524789227
N=15	0.0001363669	0.0000420771	0.0001363669	0.0158820466
	0.0036430164	0.0011121661	0.0036430164	0.0469745103
	0.0165106491	0.0049330560	0.0165106491	0.0760804874
	0.0438766877	0.0126850229	0.0438766877	0.1019691269
	0.0893348669	0.0246892597	0.0893348669	0.1235456351
	0.1545292284	0.0402856712	0.1545292284	0.1398975728
	0.2389353093	0.0578863044	0.2389353093	0.1503334422
	0.3398499201	0.0751978863	0.3398499201	0.1544119303
	0.4525911184	0.0895780514	0.4525911184	0.1519605744
	0.5708900305	0.0984668091	0.5708900305	0.1430830619
	0.6874379622	0.0998215043	0.6874379622	0.1281548666
	0.7945374945	0.0924822397	0.7945374947	0.1078074419
	0.8847964536	0.0764058223	0.8847964536	0.0829018468
	0.9518002042	0.0527291649	0.9518002042	0.0544946286
	0.9907093735	0.0236824889	0.9907093735	0.0238303175

N=20	0.0000582865	0.0000135526	0.0000582865	0.0090152789
	0.0015642922	0.0003615383	0.0015642922	0.0268292870
	0.0071553154	0.0016337809	0.0071553154	0.0439988478
	0.0192816422	0.0043223562	0.0192816422	0.0601115431
	0.0399996145	0.0087463053	0.0399996145	0.0747803414
	0.0708444740	0.0150073350	0.0708444740	0.0876528938
	0.1127316464	0.0229675102	0.1127316464	0.0984199979
	0.1658918843	0.0322513797	0.1658918843	0.1068230249
	0.2298438595	0.0422725901	0.2298438595	0.1126601318
	0.3034057701	0.0522826721	0.3034057701	0.1157911104
	0.3847454252	0.0614375118	0.3847454252	0.1161407543
	0.4714661878	0.0688752803	0.4714661878	0.1137006668
	0.5607242209	0.0737984424	0.5607242209	0.1085294624
	0.6493707878	0.0755520209	0.6493707878	0.1007513607
	0.7341120036	0.0736905930	0.7341120036	0.0905532067
	0.8116774900	0.0680275191	0.8116774900	0.0781799938
	0.8789889108	0.0586615642	0.8789889108	0.0639290193
	0.9333194204	0.0459782780	0.9333194204	0.0481429280
	0.9724359264	0.0306267271	0.9724359264	0.0312027778
	0.9947224406	0.0134922473	0.9947224406	0.0135399278
N=50	0.0000038176	3.579636×10^{-7}	0.0000038176	0.0014654715
	0.0001029749	0.0000096461	0.0001029749	0.0043906885
	0.0004758048	0.0000444837	0.0004758048	0.0072987494
	0.0013017844	0.0001213484	0.0013017844	0.0101782913
	0.0027559618	0.0002558950	0.0027559618	0.0130180627
	0.0050072391	0.0004626445	0.0050072391	0.0158069674
	0.0082167115	0.0007546931	0.0082167115	0.0185341083
	0.0125360740	0.0011434433	0.0125360740	0.0211888293
	0.0181061128	0.0016383641	0.0181061128	0.0237607575
	0.0250552936	0.0022467818	0.0250552936	0.0262398431
	0.0334984591	0.0029737072	0.0334984591	0.0286163997
	0.0435356479	0.0038217007	0.0435356479	0.0308811409
	0.0552510450	0.0047907777	0.0552510450	0.0330252177
	0.0687120740	0.0058783566	0.0687120740	0.0350402522
	0.0839686375	0.0070792503	0.0839686375	0.0369183710
	0.1010525144	0.0083857008	0.1010525144	0.0386522356
	0.1199769193	0.0097874579	0.1199769196	0.0402350710
	0.1407362246	0.0112718988	0.1407362246	0.0416606925
	0.1633058565	0.0128241897	0.1633058565	0.0429235296
	0.1876423534	0.0144274844	0.1876423534	0.0440186480
	0.2136835986	0.0160631587	0.2136835982	0.0449417687
	0.2413492148	0.0177110762	0.2413492148	0.0456892846
	0.2705411313	0.0193498810	0.2705411313	0.0462582748
	0.3011442981	0.0209573151	0.3011442981	0.0466465162
	0.3330275618	0.0225105520	0.3330275618	0.0468524918
	0.3660446827	0.0239865446	0.3660446827	0.0468753966
	0.4000354894	0.0253623805	0.4000354894	0.0467151413
	0.4348271593	0.0266156390	0.4348271593	0.0463723519
	0.4702356141	0.0277247457	0.4702356141	0.0458483680
	0.5060670189	0.0286693134	0.5060670189	0.0451452368
	0.5421193710	0.0294304865	0.5421193713	0.0442657059
	0.5781841646	0.0299912114	0.5781841646	0.0432132119
	0.6140481174	0.0303365919	0.6140481174	0.0419918672
	0.6494949437	0.0304540795	0.6494949437	0.0406064442
	0.6843071586	0.0303337306	0.6843071586	0.0390623563
	0.7182678971	0.0299683863	0.7182678971	0.0373656368
	0.7511627332	0.0293538198	0.7511627332	0.0355229154
	0.7827814825	0.0284888412	0.7827814825	0.0335413925
	0.8129199698	0.0273753580	0.8129199698	0.0314288106
	0.8413817565	0.0260183911	0.8413817565	0.0291934245
	0.8679797952	0.0244260441	0.8679797952	0.0268439689
	0.8925380135	0.0226094288	0.8925380135	0.0243896242
	0.9148928065	0.0205825445	0.9148928065	0.0218399809
	0.9348944228	0.0183621167	0.9348944221	0.0192050023
	0.9524082324	0.0159673945	0.9524082324	0.0164949863
	0.9673158797	0.0134199119	0.9673158797	0.0137205268
	0.9795162856	0.0107432206	0.9795162856	0.0108924787
	0.9889265314	0.0079626150	0.9889265314	0.0080219453
	0.9954826711	0.0051050122	0.9954826711	0.0051204444
	0.9991414262	0.0022020112	0.9991414262	0.0022032725

References

[1]. O .C. Zienkiewicz, The Finite Element Method, 3rd. Edition, Mcgraw - Hill Inc., New York (1977).

[2]. W. B. Bickford. (1990). A First Course In The Finite Element Method, Irwin, Homewood, (1990).

[3]. G. Yogawa, G. W. Ye And S. Yoshimura. A Numerical Integration Scheme For Finite Element Method Based On Symbolic Manipulation, Int. J. Numer. Methods. Eng 29, Pp. 1539-1549, (1990).

[4]. M. Abramowicz And I.A. Stegun. Handbook Of Mathematical Functions. Dover Publications, Inc. New York (1965).

[5]. J. N. Reddy. An Introduction To The Finite Element Method. Tata Mcgraw- Hill, Third Edition (2005).

[6]. H. T. Rathod. Gauss Legendre Over A Triangle. J. Indian Institute Of Science., Vol.84, Pp.183-188, (2004).

[7]. Farzana Hussain , M.S. Karim, Accurate Evaluation Schemes For Triangular Domain Integrals. Journal Of Mechanical And Civil Engineering, Vol.2, Pp. 38-51, (2012),

[8]. J. Ma, V. Rokhlin, S. Wandzura, Generalized Gaussian Quadrature Rules For Systems Of Arbitrary Functions, Siam, J. Numer. Anal. Vol. 33, No. 3,Pp. 971-996, (1996)

[9]. K. T. Shivaram. Generalised Gaussian Quadrature Over Triangle, American Journals Of Engineering Research..Vol.2,Pp.290-293 (2013),

[10]. Bharath Rathod, K.T. Shivaram And H.T. Rathod, Some Generalized Gaussian Quadrature Rule For Integrands With Singularities.

[11]. International E-Journal Of Numerical Analysis And Related Topic, Vol.6,Pp.12-36, .(2011),

[12]. K.T.Shivaram, Numerical Integration Over An Arbitrary Rectangle And Square Region Using Generalised Gaussian Quadrature Rules,

[13]. International Journal Of Mathematical Archive, Vol.5(5), May 2014.